Regularity properties of graph products of semigroups and monoids

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Outline of talk

- A bit of motivation
- Some background on semigroup and monoid presentations
- Graph products
- Regularity properties
- The results (joint work with Nouf Alqahtani and Dandan Yang)
- Some further thoughts

A bit of motivation

The construction of a **graph product** is essentially multiplicative - hence in the framework of **semigroup theory**.

It allows us to **glue together** semigroups/monoids/groups/inverse semigroups to produce new ones.

The notions of graph product of **semigroups** and graph product of **monoids** are *essentially* different.

What **algebraic properties** do graph products have, depending on the same properties of the ingredients?

Example: Fountain and Kambites (2009)

The graph product of right cancellative monoids is right cancellative.

Some background on semigroup and monoid presentations

Presentations

A **presentation** for an algebra A lying in a variety of algebras $\mathbf{V}$ is given by

$$\langle X \mid R \rangle$$

where X is a set and R is a set of relations

$$t(x_1, \cdots, x_n) = s(x_1, \cdots, x_n)$$

where s, t are term operations. This means that A is isomorphic to the free algebra in $\mathbf{V}$ on X, factored by the congruence generated by R.

We are interested in **semigroup** and **monoid** presentations.

Some background on semigroup and monoid presentations

Free algebras are defined by universal conditions.

Free semigroup X^+ on X

The elements of X^+ are all non-empty words over X, that is all sequences

$$x_1 \circ \cdots \circ x_n,\ n \in \mathbb{N}$$

with binary operation $\circ$ given by

$$(x_1 \circ \cdots \circ x_n) \circ (y_1 \circ \cdots \circ y_m) = x_1 \circ \cdots \circ x_n \circ y_1 \circ \cdots \circ y_m.$$

Some background on semigroup and monoid presentations

Free monoid X^* on X

The elements of X^* are all words over X, that is all sequences

$$x_1 \circ \cdots \circ x_n,\ n \in \mathbb{N}^0$$

with binary operation $\circ$ given by

$$(x_1 \circ \cdots \circ x_n) \circ (y_1 \circ \cdots \circ y_m) = x_1 \circ \cdots \circ x_n \circ y_1 \circ \cdots \circ y_m.$$

The empty word is usually denoted by ϵ.

Some background on semigroup and monoid presentations

Let $\langle X \mid R \rangle$ be a monoid presentation, and let M be the monoid it presents. Then

$$M = X^*/R^\sharp$$

where for any $u, v \in X^*$ we have $[u] = [v]$ if and only if $u = v$ or there exists a sequence

$$u = w_0,\, w_1,\, \cdots, w_n = v,$$

where for $1 \leq i \leq n$ we have

$$w_{i-1} = c_i p_i d_i,\, w_i = c_i q_i d_i,$$

where $p_i = q_i$ or $q_i = p_i \in R$ and $c_i, d_i \in X^*$.

Example

Let $X = \{a, b, c\}$ and let $R = \{ab = ba\}$.

We have

$$ccabacbab = cca(ba)cbab \to cca(ab)cbab = ccaabc(ba)b$$

$$\to ccaabc(ab)b = c^2a^2bcab^2.$$

In M we have $[u] = [v]$ iff

$$u = s_0t_1s_1 \cdots t_ns_n,\ v = s_0t_1's_1 \cdots t_n's_n$$

where $s_0, s_n \in c^*$, $s_i \in c^+$ for $1 \leq i \leq n-1$ and t_i, t_i' each have the same number of as and the same number of bs for $1 \leq i \leq n$.

We could declare a word such as $c^2a^2bcab^2$ to be in normal form.

Some background on semigroup and monoid presentations
Usually, things are more complicated

Gray (Feb 2019)

There is a **inverse monoid** with one-relator presentation of the form

$$\langle X : u = 1 \rangle$$

with **undecidable** word problem.

The importance of graph products

Graph products include:

1. Graph groups and monoids, also known as right-angled Artin groups/monoids, free partially commutative groups/monoids, trace groups(!)/monoids.

   (Finitely generated monoids and groups defined by presentations in which the only relations have forms $a \circ b = b \circ a$ for generators a, b.)

2. Free products of semigroups/monoids/groups.
3. Restricted direct products of semigroups/monoids/groups.

Let $\Gamma = \Gamma(V, E)$ be a simple and undirected graph with no loops.

Let $\mathcal{S} = \{S_\alpha : \alpha \in V\}$ be a set of mutually disjoint semigroups, called **vertex semigroups**.

The **graph product**

$\mathscr{G}\mathscr{P} = \mathscr{G}\mathscr{P}(\Gamma, \mathcal{S})$ of $\mathcal{S}$ with respect to Γ is defined by the presentation

$$\mathscr{G}\mathscr{P} = \langle X \mid R \rangle$$

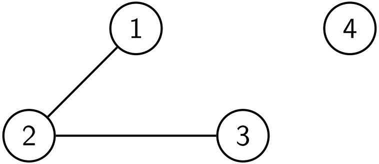
where $X = \bigcup_{\alpha \in V} S_\alpha$ and with defining relations R:

(R_1) $x \circ y = xy$ $(x, y \in S_\alpha, \alpha \in V)$;

(R_2) $x \circ y = y \circ x$ $(x \in S_\alpha, y \in S_\beta, (\alpha, \beta) \in E)$.

Two elements of X^+ are equal in $\mathscr{G}\mathscr{P}$ if and only if one can get from one to the other by substituting a LHS/RHS of a relation by a RHS/LHS.

What does this mean?



In the corresponding $\mathscr{GP}$ how do we handle (with natural convention for labelling)

$$s_1 \circ s_2 \circ s_4 \circ s_4' \circ s_2' \circ s_3 \circ s_2''?$$

We have

$$\begin{aligned}
s_1 \circ s_2 \circ s_4 \circ s_4' \circ s_2' \circ s_3 \circ s_2'' &\to s_1 \circ s_2 \circ s_4 s_4' \circ (s_2' \circ s_3) \circ s_2'' \\
&\to s_1 \circ s_2 \circ s_4 s_4' \circ (s_3 \circ s_2') \circ s_2'' \\
&= s_1 \circ s_2 \circ s_4 s_4' \circ s_3 \circ (s_2' \circ s_2'') \\
&\to s_1 \circ s_2 \circ s_4 s_4' \circ s_3 \circ s_2' s_2''.
\end{aligned}$$

Graph products - of monoids

Here we begin with **vertex monoids** $\mathcal{M} = \{M_\alpha : \alpha \in V\}$, where 1_α is the identity of M_α, and we have the additional relation

(R_3) $1_\alpha = \epsilon$, $\alpha \in V$.

(R_3) has the power to **really** cause trouble. In

$$s_1 \circ s_2 \circ s_4 s_4' \circ s_3 \circ s_2' s_2''$$

if s_4 and s_4' are mutually inverse, then

$$s_1 \circ s_2 \circ s_4 s_4' \circ s_3 \circ s_2' s_2'' = s_1 \circ s_2 \circ I_4 \circ s_3 \circ s_2' s_2'' \rightarrow$$

$$s_1 \circ s_2 \circ \epsilon \circ s_3 \circ s_2' s_2'' = s_1 \circ s_2 \circ s_3 \circ s_2' s_2'' \rightarrow s_1 \circ s_2 \circ s_2' s_2'' \circ s_3 \rightarrow s_1 \circ s_2 s_2' s_2'' \circ s_3.$$

Graph products: left Foata normal form

One first shows that any word may be expressed by a **reduced form**. This is a word of shortest length in an equivalence class.

Green 1990 - for groups

Any two reduced words in the same equivalence class are shuffle equivalent.

The same argument works for monoids and an easier argument for semigroups.

We then show that elements in $\mathscr{G}\mathscr{P}$ may be written in a normal form we refer to as **left Foata normal form.**

In the (semigroup) example above,

$$(s_1 \circ s_2) \circ s_4 s_4' \circ (s_3 \circ s_2' s_2'')$$

is in left Foata normal form.

Multiplying in graph products

In a **semigroup** graph product, if u, v are reduced forms, then if w is any reduced form of $u \circ v$ then

$$|w| \geq \max\{|u|, |v|\}.$$

For **monoids** things are more complicated but not entirely out of control.

Let $x \in X$ and $u = u_1 \circ \cdots \circ u_n \in X^+$ be reduced.
Then $x \circ u$ reduces to one of reduced forms

$$x \circ u, \text{ or } u_1 \circ \cdots \circ xu_k \circ \cdots u_n$$

or

$$u_1 \circ \cdots u_{k-1} \circ u_{k+1} \circ \cdots u_n.$$

Hence in a **monoid** graph product, if u, v are reduced forms, then if w is any reduced form of $u \circ v$ then

$$|w| \geq \max\{|u| - |v|, |v| - |u|\}.$$

Idempotents

An **idempotent** is an element $e \in S$ such that $e = e^2$.

We let $E(S)$ denote the set of idempotents of S.

Idempotents are everywhere

in...

- functional analysis
- classical ring theory
- Stone–Čech compactifications
- idempotent tropical matrices
- Putcha–Renner theory of reductive algebraic monoids
- representation theory of algebras, groups and semigroups

. . . . . .

Regularity properties: regularity

Regularity

A semigroup S is **regular** if for all $a \in S$ there exists a $b \in S$ such that

$$a = aba.$$

If $a = aba$, then immediately

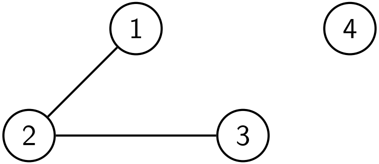
$$(ab)^2 = (ab)(ab) = (aba)b = ab,$$

so that $ab \in E(S)$.

Many natural examples of semigroups are regular (such as $\mathcal{T}_X$), but...many are not, such as X^+ and X^*.

Regularity properties: regularity

We will see that graph products are not, in general, regular, even if the ingredients are.



Note that if s_1, s_4 are not one-sided units, then $s_1 \circ s_4$ (reduced) is not regular as

$$s_1 \circ s_4 \neq s_1 \circ s_4 \circ w \circ s_1 \circ s_4.$$

Regularity properties: another view of regularity

For any semigroup S, the relation $\mathcal{R}$ is defined by the rule

$$a \,\mathcal{R}\, b \iff aS^1 = bS^1.$$

Very easy fact: mutual divisibility and $\mathcal{R}$

$$a \,\mathcal{R}\, b \iff a = bc, b = ad \text{ for some } c, d \in S^1.$$

The relation $\mathcal{L}$ is dual to $\mathcal{R}$.

Easy fact

A semigroup S is regular if and only if for any $a \in S$ we have $e, f \in E(S)$ such that

$$e \,\mathcal{R}\, a \,\mathcal{L}\, f.$$

Regularity, Abundancy and Fountainicity

We have remarked that $\mathcal{R}$ and $\mathcal{L}$ are relations of mutual divisibility.

For any semigroup S, the relation $\mathcal{R}^*$ is defined by the rule that for any $a, b \in S$, we have $a \,\mathcal{R}^*\, b$ if and only if for all $x, y \in S^1$

$$xa = ya \Longleftrightarrow xb = yb.$$

so that $\mathcal{R}^*$ is a relation of mutual cancellativity.

Easy fact

For any semigroup S we have

$$a \,\mathcal{R}\, b \Rightarrow a \,\mathcal{R}^*\, b$$

and if S is regular,

$$a \,\mathcal{R}\, b \Leftrightarrow a \,\mathcal{R}^*\, b.$$

The relation $\mathcal{L}^*$ is defined dually.

Regularity, Abundancy and Fountainicity

Definition

A semigroup S is **left abundant/abundant** if and only if for any $a \in S$ we have $e \in E(S)/e, f \in E(S)$ such that

$$e\,\mathcal{R}^*\,a\,/\,e\,\mathcal{R}^*\,a\,\mathcal{L}^*\,f.$$

Fact

A monoid M is right cancellative if and only if $E(M) = \{1\}$ and it is left abundant.

The above essentially comes from the following

$$a\,\mathcal{R}^*\,1 \Leftrightarrow \big(xa = ya \Rightarrow x = y\big).$$

Regularity, Abundancy and Fountainicity

For any semigroup S, the relation $\widetilde{\mathcal{R}}$ is defined by the rule that for any $a, b \in S$, we have $a \,\widetilde{\mathcal{R}}\, b$ if and only if for any $e \in E(S)$

$$ea = a \Longleftrightarrow eb = b.$$

The relation $\widetilde{\mathcal{L}}$ is defined dually.

It is a **fact** that $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$ and $\mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ with equalities if S is abundant.

Definition

A semigroup S is **weakly left abundant/weakly abundant** or **left Fountain/Fountain** if and only if for any $a \in S$ we have $e \in E(S)/e, f \in E(S)$ such that

$$e \,\widetilde{\mathcal{R}}\, a \,/\, e \,\widetilde{\mathcal{R}}\, a \,\widetilde{\mathcal{L}}\, f.$$

Regularity, abundancy and Fountainicity

Examples Regular semigroups, cancellative monoids, $M_n(\mathbb{Z})$, restriction monoids, partial transformation monoids, $\cdots$, Ehresmann semigroups, ...

These ideas arise from many sources and have many names.

Abundancy originally from notions of projectivity for acts

Fountainicity from various studies of small ordered categories, some in the context of Ehresmann's work on pseudo-groups.

See the work of Fountain, Lawson, Cockett, Manes, Jackson, Stokes, El Qallali, Gomes, Szendrei and many others.

The results

Theorem: Alqahtani, Gould and Yang 2020

Any graph product of left abundant (left Fountain) semigroups is left abundant (left Fountain).

Theorem: Gould and Yang 2021

Any graph product of left abundant (left Fountain) monoids is left abundant (left Fountain).

The proofs are **very** different for monoids, since the extra relation (identifying the monoid identities!) increases the complexity of the reductions.

The results
applications and extensions

Corollary

Any free product of abundant (Fountain) semigroups or monoids is abundant (Fountain).

Corollary

Any restricted direct product of abundant (Fountain) semigroups or monoids is abundant (Fountain).

Corollary: Fountain, Kambites, 2009

Any graph product of right cancellative monoids is right cancellative.

Further thoughts

- Decidability questions
- Calculating other algebraic properties such as centers
- Biordered sets of graph products
- Extensions of our ideas to other presentations (e.g. free idempotent generated semigroups)

Thank you for listening.