INDEPENDENCE ALGEBRAS or v^*-ALGEBRAS

Victoria Gould

1. Motivation

For further reading on the background for v^*-algebras I recommend 'The origins of independence algebras' by J. Araujo and J. Fountain, DOI No: 10.1142/9789812702616_0004

Similarities between End $\mathbf{V}$ (V vector space over division ring) and $\mathcal{T}(X)$:

$\exists$ a natural definition of *rank* : for $\alpha \in$ End $\mathbf{V}$, *rank* α is dim(Im α) and for $\alpha \in \mathcal{T}(X)$, *rank* α is $|\text{Im } \alpha|$.

Let $m \leq n \in \mathbb{N}$ and let $T =$ End $\mathbf{V}$ (dim $V = n$) or $\mathcal{T}_X$, ($|X| = n$). Let

$$T_m = \{\alpha \in T : \dim \alpha \leq m\}.$$

Then T_m is an ideal. We have

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

and these are the *only* ideals of T and for $m \geq 1$ the Rees quotient

$$T_m/T_{m-1}$$

is completely 0-simple. Further, if

$$E = \{\alpha = \alpha^2 : \text{rank } \alpha < n\}$$

then

$$\langle E \rangle = T_{n-1}$$

(for $\mathbf{V}$ this is due to Erdös and Reynolds and Sullivan, for $\mathcal{T}_X$ due to Howie.)

2. Closure operators and matroids

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Then C is a *closure operator* on A if for all $X, Y \in \mathcal{P}(A)$:
(i) $X \subseteq C(X)$;
(ii) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$;
(iii) C(X)=C(C(X)).
Condition (i) says that C is extensive*, condition (ii) that C is* order preserving *and condition (ii) that C is* idempotent.

Let $\mathbf{A}$ be any algebra, then

$$\text{Sg}^{\mathbf{A}} : \mathcal{P}(A) \to \mathcal{P}(A)$$

is a closure operator on A, where

$$X \mapsto \text{Sg}^{\mathbf{A}}(X) = \langle X \rangle.$$

$\text{Sg}^{\mathbf{A}}(\emptyset) = \emptyset$ iff $\mathbf{A}$ has no constants.

A closure operator C on a set B is *algebraic* if for all $X \subseteq B$

$$C(X) = \bigcup \{C(Y) : Y \subseteq X, |Y| < \infty\}.$$

Clearly $\mathrm{Sg}^{\mathbf{A}}$ is an *algebraic closure operator.*

Definition The *exchange property* (EP) for a closure operator C on a set A is defined as follows:

(EP) For all $X \subseteq A$ and $x, y \in A$, if $x \notin C(X)$ but $x \in C(X \cup \{y\})$, then $y \in C(X \cup \{x\})$.

Definition A set A with an algebraic closure operator with (EP) is a *matroid.*

there are many equivalent ways of defining a matroid.

Examples (1) $\mathbf{V}$ vector space. We have

$$\langle X \rangle = \{\lambda_1 x_{i_1} + \dots \lambda_n x_{i_n} : n \in \mathbb{N}, \lambda_i \in D, x_{i_j} \in X\}.$$

Let

$$x \in \langle X \cup \{y\} \rangle \text{ but } x \notin \langle X \rangle.$$

Then

$$x = \lambda_1 x_{i_1} + \dots \lambda_n x_{i_n} + \lambda y$$

for $\lambda, \lambda_i \in D, x_{i_j} \in X$. Note $\lambda \neq 0$. Then

$$y = \frac{1}{\lambda} x + \frac{-\lambda_1}{\lambda} x_{i_1} \dots + \frac{-\lambda_n}{\lambda} x_{i_n} \in \langle X \cup \{x\} \rangle.$$

(2) Let X be a set - it is an algebra with no operations. Then

$$\langle X \rangle = X$$

so if

$$x \in \langle X \cup \{y\} \rangle \text{ but } x \notin \langle X \rangle$$

then

$$y = x \in \langle X \cup \{x\} \rangle.$$

(3) $\mathbf{A}$ a G-act on a set A over a group G. We have

$$\langle X \rangle = \bigcup_{x \in X} Gx$$

so if

$$x \in \langle X \cup \{y\} \rangle \text{ but } x \notin \langle X \rangle$$

then

$$x = gy$$

for some $g \in G$, hence

$$y = g^{-1} x \in \langle X \cup \{y\} \rangle.$$

If G is trivial, then effectively, $\mathbf{A}$ *is a set*

3. Independence algebras

The next ingredient of our approach to the definition of an independence algebra is that of an independent subset.

Definition Let C be a closure operator on a set A, and let $X \subseteq A$. Then X is C-*independent* if for all $x \in X$, $x \notin C(X \setminus \{x\})$.

If $\mathbf{A}$ is an algebra then we refer to $\mathrm{Sg}^{\mathbf{A}}$-independent sets more simply as *independent* sets.

Example (1) The independent subsets of $\mathbf{V}$ are the linearly independent subsets.
(2) For sets, every subset is independent.
(3) $\mathbf{A}$ a G-set, then X is independent iff X contains at most one element from each orbit, that is, $|X \cap Gx| \leq 1$ for each $x \in S$.

Fact Let C be an algebraic closure operator with (EP) on a set A, and let $Y \subseteq X \subseteq A$. Then the following conditions are equivalent:
(i) Y is a maximal C-independent subset of X;
(ii) Y is C-independent and $C(Y) = C(X)$;
(iii) Y is minimal with respect to $C(Y) = C(X)$.

Definition Let $\mathbf{A}$ be an algebra. A *basis* of A is a minimal generating set.
So, if (EP) holds for $\langle , \rangle$, *them a basis is also a maximum independent set.*

Definition An algebra $\mathbf{A}$ algebra has the *free basis* property if (F) holds:
(F) For any basis X of A and function $\alpha : X \to A$, α can be extended to an element of End $\mathbf{A}$.

Definition An algebra $\mathbf{A}$ is an independence algebra if it has (EP) and (F).

Examples of Independence algebras (1) $\mathbf{V}$
(2) X a set - a basis is just X.
(3) A basis of a G-set $\mathbf{A}$ is a transversal X of orbits. If $\alpha : X \mapsto A$, can extend $\alpha : A \mapsto A$ by putting $gx \mapsto g(x\alpha)$.

Definition Let C be an algebraic closure operator with (EP) on a set X. The C-*rank* of X, written $\rho_C(X)$, is $|Y|$, where Y is a maximal C-independent subset of X.

Fact ρ_C is well-defined.

4. The results

Let $\mathbf{A}$ be an independence algebra. Write ρ for $\rho_{\langle\,,\rangle}$.

Definition Let $\alpha \in \text{End } \mathbf{A}$. The *rank* of α is $\rho(\text{Im } \alpha)$.

Theorem (G) Let $\mathbf{A}$ be an independence algebra of rank n. Let $m \leq n \in \mathbb{N}$ and let

$$T_m = \{\alpha \in \text{ End } \mathbf{A} : \dim \alpha \leq m\}.$$

Then T_m is an ideal of End $\mathbf{A}$. We have

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

and these are the *only* ideals of End $\mathbf{A}$. For $m \geq 1$ the Rees quotient

$$T_m/T_{m-1}$$

is completely 0-simple.

Theorem (Fountain and Lewin) Further, if

$$E = \{\alpha = \alpha^2 : \text{rank } \alpha < n\}$$

then

$$\langle E \rangle = T_{n-1}.$$