Inverse semigroups and inductive groupoids

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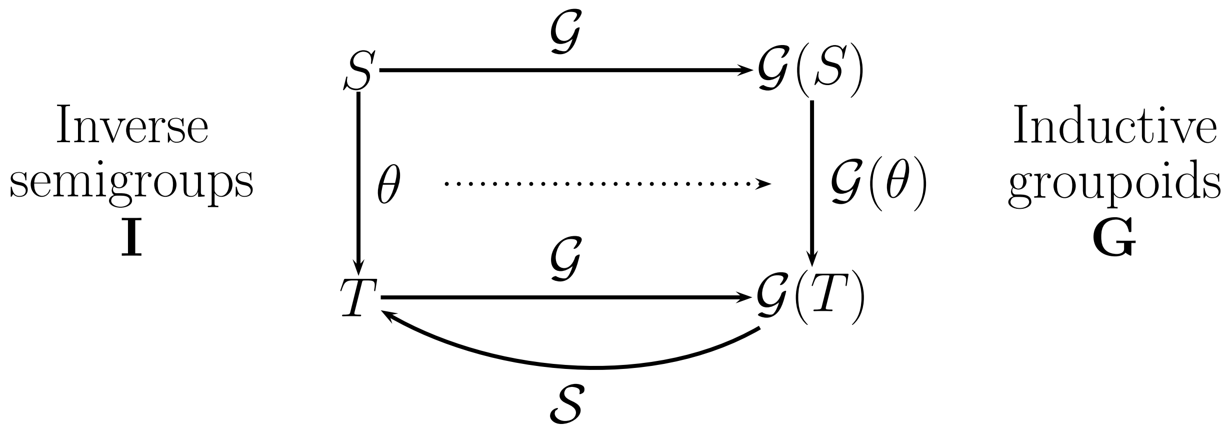
The Ehresmann-Schein-Nambooripad Theorem

The category **I** *of inverse semigroups and morphisms is isomorphic to the category* **G** *of inductive groupoids and inductive functors.*

We have mutually inverse functors

$$\mathcal{G} : \mathbf{I} \to \mathbf{G} \text{ and } \mathcal{S} : \mathbf{G} \to \mathbf{I}.$$



Inverse semigroups

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$$a = aa'a \text{ and } a' = a'aa'.$$

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Let S ***be inverse***

Fact 1 Let $a \in S$; then $aa', a'a \in E(S) = \{e \in S : e = e^2\}$ and

$$a \,\mathcal{R}\, aa', a \,\mathcal{L}\, a'a.$$

Fact 2 For all $e, f \in E(S)$, $ef = fe$ so $E(S)$ is a semilattice (commutative semigroup of idempotents), partially ordered by

$$e \leq f \Leftrightarrow e = ef;\ \ gh = g \wedge h.$$

Fact 3 $(ab)' = b'a'$.

Fact 4 Wagner-Preston Representation There exists an embedding $\theta : S \longrightarrow \mathcal{I}_S$.
Specifically For any $s \in S$ we put $s\theta = \rho_s$ where $\rho_s \in \mathcal{I}_S$ is given by

$$\text{dom } \rho_s = Sss', \; x\rho_s = xs \text{ for all } x \in \text{ dom } \rho_s.$$

Fact 5 The semigroup S is ***partially ordered*** by $\leq$ where

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S).$$

In fact, the definition of $\leq$ is not left biassed:

$$a \leq b \Leftrightarrow a = bf \text{ for some } f \in E(S) \Leftrightarrow a = aa'b \Leftrightarrow a = ba'a.$$

Fact 6 $\leq$ is a ***natural partial order*** by which we mean

$$a \leq b, c \leq d \Rightarrow ac \leq bd$$

and $\leq$ restricts to the usual partial order on $E(S)$.

Fact 7 In $\mathcal{I}_X$,

$$\alpha \leq \beta \text{ if and only if } \alpha \subseteq \beta.$$

Proof

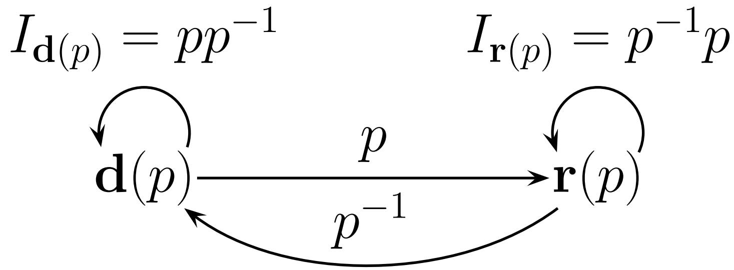
$$\alpha \leq \beta \Leftrightarrow \alpha = \alpha\alpha'\beta \Leftrightarrow \alpha\alpha^{-1}\beta \Leftrightarrow \alpha = I_{\mathrm{dom}\,\alpha}\beta \Leftrightarrow \alpha \subseteq \beta.$$

Fact 8 (a) An inverse semigroup S with $E(S) = S$ is ***precisely*** a semilattice;
(b) An inverse semigroup S with $|E(S)| = 1$ is ***precisely*** a group.

Groupoids

A ***groupoid*** is a category G in which for every $p \in \operatorname{Mor} G$ we have $p^{-1} \in \operatorname{Mor} G$ with

$$pp^{-1} = I_{\mathbf{d}(p)} \text{ and } p^{-1}p = I_{\mathbf{r}(p)}.$$



A one-object small groupoid is precisely a group.

Inverse semigroups → Groupoids

Let S be an inverse semigroup. We construct a groupoid $\mathcal{G}(S)$ from S as follows:

$$\operatorname{Mor}\mathcal{G}(S) = S, \;\; \operatorname{Ob}\mathcal{G}(S) = E(S)$$

and for $a \in \operatorname{Mor}\mathcal{G}(S)$,

$$\mathbf{d}(a) = aa', \mathbf{r}(a) = a'a$$

and when $\mathbf{r}(a) = \mathbf{d}(b)$,

$$a \cdot b = ab.$$

It is then easy to check $\mathcal{G}(S)$ is a groupoid, the ***trace groupoid of*** S***,*** with

$$I_{\mathbf{d}(a)} = aa', \; I_{\mathbf{r}(a)} = a'a \text{ and } a^{-1} = a'.$$

In a small category C we cease to distinguish between an object α and I_α, and we identify C with $\operatorname{Mor} C$. We may write $C = (C, \cdot)$ as we are thinking of C as a set with a partial binary operation.

Some justification for $\mathcal{G}(S) = (S, \cdot)$ being a groupoid

Suppose $\exists a \cdot b$. Then $a'a = bb'$. We have

$$\mathbf{d}(a \cdot b) = \mathbf{d}(ab) = (ab)(ab)' = abb'a' = aa'aa' = aa' = \mathbf{d}(a)$$

and similarly $\mathbf{r}(a \cdot b) = \mathbf{r}(b)$. So,

$$\exists (a \cdot b) \cdot c \Leftrightarrow \exists a \cdot (b \cdot c).$$

It is then easy to see we have a s. groupoid, with

$$I_{\mathbf{d}(a)} = aa', I_{\mathbf{r}(a)} = a'a, a^{-1} = a'.$$

Groupoids → Inverse semigroups

Let $G = (G, \cdot)$ be a small groupoid. We construct an inverse semigroup S as follows:

$S = G \cup \{0\}$; the binary operation in S is obtained from that in G by declaring all undefined products to be 0. Then S is an inverse semigroup with 0 such that $E(S) = E(G) \cup \{0\}$ and $a' = a^{-1}$ $(a \in G)$ and $0' = 0$.

Groupoids → Inverse semigroups

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Problem The semigroup S is ***primitive***, that is, for $e, f \in E(S)$,

$$0 \neq e \leq f \text{ implies that } e = f.$$

For, suppose that $0 \neq e = ef$. Then $e, f \in E(G)$ and $\exists e \cdot f$ in G, so that $e = f$.

Any primitive inverse semigroup is a 0-disjoint union of 0-simple primitive inverse semigroups, or ***Brandt semigroups***.

Inductive Groupoids

Let $G = (G, \cdot)$ be a small groupoid with $E(G) = E$. Suppose that $\leq$ is a partial order on G. Suppose also:

(1) $x \leq y$ implies that $x^{-1} \leq y^{-1}$;

(2) $x \leq y, u \leq v, \exists xu, \exists yv$ implies that $xu \leq yv$;

(3) if $a \in G$ and $e \in E$ with $e \leq \mathbf{d}(a)$, then there exists a unique ***restriction*** $(e|a) \in G$ with $\mathbf{d}(e|a) = e$ and $(e|a) \leq a$;

(4) if $a \in G$ and $e \in E$ with $e \leq \mathbf{r}(a)$, then there exists a unique ***co-restriction*** $(a|e) \in G$ with $\mathbf{r}(a|e) = e$ and $(a|e) \leq a$.

Then $G = (G, \cdot, \leq)$ is called an ***ordered groupoid***. If in addition

(5) E is a semilattice

then G is an ***inductive groupoid***.

Inverse semigroups → Inductive groupoids

Let S be an ***inverse semigroup*** with usual partial ordering $\leq$. Let $\mathcal{G}(S) = (S, \cdot)$ be the trace groupoid of S defined as above. Then $\mathcal{G}(S) = (S, \cdot\,, \leq)$ is an ***inductive groupoid*** with

$$(e|a) = ea, \ \ (a|e) = ae.$$

Proof Straightforward.

Inductive groupoids $\rightarrow$ Inverse semigroups

Let $G = (G, \cdot, \leq)$ be an inductive groupoid. We define a ***pseudo-product*** $\otimes$ on G by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Then $\mathcal{S}(G) = (G, \otimes)$ is an inverse semigroup (having the same partial order as G).

Proof Tricky - difficulty is showing associativity of $\otimes$; see Lawson *Inverse semigroups*, World Scientific (1998).

The category $\mathbf{G}$

The ***objects*** of $\mathbf{G}$ are inductive groupoids.

A ***morphism*** $\theta : G \longrightarrow H$ of $\mathbf{G}$ is a functor such that for any $u, v \in G$,

$$u \leq v \text{ in } G \text{ implies that } u\theta \leq v\theta \text{ in } H$$

and for any $e, f \in E(G)$,

$$(e \wedge f)\theta = e\theta \wedge f\theta.$$

The Ehresmann-Schein-Nambooripad Theorem

The category $\mathbf{I}$ *of inverse semigroups and morphisms is isomorphic to the category* $\mathbf{G}$ *of inductive groupoids and inductive functors.*

For $\theta : S \to T$ in $\mathbf{I}$ and $F : G \to H$ in $\mathbf{G}$ put $\mathcal{G}(\theta) = \theta$ and $\mathcal{S}(F) = F$ (on the underlying sets).

We have mutually inverse functors $\mathcal{G} : \mathbf{I} \to \mathbf{G}$ and $\mathcal{S} : \mathbf{G} \to \mathbf{I}$.