GRAPH EXPANSIONS OF RIGHT CANCELLATIVE MONOIDS

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ABSTRACT. The relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on a monoid M are natural generalisations of Green's relations $\mathcal{R}$ and $\mathcal{L}$, which coincide with $\mathcal{R}$ and $\mathcal{L}$ if M is regular. A monoid M in which every $\mathcal{R}^*$-class ($\mathcal{L}^*$-class) contains an idempotent is called *left (right) abundant*; if in addition the idempotents of M commute, that is, $E(M)$ is a semilattice, then M is *left (right) adequate*. Regular monoids are obviously left (and right) abundant and inverse monoids are left (and right) adequate. Many of the well known results of regular and inverse semigroup theory have analogues for left abundant and left adequate monoids, or at least to special classes thereof.

The aim of this paper is to develop a construction of left adequate monoids from the Cayley graph of a presentation of a right cancellative monoid, inspired by the construction of inverse monoids from group presentations, given by Margolis and Meakin in [10]. This technique yields in particular the free left ample (formerly *left type A*) monoid on a given set X.
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1. INTRODUCTION

The relation $\mathcal{R}^*$ is defined on a monoid M by the rule that $a\mathcal{R}^*b$ if and only if the elements a, b of S are related by Green's relation $\mathcal{R}$ in some overmonoid of M. The relation $\mathcal{L}^*$ is defined dually; clearly $\mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{L} \subseteq \mathcal{L}^*$. It is easy to see that if M is regular, then $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{L}^* = \mathcal{L}$, but in general, the inclusions are strict.

A monoid M is *left adequate* if every $\mathcal{R}^*$-class contains an idempotent and the idempotents $E(M)$ of M form a semilattice. In this case every $\mathcal{R}^*$-class of M contains a *unique* idempotent. We denote the idempotent in the $\mathcal{R}^*$-class of a by a^+ (formerly $a^\dagger$). Regarded as algebras of type $(2, 1, 0)$, where the unary operation is given by $a \mapsto a^+$, left adequate monoids are a quasi-variety [6].

In this paper we are concerned with *left ample* monoids. These are left adequate monoids in which $ae = (ae)^+a$ for each $a \in S$ and each $e \in E(S)$; they were previously named left type A monoids and are referred to as such in the literature. Any inverse monoid is left ample, however, the class of left ample monoids is much larger than the class of inverse monoids. For example, every right cancellative monoid is a left ample monoid. Not all left adequate semigroups are left ample, but left ample semigroups are those that are amenable to description in terms of semidirect products (compare [6] with [7] and [8]). Left ample monoids are a sub-quasi-variety of the quasi-variety of left adequate monoids and thus free left ample monoids exist; they are described in [7], where they are also shown to be *proper*, in the following sense.

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The diagrams in this paper are drawn using Paul Taylor's commutative diagrams package; I would like to thank Simon Eveson for advice on its use.

The least right cancellative congruence σ on a left ample monoid plays a role analogous to that of the least group congruence on an inverse monoid. We say that a left ample monoid is *proper* if $\sigma \cap \mathcal{R}^* = \iota$. Analogously to the celebrated results of McAlister for inverse monoids, proper left ample monoids may be described in terms of right cancellative monoids acting on partially ordered sets [4] and further, every left ample monoid is the image of a proper left ample monoid under an idempotent separating homomorphism [4]. An alternative characterisation of proper left ample monoids in terms of right cancellative monoids acting on categories is given in [8].

By a *monoid presentation* we mean a triple (X, f, S) where X is a set, S is a monoid and $f : X \to S$ is a function such that Xf generates S as a monoid. In this paper we make use of the *Cayley graph of a monoid presentation* and use this to construct monoids which we call *graph expansions*. If (X, f, S) is a monoid presentation where S is a right cancellative monoid, then the corresponding graph expansion $\mathcal{M}(X, f, S)$ is a proper left ample monoid which is the initial object in a suitable category $\mathbf{PLA}(X, f, S)$ of X-generated proper left ample monoids having maximum right cancellative image S. Full definitions are given in the next section. This result is analogous to Theorem 2.2 of [10], in which Margolis and Meakin show that the corresponding category of X-generated proper (E-unitary) inverse monoids with maximum group image G has an initial object, constructed from the Cayley graph of the *group* presentation of G with set of generators X; we remark that Ash gives an alternative construction of the initial object in [1]. If $\iota : X \to X^*$ is the natural embeddding then $\mathcal{M}(X, \iota, X^*)$ is the free left ample monoid on (a set in 1:1 correspondence with) X.

The latter part of the paper concentrates on the larger category $\mathbf{PLA}(X)$ of *all* X-generated proper left ample monoids, and the corresponding category $\mathbf{RC}(X)$ of X-generated right cancellative monoids. Using graph expansions we construct a functor $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ and show that F^e is an *expansion* in the accepted sense of semigroup theory, as defined in [2]. Further, F^e is a left adjoint of $F^\sigma : \mathbf{PLA}(X) \to \mathbf{RC}(X)$, where F^σ takes an X-generated proper left ample monoid to its maximum right cancellative image.

In a subsequent paper [9] we show that our techniques yield an expansion, also denoted by F^e, from the category $\mathbf{RC}$ of *all* right cancellative monoids to the category $\mathbf{PLA}$ of *all* proper left ample monoids. As F^e is 1:1 on objects, the image of F^e is a subcategory $\boldsymbol{\mathcal{M}PLA}$of $\mathbf{PLA}$. Regarded as a functor $\mathbf{RC} \to \boldsymbol{\mathcal{M}PLA}$, F^e has a left adjoint, again denoted F^σ.

Section 2 consists of some preliminary definitions and results concerning left adequate and left ample monoids. We also define the categories $\mathbf{PLA}(X, f, S), \mathbf{RC}(X)$ and $\mathbf{PLA}(X)$, where (X, f, S) is a monoid presentation of a right cancellative monoid S.

In Section 3 we consider the Cayley graph of a monoid presentation (X, f, S) and use this to construct a monoid $\mathcal{M}(X, f, S)$ called a graph expansion. We show that if (X, f, S) is a monoid presentation then $\mathcal{M}(X, f, S)$ is a left ample monoid if and only if S is right cancellative. In this case, $\mathcal{M}(X, f, S)$ is proper and has maximum right cancellative image S. Further, if S is freely generated by Xf where f is 1:1, then $\mathcal{M}(X, f, S)$ is free on a set in 1:1 correspondence with X.

Section 4 concentrates on proving that if (X, f, S) is a monoid presentation of a right cancellative monoid S, then $\mathcal{M}(X, f, S)$ is an initial object in $\mathbf{PLA}(X, f, S)$. In the final section we consider the functors $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ and F^σ :

$\mathbf{PLA}(X) \to \mathbf{RC}(X)$. We show that F^e is an expansion and F^e is a left adjoint of F^σ.

2. Preliminaries

We begin with the following alternative characterisation of the relation $\mathcal{R}^*$, which we use without further mention.

Lemma 2.1. [5] *Elements a, b of a monoid M are $\mathcal{R}^*$-related if and only if for all $x, y \in M$,*

$$xa = ya \text{ if and only if } xb = yb.$$

From Lemma 2.1 it is clear that $\mathcal{R}^*$ is an equivalence relation on any monoid M, indeed a left congruence. Further, a monoid is right cancellative if and only if it is single $\mathcal{R}^*$-class.

Recall that a monoid M is *left adequate* if every $\mathcal{R}^*$-class contains an idempotent and the idempotents of M form a semilattice; the unique idempotent in the $\mathcal{R}^*$-class of a is denoted by a^+.

Lemma 2.2. *Let M be a left adequate monoid. Then*
(1) $(ab)^+ = (ab^+)^+$ for all $a, b \in M$;
(2) $(ea)^+ = ea^+$ for all $a \in M$ and $e \in E(M)$;
(3) $(ab)^+ \leq a^+$ for all $a, b \in M$, where $\leq$ is the natural partial order on $E(M)$.

We consider left adequate monoids as algebras of type $(2, 1, 0)$. As pointed out in the introduction, they form a quasi-variety of algebras. A *left ample monoid* is a left adequate monoid M in which $ae = (ae)^+a$ for each $a \in M$ and $e \in E(M)$. Thus left ample monoids form a sub-quasi-variety of the quasi-variety of left adequate monoids.

We regard *arbitrary* monoids as varieties of algebras of type $(2, 0)$. Now, any right cancellative monoid is a left ample monoid and later in the paper we consider right cancellative monoids S with a given set of generators. The next lemma shows that no ambiguity arises whether we regard such an S as an algebra of type $(2, 1, 0)$ or of type $(2, 0)$.

Lemma 2.3. *Let S be a right cancellative monoid. Then S is a left ample monoid. A subset X of S is a set of generators of S as an algebra of type $(2, 0)$ if and only if it is a set of generators of S as an algebra of type $(2, 1, 0)$. Further, a function ϕ from a left ample monoid M to S is a monoid homomorphism, that is, a $(2, 0)$-morphism, if and only if it is morphism where S is regarded as a left ample monoid, that is, a $(2, 1, 0)$-morphism.*

For a left ample monoid M, the least right cancellative congruence has the same description as that of the least group congruence on an inverse monoid.

Lemma 2.4. [4] *Let M be a left ample monoid and define the relation σ on M by the rule that for $a, b \in M$, $a \sigma b$ if and only if $ea = eb$ for some $e \in E(M)$. Then σ is the least right cancellative monoid congruence on M and $E(M)$ is contained in a σ-class.*

Where there is danger of ambiguity, the relation σ on a left ample monoid M is denoted by σ_M.

We say that a left ample monoid is *proper* if $\sigma \cap \mathcal{R}^* = \iota$. For an inverse monoid, being proper is the same as being E-unitary. In the general case, a proper left ample monoid M is E-unitary but the converse is not true [4]. Note that if M is E-unitary then $E(M)$ is a σ-class.

Lemma 2.5. *Let M be a proper left ample monoid. If $a, b \in M$, then $a\sigma b$ if and only if $b^+a = a^+b$.*

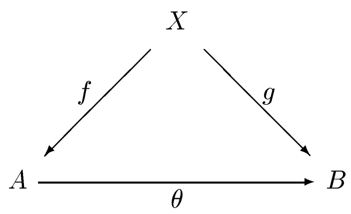
Proof. For any $a, b \in M$,

$$a^+b\mathcal{R}^*a^+b^+ = b^+a^+\mathcal{R}^*b^+a$$

since $\mathcal{R}^*$ is a left congruence. If $a\sigma b$ then as also $b^+\sigma a^+$ we have $b^+a\sigma a^+b$. Now $\sigma \cap \mathcal{R}^* = \iota$ so that $b^+a = a^+b$. The converse is clear. □

The latter sections of this paper are concerned with various categories of X-generated monoids, obtained in the following manner.

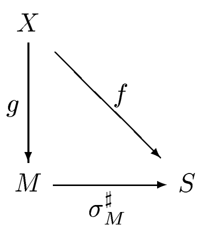
Let X be a set and $\mathcal{A}$ a class of algebras of a given fixed type. Then $\mathbf{A}(X)$ is the category which has objects pairs (f, A) where $A \in \mathcal{A}, f : X \to A$ and $< Xf >= A$; a morphism in $\mathbf{A}(X)$ from (f, A) to (g, B) is a homomorphism $\theta : A \to B$ such that



commutes. From $< Xf >= A$ we deduce that if such a θ exists, it must be unique; from $< Xg >= B$ we deduce that such a θ must be onto. Clearly $I_A \in$ Mor $((f, A), (f, A))$ for all objects (f, A) of $\mathbf{A}(X)$. Further, if $\theta \in$ Mor $((f, A), (g, B))$ and $\psi \in$ Mor $((g, B), (h, C))$, then $\theta\psi : A \to C$ is a homomorphism and $f\theta\psi = g\psi = h$, so that $\theta\psi \in$ Mor $((f, A), (h, C))$. Thus $\mathbf{A}(X)$ is a category.

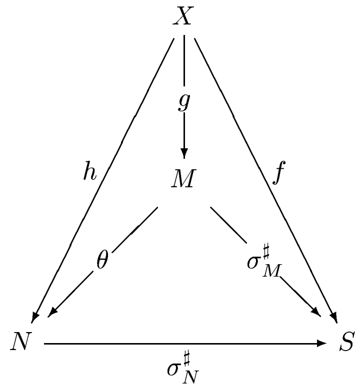
Section 5 considers the categories $\mathbf{RC}(X)$ and $\mathbf{PLA}(X)$ where $\mathcal{RC}$ is the class of right cancellative monoids and $\mathcal{PLA}$ the class of proper left ample monoids. We rely heavily on the above comments concerning the categories $\mathbf{A}(X)$, in particular that the Mor sets of these categories have at most one element, which must be an onto homomorphism. In view of Lemma 2.3 we may if we wish regard a right cancellative monoid as an algebra of type $(2, 1, 0)$ and then $\mathbf{RC}(X)$ is a full subcategory of $\mathbf{PLA}(X)$.

For the remainder of this section, (X, f, S) denotes a monoid presentation of a fixed right cancellative monoid S. To define the full subcategory $\mathbf{PLA}(X, f, S)$ of $\mathbf{PLA}(X)$, it is enough to specify the objects. An object (g, M) of $\mathbf{PLA}(X)$ is an object in $\mathbf{PLA}(X, f, S)$ if the diagram



commutes, where $\sigma^\sharp_M$ is a homomorphism with kernel σ_M. As previously remarked, $\sigma^\sharp_M$ must then be the only homomorphism making the above diagram commute, and $\sigma^\sharp_M$ must be onto, so that S is the maximum right cancellative image of M.

Lemma 2.6. *If $\theta \in$ Mor $((g, M), (h, N))$ in the category* $\mathbf{PLA}(X, f, S)$ *then θ is idempotent pure and all the triangles in the following diagram commute.*



Proof. We first show that $\theta\sigma^\sharp_N = \sigma^\sharp_M$. Let $x \in X$. Now

$$(xg)\theta\sigma^\sharp_N = xg\theta\sigma^\sharp_N = xh\sigma^\sharp_N = xf = xg\sigma^\sharp_M = (xg)\sigma^\sharp_M,$$

so that $\theta\sigma^\sharp_N$ and $\sigma^\sharp_M$ agree on Xg. But $M =< Xg >$ and by Lemma 2.3, $\theta\sigma^\sharp_N$ and $\sigma^\sharp_M$ are $(2, 1, 0)$-morphisms, so that $\theta\sigma^\sharp_N = \sigma^\sharp_M$. Thus the above diagram is commutative.

If $m \in M$ and $m\theta$ is idempotent, then

$$1\sigma^\sharp_M = 1 = m\theta\sigma^\sharp_N = m\sigma^\sharp_M$$

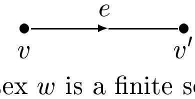
so that $1\sigma_M m$. But M is proper, hence by a previous comment, $E(M)$ is a σ_M-class, so that $m \in E(M)$ and θ is idempotent pure. □

Again using Lemma 2.3, (f, S) is an object in $\mathbf{PLA}(X, f, S)$ and if (g, M) is any other object in that category then $\sigma^\sharp_M : M \to S$ is the unique morphism in Mor $((g, M), (f, S))$. Thus (f, S) is a terminal object in $\mathbf{PLA}(X, f, S)$. In Theorem 4.2 we show that $\mathbf{PLA}(X, f, S)$ has an initial object.

3. Graph expansions

In this section we construct the graph expansion $\mathcal{M}(X, f, S)$ from a monoid presentation (X, f, S). To do so we use the Cayley graph of (X, f, S).

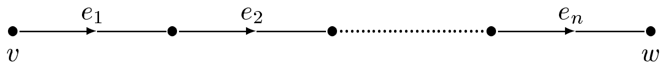
For the purposes of this paper a *graph* Γ consists of two sets $V = V(\Gamma)$ (the *vertices* of Γ) and $E = E(\Gamma)$ (the *edges* of Γ), together with two maps (written on the left), $i : E \to V$ and $t : E \to V$. The maps i and t are the *initial* and *terminal* maps, respectively. We may represent $e \in E$ with $i(e) = v$ and $t(e) = v'$ by



A *path* from a vertex v to a vertex w is a finite sequence of edges $e_1, \ldots, e_n$ with

$$i(e_1) = v,\ t(e_1) = i(e_2),\ t(e_2) = i(e_3),\ \ldots, t(e_n) = w$$

and we write this as



There is also an *empty path* I_v from any vertex v to itself. The graph Γ is v-*rooted*, where $v \in V$, if for all $w \in V$ there is a path from v to w. A *subgraph* Δ of Γ consists of a subset $V(\Delta)$ of $V(\Gamma)$ and a subset $E(\Delta)$ of $E(\Gamma)$ such that for any $e \in E(\Delta), i(e), t(e) \in V(\Delta)$. Clearly any path determines a subgraph; it is convenient at times to use the same notation for a path and the corresponding subgraph.

A *graph morphism* θ from a graph Γ to a graph Γ' consists of two functions, each denoted by θ, from $V(\Gamma)$ to $V(\Gamma')$ and from $E(\Gamma)$ to $E(\Gamma')$, such that for any $e \in E(\Gamma)$,

$$i(e)\theta = i(e\theta) \text{ and } t(e)\theta = t(e\theta).$$

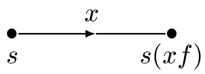
Clearly such a θ maps subgraphs to subgraphs and paths to paths.

A monoid S *acts* on a graph Γ (on the left) if V and E are left S-sets and i and t are left S-maps, that is, $i(se) = si(e)$ and $t(se) = st(e)$ for all $s \in S$ and $e \in E$. Note that if S acts on Γ, then the action of any $s \in S$ is a graph morphism so that if Δ is a subgraph of Γ, then so is $s\Delta$.

Our interest here is in the Cayley graph $\Gamma = \Gamma(X, f, S)$ of a monoid presentation (X, f, S). Here $V(\Gamma) = S$ and

$$E(\Gamma) = \{(s, x, s(xf)) : s \in S, x \in X\}$$

where $i(s, x, s(xf)) = s$ and $t(s, x, s(xf)) = s(xf)$. We may write the edge $(s, x, s(xf))$, or the corresponding subgraph, as



The monoid S acts on Γ where for $s \in S, v \in V, (t, x, t(xf)) \in E$ we have

$$s.v = sv, s.(t, x, t(xf)) = (st, x, st(xf)).$$

The *graph expansion* $\mathcal{M} = \mathcal{M}(X, f, S)$ of (X, f, S) is given by

$$\mathcal{M} = \{(\Delta, s) : \Delta \text{ is a finite 1-rooted subgraph of } \Gamma \text{ and } 1, s \in V(\Delta)\}.$$

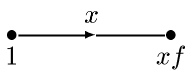
We define a multiplication on $\mathcal{M}$ by

$$(\Delta, s)(\Sigma, t) = (\Delta \cup s\Sigma, st).$$

The following is easy to check.

Lemma 3.1. *With $\mathcal{M} = \mathcal{M}(X, f, S)$ and multiplication as above, $\mathcal{M}$ is a monoid with identity $(\bullet_1, 1)$.*

Clearly

$\Gamma_x =$ 

is a 1-rooted subgraph of Γ and $(\Gamma_x, xf) \in \mathcal{M}$. We define $\tau_{\mathcal{M}} : X \to \mathcal{M}$ by $x\tau_{\mathcal{M}} = (\Gamma_x, xf)$.

In [10] Margolis and Meakin use an analogous construction to study proper (i.e. E-unitary) inverse monoids. Here our aim is to investigate proper ample monoids.

Proposition 3.2. *Let (X, f, S) be a monoid presentation. Then $\mathcal{M} = \mathcal{M}(X, f, S)$ is left abundant if and only if S is right cancellative.*

Proof. Suppose first that S is right cancellative. If Δ is a finite 1-rooted subgraph of Γ, then $(\Delta, 1) \in \mathcal{M}$ and clearly $(\Delta, 1)$ is idempotent. Moreover any idempotent of $\mathcal{M}$ must have this form. It is easy to see that if $(\Delta, s) \in \mathcal{M}$ then $(\Delta, s)\mathcal{R}^*(\Delta, 1)$ so that $\mathcal{M}$ is left abundant. In Proposition 3.3 we show that $\mathcal{M}$ is a proper left ample monoid.

Conversely, suppose that $\mathcal{M}$ is left abundant. Let $x \in X$ and consider $x\tau = (\Gamma_x, xf)$. Since $\mathcal{M}$ is left abundant there is an idempotent $(\Delta, e) \in \mathcal{M}$ with $(\Delta, e)\mathcal{R}^*(\Gamma_x, xf)$. Hence

$$(\Gamma_x, xf) = (\Delta, e)(\Gamma_x, xf) = (\Delta \cup e\Gamma_x, e(xf))$$

and so $\underset{1}{\bullet} \xrightarrow{\;x\;} \underset{xf}{\bullet}$ and $\underset{e}{\bullet} \xrightarrow{\;x\;} \underset{e(xf)}{\bullet}$ coincide. Thus $e = 1$.

Supppose now that $x \in X, xf \neq 1$ and $s(xf) = t(xf)$ where $s, t \in S$. Since $S =< Xf >$ there are paths P_s and P_t from 1 to s and from 1 to t, respectively. Let Σ be the subgraph

$$P_s \cup P_t \cup \{ \underset{s}{\bullet} \xrightarrow{\;x\;} \underset{s(xf)}{\bullet} \;, \; \underset{t}{\bullet} \xrightarrow{\;x\;} \underset{t(xf)}{\bullet} \}$$

so that $(\Sigma, s), (\Sigma, t) \in \mathcal{M}$. Now

$$(\Sigma, s)(\Gamma_x, xf) = (\Sigma, t)(\Gamma_x, xf)$$

giving

$$(\Sigma, s)(\Delta, 1) = (\Sigma, t)(\Delta, 1)$$

whence $s = t$. It follows that S is right cancellative. □

Proposition 3.3. *Let (X, f, S) be a monoid presentation of a right cancellative monoid S. Then $\mathcal{M} = \mathcal{M}(X, f, S)$ is a proper left ample monoid. Further, for any $(\Delta, s), (\Sigma, t) \in \mathcal{M}$,*
(i) $(\Delta, s) \in E(\mathcal{M})$ if and only if $s = 1$;
(ii) $(\Delta, s)^+ = (\Delta, 1)$;
(iii) $(\Delta, s)\mathcal{R}^*(\Sigma, t)$ if and only if $\Delta = \Sigma$;
(iv) $(\Delta, s)\sigma(\Sigma, t)$ if and only if $s = t$.

Proof. From Proposition 3.2, $\mathcal{M}$ is left abundant and (i) holds. Moreover $(\Delta, s)\mathcal{R}^*(\Delta, 1)$.

Given (i) it is clear that $E(\mathcal{M})$ is a semilattice. Thus $\mathcal{M}$ is left adequate and $(\Delta, s)^+ = (\Delta, 1)$. Now $(\Delta, s)\mathcal{R}^*(\Sigma, t)$ if and only if $(\Delta, s)^+ = (\Sigma, t)^+$, that is, $\Delta = \Sigma$.

If $(\Delta, s)\sigma(\Sigma, t)$ then

$$(\Theta, 1)(\Delta, s) = (\Theta, 1)(\Sigma, t)$$

for some $(\Theta, 1) \in E(\mathcal{M})$, giving $s = t$. Conversely, if $s = t$ then

$$(\Delta \cup \Sigma, 1)(\Delta, s) = (\Delta \cup \Sigma, 1)(\Sigma, t)$$

so that $(\Delta, s)\sigma(\Sigma, t)$ and (iv) holds.

Let $(\Theta, 1) \in E(\mathcal{M})$. Using (ii) we have

$$((\Delta, s)(\Theta, 1))^+(\Delta, s) = (\Delta \cup s\Theta, s)^+(\Delta, s) =$$

$$(\Delta \cup s\Theta, 1)(\Delta, s) = (\Delta \cup s\Theta, s) = (\Delta, s)(\Theta, 1)$$

for any $(\Delta, s) \in \mathcal{M}$, so that $\mathcal{M}$ is a left ample monoid. From (iii) and (iv) it is immediate that $\mathcal{M}$ is proper. □

Our next aim is to show that if (X, f, S) is a monoid presentation of a right cancellative monoid S, then $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLA}(X, f, S)$, where $\mathcal{M} = \mathcal{M}(X, f, S)$. Recall that we are regarding left ample monoids as algebras of type $(2, 1, 0)$ and so we need in particular to show that $X\tau_{\mathcal{M}}$ generates $\mathcal{M}$ as as algebra of this type.

Proposition 3.4. *Let (X, f, S) be a monoid presentation of a right cancellative monoid S. Putting $\mathcal{M} = \mathcal{M}(X, f, S)$ we have $\mathcal{M} =< X\tau_{\mathcal{M}} >$ and $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLA}(X, f, S)$.*

Proof. Let $(\Delta, s) \in \mathcal{M}$. If Δ is the trivial graph $\bullet_1$ then as s is a vertex of $\Delta, s = 1$ and $(\Delta, s) = (\bullet_1, 1)$ is the identity of $\mathcal{M}$, hence $(\Delta, s) \in< X\tau_{\mathcal{M}} >$. Suppose now that Δ is not trivial. Then there is an edge $e = \underset{u}{\bullet} \xrightarrow{\ x\ } \underset{u(xf)}{\bullet}$ in Δ. By definition, Δ is 1-rooted, so there is some path

$$\underset{1}{\bullet} \xrightarrow{x_1} \underset{x_1f}{\bullet} \xrightarrow{x_2} \underset{(x_1f)(x_2f)}{\bullet} \cdots \bullet \xrightarrow{x_n} \underset{(x_1f)\dots(x_nf)=u}{\bullet}$$

from 1 to u in Δ, so that

$$P_e = \underset{1}{\bullet} \xrightarrow{x_1} \underset{x_1f}{\bullet} \xrightarrow{x_2} \bullet \cdots \bullet \xrightarrow{x_n} \underset{u}{\bullet} \xrightarrow{x} \underset{u(xf)}{\bullet}$$

is a subgraph of Δ. Note that

$$(P_e, 1) = (x_1\tau_{\mathcal{M}}x_2\tau_{\mathcal{M}} \dots x_n\tau_{\mathcal{M}}x\tau_{\mathcal{M}})^+ \in< X\tau_{\mathcal{M}} > .$$

As Δ is 1-rooted, $\Delta = \bigcup_{e\in E(\Delta)} P_e$ and we have that

$$(\Delta, 1) = \prod_{e\in E(\Delta)} (P_e, 1) \in< X\tau_{\mathcal{M}} > .$$

Thus if $s = 1, (\Delta, s) \in< X\tau_{\mathcal{M}} >$. If $s \neq 1$ then as $s \in V(\Delta)$ and Δ is 1-rooted, we have some edge $e \in E(\Delta)$ with $t(e) = s$. Then s is a vertex of P_e so that $(P_e, s) \in \mathcal{M}$ and moreover

$$(P_e, s) = x_1\tau_{\mathcal{M}}x_2\tau_{\mathcal{M}} \dots x_n\tau_{\mathcal{M}}x\tau_{\mathcal{M}}$$

for some $x_1, \dots, x_n, x \in X$. Now

$$(\Delta, 1)(P_e, s) = (\Delta \cup P_e, s) = (\Delta, s)$$

so that $(\Delta, s) \in< X\tau_{\mathcal{M}} >$ as required.

The above shows that $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLA}(X)$. To show that $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in the subcategory $\mathbf{PLA}(X, f, S)$ we must show that

$$\begin{array}{ccc} X & & \\ {\scriptstyle \tau_{\mathcal{M}}}\downarrow & \searrow^{f} & \\ \mathcal{M} & \xrightarrow[\sigma^\sharp_{\mathcal{M}}]{} & S \end{array}$$

commutes, where $\sigma^\sharp_{\mathcal{M}}$ is a homomorphism with kernel $\sigma_{\mathcal{M}}$. Defining $\sigma^\sharp_{\mathcal{M}} : \mathcal{M} \to S$ by $(\Delta, s)\sigma^\sharp_{\mathcal{M}} = s$ it is clear that $\sigma^\sharp_{\mathcal{M}}$ is a homomorphism. By Proposition 3.3, Ker $\sigma^\sharp_{\mathcal{M}} = \sigma_{\mathcal{M}}$; clearly $\tau_{\mathcal{M}}\sigma^\sharp_{\mathcal{M}} = f$. $\square$

The final result of this section shows that we may use the technique of graph expansions to construct free left type A monoids. Fountain's paper [7] considers free *right* ample semigroups. In particular, Proposition 4.2 of [7] gives a set of necessary and sufficient conditions for a subset Y of a right ample semigroup N to be a set of free generators for the $^+$-subsemigroup generated by Y. We make use of the dual of this result.

For any left ample semigroup or monoid N, the relation $\leq$ is defined by the rule

$$a \leq b \text{ if and only if } a = a^+ b.$$

As noted in [7], $a \leq b$ is equivalent to $a = eb$ for some $e \in E(N)$. The relation $\leq$ is a compatible partial order on N which extends the natural partial order on the semilattice $E(N)$ [3].

Proposition 3.5. [7] *Let Y be a subset of a left ample* semigroup N. *Regarding N as an algebra of type $(2,1)$ let $< Y >_S$ denote the subalgebra of N generated by Y. Then Y is a set of free generators of $< Y >_S$ if and only if*
(1) no pair of elements that are products of elements of Y have a lower bound in $< Y >_S$
and
(2) if

$$(y_1 \dots y_t)^+ \geq \prod_{j=1}^{m} (y_{j1} \dots y_{jp(j)})^+$$

where $y_i, y_{jk} \in Y, 1 \leq i \leq t, 1 \leq j \leq m, 1 \leq k \leq p(j)$, then there is a $j \in \{1, \dots, m\}$ such that $y_i = y_{ji}, 1 \leq i \leq t$.

In this paper we are dealing with left ample *monoids*.

Lemma 3.6. *Let M be a left ample monoid and Y a subset of M. If Y is a set of free generators of $< Y >_S$ and $1 \notin < Y >_S$, then $< Y > = < Y >_S \cup \{1\}$ and Y is a set of free generators for $< Y >$.*

Proof. Clear. □

We use Proposition 3.5 and Lemma 3.6 to prove the following.

Theorem 3.7. *Let (X, f, S) be a monoid presentation of a right cancellative monoid S, where f is 1:1 and Xf is a set of free generators of S. Then $\tau_{\mathcal{M}} : X \to \mathcal{M}$ is 1:1 and $\mathcal{M}$ is the free left ample monoid on $X\tau_{\mathcal{M}}$.*

Proof. It is clear that $\tau_{\mathcal{M}}$ is 1:1 and we know from Proposition 3.4 that $\mathcal{M} = < X\tau_{\mathcal{M}} >$. Using the definition of multiplication in $\mathcal{M}$, together with statement (ii) of Proposition 3.3, it is easy to see that the identity $(\bullet_1, 1)$ of $\mathcal{M}$ cannot be obtained from $X\tau_{\mathcal{M}}$ by applications of multiplication and the operation $^+$. That is, $1_{\mathcal{M}} \notin < X\tau_{\mathcal{M}} >_S$.

Suppose now that α, β are products of elements of $X\tau_{\mathcal{M}}$, say

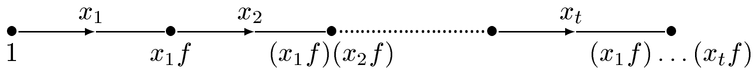
$$\alpha = x_1\tau_{\mathcal{M}} \dots x_m\tau_{\mathcal{M}}, \beta = y_1\tau_{\mathcal{M}} \dots y_n\tau_{\mathcal{M}}$$

where $x_1, \dots, x_m, y_1, \dots, y_n \in X$. Suppose in addition that $\gamma \in \mathcal{M}$ is a lower bound of α and β. Hence $\gamma = \gamma^+\alpha = \gamma^+\beta$. Using Proposition 3.3 it follows that $x_1 f \dots x_m f = y_1 f \dots y_n f$. Since Xf freely generates S we have that $m = n$ and $x_i f = y_i f, 1 \leq i \leq m$. But f is 1:1 so that $x_i = y_i, 1 \leq i \leq m$ and $\alpha = \beta$. Thus condition (1) of Proposition 3.5 holds.

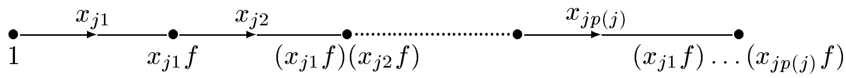
To show that condition (2) of Proposition 3.5 holds, consider $x_i, x_{jk} \in X, 1 \leq i \leq t, 1 \leq j \leq m, 1 \leq k \leq p(j)$ where

$$(x_1\tau_{\mathcal{M}} \dots x_t\tau_{\mathcal{M}})^+ \geq \prod_{j=1}^{m} (x_{j1}\tau_{\mathcal{M}} \dots x_{jp(j)}\tau_{\mathcal{M}})^+.$$

Writing $(\Delta, 1)$ for $(x_1\tau_{\mathcal{M}} \dots x_t\tau_{\mathcal{M}})^+$ and $(\Sigma, 1)$ for $\prod_{j=1}^{m}(x_{j1}\tau_{\mathcal{M}} \dots x_{jp(j)}\tau_{\mathcal{M}})^+$ we have that $(\Sigma, 1)(\Delta, 1) = (\Sigma, 1)$, so that $\Delta \subseteq \Sigma$. Now Δ is the subgraph



and as $\Delta \subseteq \Sigma$ we have in particular that $x_1 f \dots x_t f$ is a vertex of Σ. Since Σ is the union of subgraphs of the form



we must have that $x_1 f \dots x_t f = x_{j1} f \dots x_{js} f$ for some $j \in \{1, \dots, m\}$ and $s \in \{1, \dots, p(j)\}$. Again we use the fact that S is free on Xf and f is 1:1 to obtain $s = t$ and $x_i = x_{ji}, 1 \leq i \leq t$. This gives that the required condition holds. In view of Lemma 3.6 this completes the proof that $\mathcal{M}$ is the free left ample monoid on $X\tau_{\mathcal{M}}$. $\square$

4. The category $\mathbf{PLA}(X, f, S)$

In this section we show that the category $\mathbf{PLA}(X, f, S)$, where (X, f, S) is a monoid presentation of a right cancellative monoid S, has an initial object $(\tau_{\mathcal{M}}, \mathcal{M}(X, f, S))$. As remarked at the end of Section 2, (f, S) is a terminal object in $\mathbf{PLA}(X, f, S)$.

From Proposition 3.4 we have that if (X, f, S) is as above, then $\mathcal{M}(X, f, S) = \mathcal{M} =< X\tau_{\mathcal{M}} >$ and $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLA}(X, f, S)$. The next lemma gives a 'standard form' for elements of a left ample monoid with a given set of generators, which we apply to $\mathcal{M}$ in Theorem 4.2.

Lemma 4.1. *Let M be a left ample monoid and suppose that $M =< Y >$. Then any $a \in M$ can be written as*

$$a = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+ y_1 \dots y_n$$

for some $m, n \in \mathbb{N}$ where $x_j^i, y_k \in Y, 1 \leq i \leq m, 1 \leq j \leq p(i), 1 \leq k \leq n$ (n and m may be 0).

Proof. As every left ample monoid is the image of a free left ample monoid it would be possible to deduce this result from [7]. However, it is easier and relevant to later arguments to give a direct proof.

Clearly the elements of Y are of the required form. We make the inductive assumption that $q \in \mathbb{N}$ and all elements of M obtained from the elements of Y by less than q applications of fundamental operations have the required form. Suppose that $a \in M$ is obtained from Y by q applications of fundamental operations.

Case (i) $a = 1$. Putting $m = n = 0$, a has the required form.

Case (ii) $a = b^+$ where b is obtained from Y in $q - 1$ steps. By the inductive hypothesis,

$$b = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+ y_1 \dots y_n$$

for some $m, n \in \mathbb{N}, x_j^i, y_k \in Y, 1 \leq i \leq m, 1 \leq j \leq p(i), 1 \leq k \leq n$. Now $b^+ = (ey_1 \dots y_n)^+$ where $e = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+$ is idempotent, so that by Lemma 2.2, $a = b^+ = e(y_1 \dots y_n)^+$ and a has the required form.

Case (iii) $a = bc$ where b and c are obtained from Y in fewer than q steps. By the inductive hypothesis

$$b = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+ y_1 \dots y_s$$

and

$$c = (z_1^1 \dots z_{q(1)}^1)^+ \dots (z_1^n \dots z_{q(n)}^n)^+ w_1 \dots w_t$$

for some $m, n, s, t \in \mathbb{N}$ where $x_j^i, y_k \in Y, 1 \leq i \leq m, 1 \leq j \leq p(i), 1 \leq k \leq s$ and $z_j^i, w_k \in Y, 1 \leq i \leq n, 1 \leq j \leq q(i), 1 \leq k \leq t$.

If $s = 0$ or $n = 0$ then $a = bc$ has the required form. Suppose that $s \neq 0$ and $n \neq 0$. Put $y = y_1 \dots y_s$ and for $1 \leq i \leq n$ put $e_i = (z_1^i \dots z_{q(i)}^i)^+$. As M is left ample we have

$$ye_1 \dots e_n = (ye_1)^+ ye_2 \dots e_n = \dots = (ye_1)^+ \dots (ye_n)^+ y.$$

Now for any $i \in \{1, \dots, n\}$,

$$(ye_i)^+ = (y(z_1^i \dots z_{q(i)}^i)^+)^+ = (yz_1^i \dots z_{q(i)}^i)^+,$$

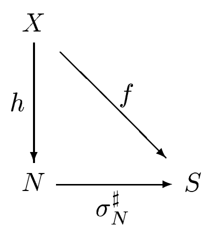
using Lemma 2.2. It follows that $a = bc$ has the required form.

Induction now gives the result. □

Theorem 4.2. *Let (X, f, S) be a monoid presentation of a right cancellative monoid S. Then putting $\mathcal{M} = \mathcal{M}(X, f, S)$, the pair $(\tau_{\mathcal{M}}, \mathcal{M})$ is an initial object in* $\mathbf{PLA}(X, f, S)$.

Proof. We need to show that for any object (h, N) in $\mathbf{PLA}(X, f, S)$, $|\text{Mor}\,((\tau_{\mathcal{M}}, \mathcal{M}), (h, N))| = 1$. From a remark in Section 2, this is equivalent to showing that Mor $((\tau_{\mathcal{M}}, \mathcal{M}), (h, N)) \neq \emptyset$.

Let (h, N) be an object in $\mathbf{PLA}(X, f, S)$. Thus $N =< Xh >$ and



commutes, where $\sigma_N^\sharp$ is a homomorphism with kernel σ_N.

Define $\theta : \mathcal{M} \to N$ by

$$((x_1^1 \tau_{\mathcal{M}} \dots x_{p(1)}^1 \tau_{\mathcal{M}})^+ \dots (x_1^m \tau_{\mathcal{M}} \dots x_{p(m)}^m \tau_{\mathcal{M}})^+ y_1 \tau_{\mathcal{M}} \dots y_s \tau_{\mathcal{M}})\theta$$
$$= (x_1^1 h \dots x_{p(1)}^1 h)^+ \dots (x_1^m h \dots x_{p(m)}^m h)^+ y_1 h \dots y_s h$$

where $m, s \in \mathbb{N}, x_j^i, y_k \in X, 1 \leq i \leq m, 1 \leq j \leq p(i), 1 \leq k \leq s$. From Lemma 4.1, this defines θ on the whole of $\mathcal{M}$; the question is whether θ is well defined.

Suppose that

$$(x_1^1 \tau_{\mathcal{M}} \dots x_{p(1)}^1 \tau_{\mathcal{M}})^+ \dots (x_1^m \tau_{\mathcal{M}} \dots x_{p(m)}^m \tau_{\mathcal{M}})^+ y_1 \tau_{\mathcal{M}} \dots y_s \tau_{\mathcal{M}}$$
$$= (z_1^1 \tau_{\mathcal{M}} \dots z_{q(1)}^1 \tau_{\mathcal{M}})^+ \dots (z_1^n \tau_{\mathcal{M}} \dots z_{q(n)}^n \tau_{\mathcal{M}})^+ w_1 \tau_{\mathcal{M}} \dots w_t \tau_{\mathcal{M}} \ (*)$$

where $m,n,s,t\in\mathbb{N}, x^i_j, y_k\in X, 1\le i\le m, 1\le j\le p(i), 1\le k\le s$ and $z^i_j, w_k\in X, 1\le i\le n, 1\le j\le q(i), 1\le k\le t$.

We aim to show

$$(x^1_1h\dots x^1_{p(1)}h)^+\dots(x^m_1h\dots x^m_{p(m)}h)^+y_1h\dots y_sh$$
$$=(z^1_1h\dots z^1_{q(1)}h)^+\dots(z^n_1h\dots z^n_{q(n)}h)^+w_1h\dots w_th\ (**).$$

Note first that if $m=s=0$ then the left hand side of $(*)$ is the identity $(\bullet_1,1)$ of $\mathcal{M}$. It follows from the definition of $\tau_{\mathcal{M}}$, the multiplication in $\mathcal{M}$, and the description of $^+$ in Proposition 3.3, that also $n=t=0$. Clearly $(**)$ holds in this case.

To proceed we need a result convenient to state as a subsidiary lemma.

Lemma 4.3. *Let $a_1,\dots,a_s,b_1,\dots,b_t\in X$ (where s or t may be 0) and suppose that*

$$a_1f\dots a_sf=b_1f\dots b_tf,$$

where the empty product is taken to be 1. *Then*

$$(a_1h\dots a_sh)\ \sigma_N\ (b_1h\dots b_th).$$

Proof. If $s\neq 0$ and $t\neq 0$ then

$$(a_1h\dots a_sh)\sigma^\sharp_N=a_1h\sigma^\sharp_N\dots a_sh\sigma^\sharp_N=a_1f\dots a_sf=$$
$$b_1f\dots b_tf=b_1h\sigma^\sharp_N\dots b_th\sigma^\sharp_N=(b_1h\dots b_th)\sigma^\sharp_N$$

so that the result is true in this case.

If $s\neq 0$ and $t=0$ then

$$(a_1h\dots a_sh)\sigma^\sharp_N=1=1\sigma^\sharp_N$$

so that $a_1h\dots a_sh\ \sigma_N\ 1$; it follows that the result is true in every case. □

Proceeding with the proof of Theorem 4.2, suppose that *not both* m and s are 0 and *not both* n and t are 0. From $(*)$ we have

$$(y_1\tau_{\mathcal{M}}\dots y_s\tau_{\mathcal{M}})\ \sigma_{\mathcal{M}}\ (w_1\tau_{\mathcal{M}}\dots w_t\tau_{\mathcal{M}})$$

so that from (iv) of Proposition 3.3 we have

$$y_1f\dots y_sf=w_1f\dots w_tf.$$

Lemma 4.3 gives that

$$(y_1h\dots y_sh)\ \sigma_N\ (w_1h\dots w_th).$$

From Lemma 2.5 we now have

$$(w_1h\dots w_th)^+y_1h\dots y_sh=(y_1h\dots y_sh)^+w_1h\dots w_th.$$

For the remainder of the proof we write

$$y_1=x^{m+1}_1,\dots,y_s=x^{m+1}_{p(m+1)}$$

and

$$w_1=z^{n+1}_1,\dots,w_t=z^{n+1}_{q(n+1)}.$$

With the usual convention for empty products we put

$$E=(x^1_1h\dots x^1_{p(1)}h)^+\dots(x^m_1h\dots x^m_{p(m)}h)^+$$
$$Y=x^{m+1}_1h\dots x^{m+1}_{p(m+1)}h$$

$$F = (z_1^1 h \dots z_{q(1)}^1 h)^+ \dots (z_1^n h \dots z_{q(n)}^n h)^+$$

and

$$W = z_1^{n+1} h \dots z_{q(n+1)}^{n+1} h;$$

note we have shown that

$$W^+ Y = Y^+ W.$$

Our next aim is to show that $EY^+ = FW^+$.

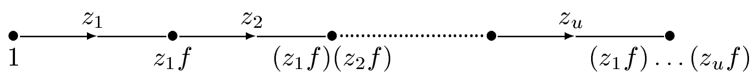
Let $i \in \{1, \dots, n+1\}$ where $q(i) \neq 0$ and write

$$z_1^i = z_1, \dots, z_{q(i)}^i = z_u.$$

Lemma 4.4. *With notation as above,*

$$EY^+ \leq (z_1 h \dots z_u h)^+.$$

Proof. If Δ denotes the graph that is the first coordinate of $(*)$, then from the expression for $(*)$ we know that



is a subgraph of Δ. It follows that there exist $i_1, \dots, i_u \in \{1, \dots, m+1\}$ and $j_1, \dots, j_u$ with $j_k \in \{1, \dots, p(i_k)\}$ for $k \in \{1, \dots, u\}$ such that

$$z_1 = x_{j_1}^{i_1} \quad \text{where} \quad x_1^{i_1} f \dots x_{j_1-1}^{i_1} f = 1$$

$$z_2 = x_{j_2}^{i_2} \quad \text{where} \quad x_1^{i_2} f \dots x_{j_2-1}^{i_2} f = z_1 f$$

$$z_3 = x_{j_3}^{i_3} \quad \text{where} \quad x_1^{i_3} f \dots x_{j_3-1}^{i_3} f = z_1 f z_2 f$$

$$\vdots$$

$$z_u = x_{j_u}^{i_u} \quad \text{where} \quad x_1^{i_u} f \dots x_{j_u-1}^{i_u} f = z_1 f \dots z_{u-1} f.$$

From Lemma 4.3 we have that

$$x_1^{i_1} h \dots x_{j_1-1}^{i_1} h \; \sigma_N \; 1.$$

Since N is proper, $E(N)$ is a σ_N-class, so that $x_1^{i_1} h \dots x_{j_1-1}^{i_1} h$ is idempotent. Using Lemma 2.2 we deduce that

$$EY^+ \leq (x_1^{i_1} h \dots x_{p(i_1)}^{i_1} h)^+ = ((x_1^{i_1} h \dots x_{j_1-1}^{i_1} h) z_1 h (x_{j_1+1}^{i_1} h \dots x_{p(i_1)}^{i_1} h))^+$$

$$\leq ((x_1^{i_1} h \dots x_{j_1-1}^{i_1} h) z_1 h)^+ = (x_1^{i_1} h \dots x_{j_1-1}^{i_1} h)(z_1 h)^+ \leq (z_1 h)^+.$$

Assume by finite induction that for $1 \leq v < u$,

$$EY^+ \leq (z_1 h \dots z_v h)^+$$

and put $w = v + 1$. We have

$$EY^+ \leq ((x_1^{i_w} h \dots x_{j_w-1}^{i_w} h)(z_w h)(x_{j_w+1}^{i_w} h \dots x_{p(i_w)}^{i_w} h))^+$$

which together with Lemma 2.2 and the induction hypothesis gives

$$EY^+ \leq (z_1 h \dots z_v h)^+ ((x_1^{i_w} h \dots x_{j_w-1}^{i_w} h)(z_w h))^+.$$

We know that

$$z_1 f \dots z_v f = x_1^{i_w} f \dots x_{j_w-1}^{i_w} f.$$

Lemmas 4.3 and 2.5 combine to give

$$(x_1^{i_w}h\dots x_{j_w-1}^{i_w}h)^+z_1h\dots z_vh=(z_1h\dots z_vh)^+x_1^{i_w}h\dots x_{j_w-1}^{i_w}h.$$

Now

$$(z_1h\dots z_vh)^+((x_1^{i_w}h\dots x_{j_w-1}^{i_w}h)(z_wh))^+\mathcal{R}^*(z_1h\dots z_vh)^+(x_1^{i_w}h\dots x_{j_w-1}^{i_w}h)(z_wh)$$

$$=(x_1^{i_w}h\dots x_{j_w-1}^{i_w}h)^+z_1h\dots z_vhz_wh\;\mathcal{R}^*(x_1^{i_w}h\dots x_{j_w-1}^{i_w}h)^+(z_1h\dots z_wh)^+$$

so that as each $\mathcal{R}^*$-class contains only one idempotent we have

$$EY^+\leq(x_1^{i_w}h\dots x_{j_w-1}^{i_w}h)^+(z_1h\dots z_wh)^+\leq(z_1h\dots z_wh)^+.$$

Finite induction now gives that

$$EY^+\leq(z_1h\dots z_uh)^+$$

as required. □

Since Lemma 4.4 holds for *any* $i\in\{1,\dots,n+1\}$ with $q(i)\neq0$ we obtain $EY^+\leq FW^+$. Together with the dual argument this gives that $EY^+=FW^+$. Then

$$EY=EY^+Y=FW^+Y=FY^+W$$

since $W^+Y=Y^+W$. But from $EY^+=FW^+$ we also have that $Y^+FW^+=FW^+$ so that

$$EY=FY^+W^+W=FW^+W=FW$$

which finishes the proof that θ is well defined.

We must now show that θ is a homomorphism. By definition, $1\theta=1$ and from (2) of Lemma 2.2 it is easy to see that $a^+\theta=(a\theta)^+$ for any $a\in M$.

Lemma 4.1 shows that for *any* expressions

$$b=(a_1^1\dots a_{p(1)}^1)^+\dots(a_1^m\dots a_{p(m)}^m)^+b_1\dots b_s$$

and

$$d=(c_1^1\dots c_{q(1)}^1)^+\dots(c_1^n\dots c_{q(n)}^n)^+d_1\dots d_t$$

in *any* left ample monoid, we have that

$$bd=(a_1^1\dots a_{p(1)}^1)^+\dots(a_1^m\dots a_{p(m)}^m)^+(b_1\dots b_sc_1^1\dots c_{q(1)}^1)^+$$

$$\dots(b_1\dots b_sc_1^n\dots c_{q(n)}^n)^+b_1\dots b_sd_1\dots d_t.$$

It is then clear that θ preserves multiplication and so is a homomorphism.

Finally, for any $x\in X$ we have

$$x\tau_{\mathcal{M}}\theta=xh$$

so that θ is (the unique morphism) in Mor $((\tau_{\mathcal{M}},\mathcal{M}),(h,N))$. This completes the proof that $(\tau_{\mathcal{M}},\mathcal{M})$ is an initial object in $\mathbf{PLA}(X,f,S)$. □

5. The functors F^σ and F^e

In this final section we introduce functors $F^\sigma : \mathbf{PLA}(X) \to \mathbf{RC}(X)$ and $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$. We show that F^e is an expansion and a left adjoint of F^σ.

We begin with the functor F^e. Suppose now that (f, S) is an object in $\mathbf{RC}(X)$. From Proposition 3.4, $(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S))$ is an object in $\mathbf{PLA}(X)$. We put $(f, S)F^e = (\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S))$. If (g, T) is another object in $\mathbf{RC}(X)$ and $\theta \in$ Mor $((f, S), (g, T))$, then we define a map, also denoted by θ, from $\Gamma(X, f, S)$ to $\Gamma(X, g, T)$ by the obvious action on vertices and action on edges given by

$$(s, x, s(xf))\theta = (s\theta, x, s\theta xg).$$

Then θ is a graph morphism. As commented in Section 3, θ maps subgraphs to subgraphs and paths to paths. Hence θ maps a 1-rooted subgraph of $\Gamma(X, f, S)$ to a 1-rooted subgraph of $\Gamma(X, g, T)$ and so we can define $\theta^e : \mathcal{M}(X, f, S) \to \mathcal{M}(X, g, T)$ by $(\Delta, s)\theta^e = (\Delta\theta, s\theta)$. It is easy to check that for any subgraph Δ of $\Gamma(X, f, S)$ and any $s \in S$, we have $(s\Delta)\theta = s\theta\Delta\theta$ from which we deduce that θ^e preserves multiplication. Further, $(\bullet_1, 1)\theta^e = (\bullet_1, 1)$ and for any $(\Delta, s) \in \mathcal{M}(X, f, S)$,

$$(\Delta, s)^+\theta^e = (\Delta, 1)\theta^e = (\Delta\theta, 1\theta) = (\Delta\theta, 1) = (\Delta\theta, s\theta)^+ = ((\Delta, s)\theta^e))^+$$

so that θ^e is a $(2, 1, 0)$-morphism. For any $x \in X$ we have

$$x\tau_{\mathcal{M}(X,f,S)}\theta^e = (\, \underset{1}{\bullet} \xrightarrow{\;x\;} \underset{xf}{\bullet}\, , xf)\theta^e = (\, \underset{1}{\bullet} \xrightarrow{\;x\;} \underset{xg}{\bullet}\, , xg) = x\tau_{\mathcal{M}(X,g,T)}$$

and this completes the argument that

$$\theta^e \in \text{ Mor } ((\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S)), (\tau_{\mathcal{M}(X,g,T)}, \mathcal{M}(X, g, T)));$$

we now put $\theta F^e = \theta^e$. It is straightforward to check that F^e is a functor from $\mathbf{RC}(X)$ to $\mathbf{PLA}(X)$.

We adapt the terminology introduced by Birget and Rhodes [2] to define an expansion from $\mathbf{RC}(X)$ to $\mathbf{PLA}(X)$. For this purpose we regard $\mathbf{RC}(X)$ as a subcategory of $\mathbf{PLA}(X)$.

We say that a functor $F : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ is an *expansion* if for any object (f, S) of $\mathbf{RC}(X)$ there is an onto homomorphism

$$\eta_{(f,S)} \in \text{ Mor } ((f, S)F, (f, S))$$

such that

(i) for each $\theta \in$ Mor $((f, S), (g, T))$ in $\mathbf{RC}(X)$, the following square commutes

$$\begin{array}{ccc} (f,S)F & \xrightarrow{\theta F} & (g,T)F \\ {\scriptstyle \eta_{(f,S)}}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{(g,T)}} \\ (f,S) & \xrightarrow[\theta]{} & (g,T) \end{array}$$

and

(ii) if $\theta \in$ Mor $((f, S), (g, T))$ is onto, then $\theta F \in$ Mor $((f, S)F, (g, T)F)$ is also onto.

Proposition 5.1. *The functor $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ is an expansion.*

Proof. From remarks made in Section 2, any Mor set in $\mathbf{PLA}(X)$ contains at most one element; further, if the element exists it must be an onto homomorphism. It follows that to show that $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ is an expansion it is enough to show that for any object (f,S) of $\mathbf{RC}(X)$, Mor $((f,S)F^e, (f,S)) \neq \emptyset$.

Let (f,S) be an object in $\mathbf{RC}(X)$. Then $(f,S)F^e = (\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S))$. Define

$$\eta_{(f,S)} : \mathcal{M}(X,f,S) \to S$$

by

$$(\Delta, s)\eta_{(f,S)} = s.$$

Then $\eta_{(f,S)}$ is a $(2,1,0)$-morphism and for any $x \in X, x\tau_{\mathcal{M}(X,f,S)}\eta_{(f,S)} = xf$; so that

$$\eta_{(f,S)} \in \text{ Mor } ((\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S)), (f,S))$$

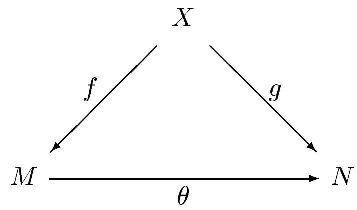
as required. □

We now define the functor $F^\sigma : \mathbf{PLA}(X) \to \mathbf{RC}(X)$ and show that F^e is a left adjoint of F^σ.

The action of F^σ on objects is given by

$$(f,M)F^\sigma = (f\sigma_M^\natural, M/\sigma_M)$$

where $\sigma_M^\natural : M \to M/\sigma_M$ is the natural homomorphism. By definition of σ_M, the monoid M/σ_M is right cancellative. It follows from Lemma 2.3 that $(f\sigma_M^\natural, M/\sigma_M)$ is an object in $\mathbf{RC}(X)$. Suppose now that $(f,M),(g,N)$ are objects in $\mathbf{PLA}(X)$ and $\theta \in$ Mor $((f,M),(g,N))$ so that



commutes. Define $\theta^\sigma : M/\sigma_M \to N/\sigma_N$ by

$$[m]\theta^\sigma = [m\theta].$$

Note that if $m, m' \in M$ and $[m] = [m']$, then $m \ \sigma_M \ m'$ so that $em = em'$ for some $e \in E(M)$. Hence $e\theta m\theta = e\theta m'\theta$ so that $m\theta \ \sigma_N \ m'\theta$ and $[m\theta] = [m'\theta]$ in N/σ_N, giving that θ^σ is well defined. It is easy to see that θ^σ is a monoid homomorphism and $f\sigma_M^\natural\theta^\sigma = g\sigma_N^\natural$. Thus $\theta^\sigma \in$ Mor $((f\sigma_M^\natural, M/\sigma_M), (g\sigma_N^\natural, N/\sigma_N))$; we put $\theta F^\sigma = \theta^\sigma$. Clearly F^σ is a functor from $\mathbf{PLA}(X)$ to $\mathbf{RC}(X)$.

Theorem 5.2. *The functor F^e is a left adjoint of the functor F^σ.*

Proof. We have to show that for any objects (f,S) in $\mathbf{RC}(X)$ and (g,M) in $\mathbf{PLA}(X)$ there is a bijection

$$\alpha_{(f,S),(g,M)} : \text{ Mor } ((f,S)F^e, (g,M)) \to \text{ Mor } ((f,S),(g,M)F^\sigma)$$

such that for any morphisms $\phi \in$ Mor $((f',S'),(f,S))$ in $\mathbf{RC}(X)$ and $\theta \in$ Mor $((g,M),(g',M'))$ in $\mathbf{PLA}(X)$, the square

$$\begin{array}{ccc}
\text{Mor }((f,S)F^e,(g,M)) & \xrightarrow{\alpha_{(f,S),(g,M)}} & \text{Mor }((f,S),(g,M)F^\sigma) \\
\Big\downarrow \text{Mor }(\phi F^e,\theta) & & \Big\downarrow \text{Mor }(\phi,\theta F^\sigma) \\
\text{Mor }((f',S')F^e,(g',M')) & \xrightarrow[\alpha_{(f',S'),(g',M')}]{} & \text{Mor }((f',S'),(g',M')F^\sigma)
\end{array}$$

is commutative. Here

$$\text{Mor }(\phi F^e,\theta):\ \text{Mor }((f,S)F^e,(g,M)) \to \ \text{Mor }((f',S')F^e,(g',M'))$$

is given by

$$\psi\ \text{Mor }(\phi F^e,\theta) = (\phi F^e)\psi\theta$$

and

$$\text{Mor }(\phi,\theta F^\sigma):\ \text{Mor }((f,S),(g,M)F^\sigma) \to \ \text{Mor }((f',S'),(g',M')F^\sigma)$$

is given by

$$\psi\ \text{Mor }(\phi,\theta F^\sigma) = \phi\psi(\theta F^\sigma).$$

Since the Mor sets of $\mathbf{PLA}(X)$ and $\mathbf{RC}(X)$ contain at most one element, this amounts to showing that for any objects (f,S) in $\mathbf{RC}(X)$ and (g,M) in $\mathbf{PLA}(X)$,

$$\text{Mor }((f,S)F^e,(g,M)) \neq \emptyset$$

if and only if

$$\text{Mor }((f,S),(g,M)F^\sigma) \neq \emptyset.$$

Suppose first that

$$\theta \in \ \text{Mor }((f,S)F^e,(g,M)).$$

Since $F^\sigma : \mathbf{PLA}(X) \to \mathbf{RC}(X)$ is a functor, we have

$$\theta F^\sigma \in \ \text{Mor }((f,S)F^eF^\sigma,(g,M)F^\sigma).$$

Now, writing $\mathcal{M} = \mathcal{M}(X,f,S)$ we have

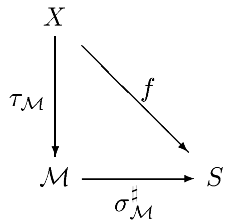
$$(f,S)F^eF^\sigma = (\tau_{\mathcal{M}},\mathcal{M})F^\sigma = (\tau_{\mathcal{M}}\sigma^\natural_{\mathcal{M}},\mathcal{M}/\sigma_{\mathcal{M}})$$

and $(g,M)F^\sigma = (g\sigma^\natural_M, M/\sigma_M)$ and so the diagram

$$\begin{array}{ccc}
 & X & \\
\swarrow{\scriptstyle \tau_{\mathcal{M}}\sigma^\natural_{\mathcal{M}}} & & \searrow{\scriptstyle g\sigma^\natural_M} \\
\mathcal{M}/\sigma_{\mathcal{M}} & \xrightarrow[\theta^\sigma]{} & M/\sigma_M
\end{array}$$

commutes.

By Proposition 3.4, $(\tau_{\mathcal{M}},\mathcal{M})$ is an object in $\mathbf{PLA}(X,f,S)$ so that



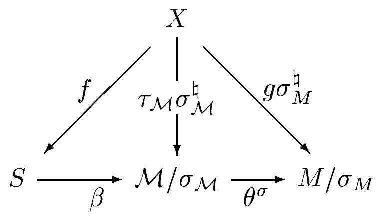
commutes, where $\sigma^{\sharp}_{\mathcal{M}}$ is a homomorphism with kernel $\sigma_{\mathcal{M}}$. As remarked in Section 2, $\sigma^{\sharp}_{\mathcal{M}}$ is onto. We now define $\beta : S \to \mathcal{M}/\sigma_{\mathcal{M}}$ by $(m\sigma^{\sharp}_{\mathcal{M}})\beta = [m]$. For any $m, m' \in \mathcal{M}$,

$$m\sigma^{\sharp}_{\mathcal{M}} = m'\sigma^{\sharp}_{\mathcal{M}} \text{ if and only if } m \; \sigma_{\mathcal{M}} \; m'$$

so that β is well defined; it is easy to check that β is a homomorphism. For any $x \in X$,

$$xf\beta = x\tau_{\mathcal{M}}\sigma^{\sharp}_{\mathcal{M}}\beta = [x\tau_{\mathcal{M}}] = x\tau_{\mathcal{M}}\sigma^{\sharp}_{\mathcal{M}}$$

so that the diagram



commutes. Thus

$$\beta\theta^{\sigma} \in \text{ Mor } ((f,S),(g,M)F^{\sigma}).$$

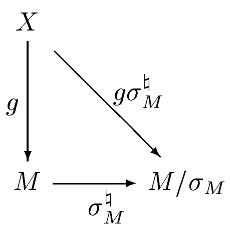
Conversely, suppose that

$$\psi \in \text{ Mor } ((f,S),(g,M)F^{\sigma}).$$

Since $F^e : \mathbf{RC}(X) \to \mathbf{PLA}(X)$ is a functor,

$$\psi F^e \in \text{ Mor } ((f,S)F^e,(g,M)F^{\sigma}F^e).$$

Now $(g,M)F^{\sigma} = (g\sigma^{\natural}_M, M/\sigma_M)$ so that $(X, g\sigma^{\natural}_M, M/\sigma_M)$ is a monoid presentation of the right cancellative monoid M/σ_M. Putting $\mathcal{M} = \mathcal{M}(X, g\sigma^{\natural}_M, M/\sigma_M)$, $(g,M)F^{\sigma}F^e = (\tau_{\mathcal{M}}, \mathcal{M})$ and by Proposition 3.4, $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLA}(X, g\sigma^{\natural}_M, M/\sigma_M)$. But



certainly commutes, so that also (g,M) is an object in $\mathbf{PLA}(X, g\sigma^{\natural}_M, M/\sigma_M)$. From Theorem 4.2, $(\tau_{\mathcal{M}}, \mathcal{M})$ is an initial object in this category, so there is a morphism $\phi \in$ Mor $((\tau_{\mathcal{M}}, \mathcal{M}),(g,M))$. Thus $(\psi F^e)\phi \in$ Mor $((f,S)F^e,(g,M))$ and Mor $((f,S)F^e,(g,M)) \neq \emptyset$. This finishes the proof of Theorem 5.2. □

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