Semigroup Theory
A suite of exercises

1. First set

Questions 1-6 are entirely routine, and you can get cracking with them just by knowing the definition of a semigroup and monoid. It is important you know the definition of a left (right) identity and of a left (right) zero. Questions 7-10 use ideas introduced in the first couple of lectures, but are not difficult. Questions 11-15 also only need the definition of a semigroup but are there to give you something to get your teeth into!

(1) Let M be a monoid. Show that the identity of M is unique.

(2) A semigroup S has a *zero* z if $z \in S$ satisfies

$$zs = z = sz$$

for all $s \in S$. Show that any semigroup has at most one zero.

Give an example of

(a) a semigroup with no identity and no zero;
(b) a semigroup with no identity but with a zero;
(c) a monoid with no zero;
(d) a monoid with zero.

(3) Let M be a monoid with identity 1 and zero z. Show that *either* M is the *trivial* semigroup/monoid/group or $z \neq 1$.

(4) Let S be a semigroup. An element $e \in S$ is a *left (right) identity* if $es = s$ $(se = s)$ for all $s \in S$. An element $a \in S$ is a *left (right) zero* if

$$ab = a \quad (ba = a)$$

for all $b \in S$. If all elements of S are left (right) zeroes, we say that S is a *left (right) zero semigroup*. Show that a semigroup is a left zero semigroup if and only if it consists entirely of right identities. Verify that the rectangular band $T = I \times J$ is a left zero semigroup if and only if $|J| = 1$. Find a necessary and sufficient condition for a semigroup to be a left and a right zero semigroup.

(5) Show that a rectangular band $T = I \times J$ is a monoid if and only if it is trivial.

(6) An element e of a semigroup S is *idempotent* if $e^2 = e$. Show that if e is idempotent, then $e^n = e$ for all $n \in \mathbb{N}$.

We denote by $E(S)$ the set of idempotents of S.

(7) Show that

$$E(B) = \{(a, a) : a \in \mathbb{N}^0\}.$$

(8) Let S be a semigroup. We say that S *satisfies (the identity)*

$$x = xyx$$

if

$$a = aba \text{ for all } a, b \in S.$$

Show that if S satisfies $x = xyx$, then $E(S) = S$.
Show that any rectangular band satisfies $x = xyx$.

(9) Show that if $\alpha \in \mathcal{T}_n$ then the following are equivalent:
(i) α is one-one;
(ii) α is a bijection;
(iii) α is onto.
Give an example of functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ such that f_1 is one-one but not onto, and f_2 is onto but not one-one.

(10) For $\alpha \in \mathcal{T}_n$, show that $\alpha \in \mathcal{S}_n$ if and only if the map diagram of α has no tails.

(11) For an element a of a semigroup S we define

$$aS = \{as : s \in S\} \text{ and } Sa = \{sa : s \in S\}$$

(more information will be given later about such subsets). Show that S is a group if and only if $aS = S = Sa$ for all $a \in S$.

(12) A semigroup S is *right reversible* if for any $a, b \in S$, there exist $c, d \in S$ with $ca = db$.
Let S be a semigroup and let G be a group. Then G is a *group of left quotients* of S if S is a subsemigroup of G and any $g \in G$ can be written as $g = a^{-1}b$ for some $a, b \in S$. Show that if S has a group of left quotients, then S is cancellative and right reversible. *Your answer should be only a few lines if you spot what to do...*

(13) If S is a right reversible, cancellative semigroup, does S have a group of left quotients? *This is tricky...think how rationals are constructed from integers...*

(14) Extra: show that $\mathcal{B}(X)$ (the semigroup of binary relations on a set) is a semigroup under the composition

$$\rho \circ \sigma = \{(u, v) : \exists w \in X \text{ such that } (u, w) \in \rho \text{ and } (w, v) \in \sigma\}.$$

(15) Extra: show that if S is a finite monoid, then $ab = 1$ implies $ba = 1$. Find an infinite monoid in which $ab = 1$ but $ba \neq 1$

2. Second set

(1) Show that any group G is embedded into $\mathcal{S}_G$.

(2) How many elements are there in $\mathcal{T}_n$, where $n \in \mathbb{N}$?
Write down the elements of $\mathcal{T}_1$ and $\mathcal{T}_2$ and the multiplication tables of these monoids.
For which X is $\mathcal{T}_X$ commutative?

(3) Show that a subsemigroup of a finite *group* is a sub*group*.

(4) Let S and T be semigroups.

(a) On the set $S \times T$ a binary operation is defined by

$$(s,t)(s',t') = (ss',tt').$$

Show that $S \times T$ is a semigroup and if S and T are monoids, then $S \times T$ is a monoid.

The semigroup (monoid) $S \times T$ is called the direct product *of S and T. It is NOT the same as the rectangular band on the set $S \times T$.*

(b) For any set X, $\mathcal{P}(X)$ denotes the set of subsets of X. Show that $\mathcal{P}(T)$ becomes a semigroup under

$$UV = \{uv : u \in U, v \in V\},$$

the *power semigroup* of T.

Show that $\mathcal{P}(T)$ has a zero and $\mathcal{P}(T)$ is a monoid if and only if T is.

(c) Suppose now $\alpha : S \to \mathcal{P}(T)$ is a morphism such that $s\alpha \neq \emptyset$ for at least one $s \in S$. Let

$$P = \{(s,t) \in S \times T : t \in s\alpha\}.$$

Show that P is a subsemigroup of $S \times T$.

(5) Let $\alpha \in \mathcal{T}_{10}$ be given by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 6 & 7 & 4 & 9 & 10 & 10 \end{pmatrix}.$$

Write down the map diagram of α. Explain why α has index 3 and period 4. Write down the multiplication table for $\langle \alpha \rangle$. Use this to explain why

$$\{\alpha^3, \alpha^4, \alpha^5, \alpha^6\}$$

is a group.

(6) Let S be a semigroup and let $a \in S$ be such that $\langle a \rangle$ is finite. Let n be the index of a and let r be the period.

(a) Let $s \in \{0, \dots, r-1\}$ be chosen with $s \equiv -n \pmod{r}$, so that a^{n+s} idempotent. Show that

$$G = \{a^n, a^{n+1}, \dots, a^{n+r-1}\}$$

is a group with identity a^{n+s}.

(b) Show that G is cyclic.

(7) Consider the functions $f, g : \mathbb{N}^0 \to \mathbb{N}^0$ given by

$$xf = x + 1 \text{ for all } x \in \mathbb{N}^0$$

and

$$xg = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } x \geq 1. \end{cases}$$

Show that

(a) $fg = I_{\mathbb{N}^0}$;
(b) (with the convention that $f^0 = g^0 = I_{\mathbb{N}^0}$) we have $f^n g^n = I_{\mathbb{N}^0}$ for any $n \in \mathbb{N}^0$;
(c) for any $n \in \mathbb{N}^0$, f^n is given by

$$xf^n = x + n$$

and g^n is given by

$$xg^n = \overline{x - n}$$

where for any integer z,

$$\overline{z} = \begin{cases} z & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

(d) for any $m, n, a, b \in \mathbb{N}$,

$$g^m f^n = g^a f^b$$

if and only if $m = a$ and $n = b$;
(e) $S = \{g^m f^n : m, n \in \mathbb{N}^0\}$ is a submonoid of $\mathcal{T}_{\mathbb{N}^0}$;
(f) $\alpha : S \to B$ given by

$$g^m f^n = (m, n)$$

is an isomorphism (consequently, α^{-1} is an embedding of B into $\mathcal{T}_{\mathbb{N}^0}$.)

3. Third set

(1) Describe the map diagrams of idempotents of $\mathcal{T}_n$.

(2) Show that the idempotents of $\mathcal{T}_3$ do not form a submonoid. Extend this result to $\mathcal{T}_X$ for any X with $|X| > 2$.

(3) Let $\leq$ be a partial order on a non-empty set X. Elements $a, b \in X$ have a *greatest lower bound* $z \in X$ if

$$z \leq a \text{ and } z \leq b$$

and if for any $t \in X$,

$$(t \leq a \text{ and } t \leq b) \Rightarrow t \leq z.$$

Explain why if a and b have a greatest lower bound, it is unique.

The partially ordered set X is called an *order semilattice* if every pair of elements has a greatest lower bound. We denote the greatest lower bound of a and b by $a \wedge b$. Show that if X is an order semilattice, then it is a (semigroup) semilattice (that is, a commutative semigroup of idempotents) under $\wedge$.

Conversely, suppose that S is a (semigroup) semilattice. Define a relation $\leq$ on S by the rule that

$$e \leq f \text{ if and only if } ef = e.$$

Show that $\leq$ is a partial order and under this partial order, S is an order semilattice with $e \wedge f = ef$.

The above allows us to move without ambiguity between the concepts of semigroup semilattice and order semilattice. We usually use the former, but make use of the partial ordering.

What is the partial order associated with $E(B)$?

(4) Let S be any semigroup, and define a relation $\leq$ on $E(S)$ by the rule that

$$e \leq f \text{ if and only if } ef = e = fe.$$

Show that $\leq$ is a partial order.

(5) A relation ρ on a semigroup S is *left (right) compatible* if

$$a \,\rho\, b \Rightarrow ca \,\rho\, cb \; (ac \,\rho\, bc)$$

for all $a, b, c \in S$. A left (right) compatible equivalence relation is called a *left (right) congruence*.

Show that a relation ρ on a semigroup S is a congruence if and only if it is a left congruence and a right congruence.

(6) Let G be a group and let N be a normal subgroup of G. Define a relation ρ_N by the rule that for $a, b \in G$,

$$a \,\rho_N\, b \Leftrightarrow a^{-1}b \in N.$$

Show that ρ_N is a congruence on G.

Let ρ be a congruence on G. Show that

$$N_\rho = \{g \in G : g \,\rho\, 1\}$$

is a normal subgroup of G (so the normal subgroups of a group G correspond to congruences on G).

(7) Let ρ be the congruence on the Bicyclic semigroup given by

$$(a, b) \,\rho\, (c, d) \text{ if and only if } a - b = c - d.$$

Show that

$$(a, b) \,\rho\, (c, d) \text{ if and only if } (u, u)(a, b) = (u, u)(c, d)$$

for some $(u, u) \in B$.

(8) Let X be a non-empty set and let X^+ be the following semigroup:

$$X^+ = \{(x_1, \ldots, x_n) : n \in \mathbb{N}, x_i \in X\}$$

with operation

$$(x_1, \ldots, x_n)(y_1, \ldots, y_m) = (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

We call X^+ the *free* semigroup on X.

Let S be any semigroup. Show that there is a free semigroup X^+ and an onto morphism $\phi : X^+ \to S$. *Hint: let $X = S$.*

4. Fourth set

(1) Show that G^0 is 0-simple for any group G.

(2) Show that if I is an ideal of a monoid M, then $I = M$ if and only if $1 \in I$.

(3) Let S be a semigroup and let $n \in \mathbb{N}$. Show that S^n is an ideal of S. Explain why $S^n = S$ for any monoid S and any $n \in \mathbb{N}$.

With $\mathbb{N}$ the natural numbers under addition, show that for $n \in \mathbb{N}$, $\mathbb{N}^n = I_n$ where $I_n = \{n, n+1, \ldots\}$.

(4) Let $T = I \times J$ be a rectangular band.

(a) Show that for any $(i, j) \in T$,

$$(i, j)T = \{(i, k) : k \in J\},$$

$$T(i, j) = \{(l, j) : l \in I\}.$$

Determine the relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{H}$ on T.

(b) Show that T is simple.

(5) Recall that a semigroup S with zero 0 is 0-*simple* if $\{0\}$ and S are the only ideals of S, and $S^2 \neq \{0\}$.

A semigroup S with zero is *null* if $S^2 = \{0\}$. Show that if S is null then any subset containing zero is an ideal.

Deduce that if S is a semigroup with zero and $\{0\}$ and S are the only ideals, then either S is 0-simple or trivial or the two element semigroup with zero $\{a, 0\}$ such that $a^2 = 0$.

(6) Show that every right ideal of B is principal.

(7) Let S be a semigroup with zero 0. Show that $\{0\}$ is a $\mathcal{K}$-class where $K = R, L$ or H.

(8) Let $\alpha, \beta \in \mathcal{T}_X$. Show that

$$\mathcal{T}_X\alpha \subseteq \mathcal{T}_X\beta$$

if and only if

$$\text{Im }\alpha \subseteq \text{ Im}\beta.$$

Hence show that $\alpha \,\mathcal{L}\, \beta$ if and only if $\text{Im }\alpha = \text{Im }\beta$.

(9) Let e, f be idempotents in a semigroup S. Show that

$$e \,\mathcal{L}\, f \text{ if and only if } ef = e \text{ and } fe = f,$$

and

$$e \,\mathcal{R}\, f \text{ if and only if } ef = f \text{ and } fe = e.$$

Suppose that $ef = fe$ for all $e, f \in E(S)$. Explain why any $\mathcal{L}$-class contains at most one idempotent; similarly for $\mathcal{R}$-classes.

(10) For any ideal I of a semigroup S, define a relation $\sim_I$ by the rule that

$$a \sim_I b \Leftrightarrow a = b \text{ or } a, b \in I.$$

Show that $\sim_I$ is a congruence on S. Describe the semigroup $S/\sim_I$.

5. Fifth set

(1) (a) Let S be a band. Explain why $\mathcal{H} = \iota$.
(b) Explain why any semigroup for which $\mathcal{H} = \iota$ has only trivial subgroups.

(2) Let S be a finite semigroup.
(a) Show that S has only trivial subgroups if and only if the period of every element in S is 1.
(b) (Harder) Show that S has only trivial subgroups if and only if $\mathcal{H} = \iota$.

(3) Let $M_2(\mathbb{R})$ be the monoid of 2×2 matrices over $\mathbb{R}$ with operation matrix multiplication. Show that

$$G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \neq 0 \right\}$$

is a maximum subgroup of $M_2(\mathbb{R})$. Is G a submonoid of $M_2(\mathbb{R})$?

(4) Let $\alpha \in \mathcal{T}_n$ and $\beta \in \mathcal{S}_n$. Show that $\alpha \,\mathcal{R}\, \alpha\beta$. Give an example such that $\alpha \not\mathcal{H} \alpha\beta$.

(5) Let S be a semigroup such that $aba = a$ for all $a, b \in S$. Show that
(a) $\mathcal{H}$ is equality on S;
(b) for all $a, b \in S$,

$$a \,\mathcal{R}\, ab \,\mathcal{L}\, b;$$

(c) for all $a, b, c \in S$, $abc = ac$;
(d) with $R = \{R_a : a \in S\}$ and $L = \{L_a : a \in S\}$, and T the rectangular band on $R \times L$,

$$\phi : T \to S$$

given by

$$(R_a, L_b)\phi = ab$$

is an isomorphism.

(6) (a) Let $\phi : S \to G$ be a semigroup morphism from a semigroup S to a group G. Show that (if $E(S) \neq \emptyset$), then $E(S)\phi = \{1\}$, where 1 is the identity of G.
(b) Let $\theta : B \to \mathbb{Z}$ be the morphism given by $(a, b)\theta = a - b$ (see your notes). Explain why an earlier exercise gives that

$$(a, b) \text{ Ker } \theta \, (c, d) \Leftrightarrow (u, u)(a, b) = (u, u)(c, d)$$

for some $(u, u) \in E(B)$.
Suppose now that $\phi : B \to G$ is a morphism, where G is a group. Show that there exists a morphism $\psi : \mathbb{Z} \to G$ such that $\theta\psi = \phi$.

6. Sixth set

(1) Which of the following elements are in a subgroup of $\mathcal{T}_6$? For those that are, write down the elements (using good notation) of the maximum subgroup in which they lie, and the group table.

(a) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 3 & 6 & 6 \end{pmatrix}$
(b) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 4 & 4 & 4 \end{pmatrix}$
(c) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 3 & 3 & 6 \end{pmatrix}$

(2) Show that if S is a semigroup with zero 0, then $\{0\}$ is a $\mathcal{D}$-class and a $\mathcal{J}$-class.

(3) Let S be a commutative semigroup. Show that in S,
$$\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J}.$$

(4) Show that if S has a zero 0 then S has the property that its only ideals are $\{0\}$ and S if and only if the $\mathcal{J}$-classes are $\{0\}$ and $S \setminus \{0\}$.

Definition A semigroup S is *bisimple* if $\mathcal{D} = S \times S$.

(5) Let B_2 be the subset of B defined by
$$B_2 = \{(m, n) : m \equiv n \pmod 2\}.$$
(a) Show that B_2 is a submonoid of B.
(b) Argue directly that for any $(m, n), (p, q) \in B_2$,
$$(m, n)\,\mathcal{R}\,(p, q) \text{ in } B_2 \Leftrightarrow m = p$$
and
$$(m, n)\,\mathcal{L}\,(p, q) \text{ in } B_2 \Leftrightarrow n = q.$$
(c) Show that B_2 is simple.
(d) Show that B_2 is not bisimple (so that in B_2, $\mathcal{D} \neq \mathcal{J}$). How many $\mathcal{D}$-classes are there?
(e) Could you conjecture a simple semigroup with n $\mathcal{D}$-classes, for a given $n \in \mathbb{N}$?

(6) Consider the full transformation monoid $\mathcal{T}_n$. This question describes all the ideals of $\mathcal{T}_n$. It is useful to define the *rank* ρ of $\alpha \in \mathcal{T}_n$ by
$$\rho(\alpha) = |\mathrm{Im}\ \alpha|.$$
For any $k \in \{1, \ldots, n\}$, define
$$I_k = \{\alpha \in \mathcal{T}_n : \rho(\alpha) \leq k\}.$$
(a) What are the elements of I_1? What are the elements of $I_n \setminus I_{n-1}$ (set difference)?

(b) Show that for any $\alpha, \beta \in \mathcal{T}_n$,

$$\rho(\alpha\beta) \leq \rho(\alpha) \text{ and } \rho(\alpha\beta) \leq \rho(\beta).$$

(c) Show that for any $k \in \{1, \ldots, n\}$, I_k is an ideal of $\mathcal{T}_n$.

(d) Suppose now that J is an ideal of $\mathcal{T}_n$. Pick $\alpha \in J$ with maximum rank. Show that $J = I_r$, where $r = \rho(\alpha)$.

(e) Is $\mathcal{T}_n$ simple?

(f) Using the above, and the description of $\mathcal{R}$ and $\mathcal{L}$ in $\mathcal{T}_n$, show that for any $\alpha, \beta \in \mathcal{T}_n$, we have

$$\alpha\,\mathcal{D}\,\beta \Leftrightarrow \alpha\,\mathcal{J}\,\beta \Leftrightarrow \rho(\alpha) = \rho(\beta).$$

(7) Let S be a commutative semigroup and let $L \subseteq S$. The relation $\sim_L$ is defined on S by the rule that $a \sim_L b$ if and only if for all $x \in S^1$,

$$xa \in L \Leftrightarrow xb \in L.$$

Assuming that $\sim_L$ is an equivalence, show that $\sim_L$ is a congruence on S.

If S is a group and L is a subgroup of S, what are the congruence classes of $\sim_L$?

7. Seventh set

(1) (a) Let $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup. Suppose that $p_{\lambda i} \neq 0$. We know that

$$H_{i\,\lambda} = \{(i, a, \lambda) : a \in G\}$$

is the $\mathcal{H}$-class of the idempotent $(i, p_{\lambda i}^{-1}, \lambda)$. Show (without use of the Maximum Subgroup Theorem) that $H_{i\,\lambda}$ is a subgroup of $\mathcal{M}^0$.

(b) Let $g \in G$ and define $\rho_g : G \to G$ by $x\rho_g = xg$. Verify that ρ_g is a bijection. *You may now assume that dually, $\lambda_g : G \to G$ given by $x\lambda_g = gx$ is a bijection.*

(c) Suppose now that $p_{\lambda i} \neq 0$ and $p_{\mu j} \neq 0$, so that

$$H_{j\,\mu} = \{(j, a, \mu) : a \in G\}$$

is also a subgroup of $\mathcal{M}^0$. Show that $H_{i\,\lambda}$ and $H_{j\,\mu}$ are isomorphic.

(2) Show that if a is an element of a semigroup S and $a^m\,\mathcal{L}\,a^{m+1}$ for some $m \geq 1$, then $a^m\,\mathcal{L}\,a^{m+t}$ for all $t \geq 0$. Now show the following conditions are equivalent:

(i) for every $a \in S$ there exists $m, n \in \mathbb{N}$ such that $a^m\,\mathcal{L}\,a^{m+1}$ and $a^n\,\mathcal{R}\,a^{n+1}$;

(ii) for every $a \in S$ there exists $m \in \mathbb{N}$ such that $a^m\,\mathcal{L}\,a^{m+1}$ and $a^m\,\mathcal{R}\,a^{m+1}$;

(iii) for every $a \in S$, there exists $m \in \mathbb{N}$ such that $a^m\,\mathcal{H}\,a^{m+1}$;

(iv) for every $a \in S$, there exists $m \in \mathbb{N}$ such that a^m lies in a subgroup.

The latter property is called *group bound*. Note therefore that any semigroup with M_R and M_L (see notes for definition) is group bound.

(3) Explain why the rectangular band $T = I \times J$ has all four chain conditions M_L, M_R, M^L and M^R.

(4) Show that the following conditions are equivalent for a band B:
  (a) B is simple;
  (b) B is bisimple;
  (c) B is isomorphic to a rectangular band.

Hint: you know that a semigroup S is isomorphic to a rectangular band if and only if $aba = a$ for all $a, b \in S$.

(5) Let B be a band.
  (a) Explain why for a band B, $\mathcal{J} = \mathcal{D}$.
  (b) Show that for any $a, b \in B$, $ab \,\mathcal{J}\, ba$.
  (c) Show that $\mathcal{J}$ is a congruence on B.
  (d) Show that $S/\mathcal{J}$ is a semilattice.

(6) Let S be the semigroup with multiplication table

	e	a	b_1	b_2	c_1	c_2	d_1	d_2	0
e	e	a	b_1	b_2	e	a	b_2	b_1	0
a	a	e	b_2	b_1	a	e	b_1	b_2	0
b_1	0	0	0	0	e	a	b_2	b_1	0
b_2	0	0	0	0	a	e	b_1	b_2	0
c_1	c_1	c_2	d_2	d_1	c_1	c_2	d_1	d_2	0
c_2	c_2	c_1	d_1	d_2	c_2	c_1	d_2	d_1	0
d_1	0	0	0	0	c_2	c_1	d_2	d_1	0
d_2	0	0	0	0	c_1	c_2	d_1	d_2	0
0	0	0	0	0	0	0	0	0	0

Write down the $\mathcal{L}$-classes and the $\mathcal{R}$-classes of S, giving explanations. Hence find the $\mathcal{H}$-classes and the $\mathcal{D}$-classes of S. Explain why S is completely 0-simple. Find a Rees matrix semigroup $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$ isomorphic to S and give explicitly an isomorphism between $\mathcal{M}^0$ and S.

(7) Show that a completely 0-simple semigroup need not be inverse.

(8) A Brandt semigroup

$$\mathcal{B}^0 = \mathcal{B}^0(G; I)$$

is a Rees matrix semigroup

$$\mathcal{M}^0 = \mathcal{M}^0(G; I, I; P),$$

where P is the $I \times I$ identity matrix over $G \cup \{0\}$, that is, $p_{ij} = 0$ for all $i \neq j$ and $p_{ii} = 1$ (the identity of G) for all i.

Show that every Brandt semigroup is inverse.

(9) Let S be an inverse semigroup. Show that each $\mathcal{R}$-class contains a unique idempotent (dually, each $\mathcal{L}$-class contains a unique idempotent).

(10) Let S be an inverse semigroup. Show that for any $a, b \in S$,

$$a \,\mathcal{R}\, b \Leftrightarrow aa' = bb'$$

(dually,

$$a \,\mathcal{L}\, b \Leftrightarrow a'a = b'b.)$$

(11) *Harder* Let S be an inverse completely 0-simple semigroup. Show that S is isomorphic to a Brandt semigroup.