INDEPENDENCE STRUCTURES

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1 Introduction

The study of the monoid of endomorphisms of an *independence algebra* was initiated in [G] and developed in [FL1,2],[F2],[L] and [Li]. Independence algebras provide a common generalisation of vector spaces and sets. The first key idea is that Sg^A has the *exchange property*, where Sg^A is the closure operator given by $Sg^A(B) =< B >$ for all subsets B of A. This ensures the existence of a basis of A. The second requirement for A to be an independence algebra is that every map on a basis may be extended to an element of End A.

An *independence space* is a set A together with an algebraic closure operator satisfying the exchange property. Independence algebras are of course independence spaces. Careful study of [G] reveals the gist of the approach to be that given any basis X and any map $\alpha : X \to A$, then α may be extended to a map $\overline{\alpha} : A \to A$ such that $Sg^A(B)\overline{\alpha} = Sg^A(B\alpha)$ for all $B \subseteq A$. This motivates the following definitions.

Let A be an independence space with closure operator C. A map α such that $C(B)\alpha = C(B\alpha)$ for all $B \subseteq A$ we call a *C-morphism.* Such maps are precisely the closed continuous maps of A. Clearly the composition of C-morphisms is again a C-morphism. The independence space A is a *weak independence structure* if any map on a basis may be extended to a C-morphism of A. If we have the stronger property that given any subset X of a basis, any map from X to A can be *uniquely* extended to a C-morphism from $C(X)$ to A, we say that A is an *independence structure.*

If A is an independence space then the notion of the *rank* ρ of a subset B is well defined. We then define the *rank* $\rho(\gamma)$ of a C-morphism γ to be $\rho(Im\gamma)$. If α, β are C-morphisms of A we show that $\rho(\alpha\beta) \leq \min\ \{\rho(\alpha), \rho(\beta)\}$, hence if $n \in \mathbf{N}$ and $T_n = \{\alpha : \rho(\alpha) \leq n\}$, then T_n is an ideal of $\mathrm{Mor}_C(A)$, the monoid of all C-morphisms of A. The main result of this paper shows that, in the case where A is an independence structure, the Rees quotients T_n/T_{n-1} are completely 0-simple and we give a specific construction for the corresponding Rees matrix semigroups. The situation is parallel to that for independence algebras.

Section 2 consists of definitions and some preliminary results required for the rest of the paper. In Section 3 we consider what can be said of the Rees quotients T_n/T_{n-1} of $\mathrm{Mor}_C(A)$ where A is an independence space or weak independence structure but not necessarily an independence structure. If A is a weak independence structure then T_n/T_{n-1} is primitive and has 0 as a prime ideal. It follows that if $\mathrm{Mor}_C(A)$ is regular, then T_n/T_{n-1} is completely 0-simple. This we discuss in Section 4. We give an example of a weak independence structure A (*not* an independence structure) such that $\mathrm{Mor}_C(A)$ is regular.

The final section concentrates on the case where A is an independence structure. Here we show that $\mathrm{Mor}_C(A)$ is regular so that the quotients T_n/T_{n-1} are completely 0-simple, and describe the corresponding Rees matrix semigroups. In fact, our results are true in a wider context and we state them for such. This allows us to deduce in particular the results of [G] for the endomorphism monoid of an independence algebra.

The work is interspersed with a number of examples of independence spaces and (weak) independence structures. We assume the reader has a basic knowledge of semigroup theory and universal algebra and recommend [H] and [MMT] as references. As far as possible we follow the notation of [H] and [MMT] with the exception that we do not distinguish between an algebra and its universe, unless absolutely necessary.

I would like to thank Marcel Wild for his help with the early stages of this paper, and Matthew Gould for his careful reading of the text.

2 Preliminaries

Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a function, where $\mathcal{P}(A)$ is the set of all subsets of A. Then C is a *closure operator* on A if C satisfies the following conditions for all $X, Y \subseteq A$:

(i) $X \subseteq C(X)$;

(ii) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$;

(iii) $C(X) = C(C(X))$.

Condition (i) says that C is *extensive*, condition (ii) that C is *order preserving* and condition (iii) that C is *idempotent*. If $B \subseteq A$ and $B = C(X)$ for some $X \subseteq A$ (equivalently, $B = C(B)$), then B is *closed*.

A natural example of a closure operator is Sg^A for any algebra A, where $Sg^A(B)$ is the subuniverse generated by B, for any $B \subseteq A$. The closure operators Sg^A are precisely the *algebraic closure operators*, where a closure operator C on a set A is *algebraic* if for all $X \subseteq A$

$$C(X) = \bigcup\{C(Y) : Y \subseteq X, |Y| < \aleph_0\}.$$

The *exchange property* (EP) for a closure operator C on a set A is defined as follows.

(EP) For all $X \subseteq A$ and $x, y \in A$, if $x \notin C(X)$ but $x \in C(X \cup \{y\})$, then $y \in C(X \cup \{x\})$.

If A is a finite set then (EP) says that C is the closure operator associated with a *matroid*. An *independence space* is a set A together with an algebraic closure operator satisfying (EP). Thus independence spaces may be thought of as possibly infinite matroids. By definition an independence algebra is an independence space.

A map $\alpha : A \to A$ where A is an independence space is a *C-morphism* if $C(B\alpha) = C(B)\alpha$ for all $B \subseteq A$. Clearly this definition is satisfied by homomorphisms of an independence algebra. Moreover, given an independence space A, a C-morphism of A is a homomorphism of the algebra on $\mathcal{P}(A)$ whose single operation is the unary operation $B \to C(B)$. However it is not difficult to see that the closure operator $\overline{C} = Sg^{\mathcal{P}(A)}$ associated with this unary algebra need not have (EP). For consider any subset B of A with $C(\emptyset) \subset B \subset C(B)$. (For a concrete example, take a non-empty subset B of a free left G-set S over a group G such that B is not a union of orbits. From [G] S is an independence algebra, thus an independence space). Then $C(B) \in \overline{C}(\{B\}) = \{B, C(B)\}$ but $C(B) \notin \overline{C}(\{\emptyset\}) = \{\emptyset, C(\emptyset)\}$. If $\overline{C}$ had (EP) then $B \in \overline{C}(\{C(B)\}) = \{C(B)\}$, a contradiction.

Clearly the set $\mathrm{Mor}_C(A)$ of C-morphisms of an independence space A is a submonoid of the full transformation monoid $\mathcal{T}_A$ of A. Our aim is to describe the behaviour of $\mathrm{Mor}_C(A)$ in various special cases.

Let C be a closure operator on a set A and let $X \subseteq A$. Then X is *independent* if, for all $x \in X, x \notin C(X \setminus \{x\})$. A *basis* of a subset B of an independence space A is a subset X of B minimal with respect to the property that $C(X) = C(B)$. In view of the following result, bases of A are precisely maximal independent subsets of A.

Proposition 2.1 *Let A be an independence space and $Y \subseteq X \subseteq A$. Then the following conditions are equivalent :*

(i) Y is a maximal independent subset of X;

(ii) Y is independent and $C(Y) = C(X)$;

(iii) Y is minimal with respect to $C(Y) = C(X)$.

Proposition 3.1 is essentially folklore; a proof can be found in [L] or [G]. Let X be an independent subset of a subset B of an independence space A. A straightforward argument using Zorn's Lemma shows that $X \subseteq Y$ for a basis Y of B. Now $Y = X \cup (Y \setminus X)$ and $X \cap (Y \setminus X) = \emptyset$. If Z is any subset of B such that $X \cup Z$ is a basis of B and $X \cap Z = \emptyset$, then we say that Z *extends* X to a basis of B. Since $\emptyset$ is always independent, the above implies that every subset of A has a basis.

A *weak independence structure* is an independence space A such that for any basis X of A, any map $\alpha : X \to A$ may be extended to a C-morphism of A. An *independence structure* is a weak independence structure A such that for any independent set Y of A, any map $\beta : Y \to A$ may be extended to a *unique* C-morphism from $C(Y)$ to A. Independence algebras are weak independence structures, but not necessarily independence structures, as we now show.

Example Let S be a free left G-set on X over a group G, and let α be a non-identity automorphism of G. Define $\beta_\alpha : S \to S$ by $(ax)\beta_\alpha = (a\alpha)x$ for all $a \in G, x \in X$. It is easy to see that $(gs)\beta_\alpha = (g\alpha)(s\beta_\alpha)$ for any $g \in G$ and $s \in S$. Thus if $T \subseteq S$, $Sg^S(T)\beta_\alpha = \{gt : g \in G, t \in T\}\beta_\alpha = \{(g\alpha)(t\beta_\alpha) : g \in G, t \in T\} = \{g(t\beta_\alpha) : g \in G, t \in T\} = Sg^S(T\beta_\alpha)$ so that β_α is an Sg^S-morphism. However, β_α is not a homomorphism of S. As commented earlier, S is an independence algebra. Clearly S is not an independence structure.

Before giving further examples, we introduce the central concept of rank. Let X be a subset of an independence space A. Then the *rank* of X, written $\rho(X)$, is $|Y|$, where Y is a basis of X. It is a fact that $\rho(X)$ is well-defined and $\rho(X) = \rho(C(X))$. The details may be found in [L]. We remark that if Z is an independent subset of a closed set B where $\rho(B) < \aleph_0$, then $|Z| \leq \rho(B)$ and $|Z| = \rho(B)$ if and only if $C(Z) = B$.

Example Let $U_{k,n}$ be the uniform matroid of rank k on a set A_n of cardinality n. The independent sets of $U_{k,n}$ are all subsets of A_n of cardinality less than or equal to k. If $B \subseteq A_n$ then

$$C(B) = \begin{cases} B & |B| < k \\ A & |B| \geq k \end{cases}$$

Further details may be found in [W]. Note that for any map β on A_n, $C(\emptyset)\beta = C(\emptyset\beta)$.

Consider $U_{2,n}$ for some $n \geq 3$. Let $X = \{x, y\}$ be a basis of $U_{2,n}$ and $\alpha : X \to A_n$. Suppose that $x\alpha = y\alpha = u$. Let κ_u be the constant map on A_n with image $\{u\}$. Then for any non-empty subset B of A_n, $C(B)\kappa_u = \{u\} = C(\{u\}) = C(B\kappa_u)$. On the other hand, if α is one-one then we may extend α to a bijection $\overline{\alpha}$ of A_n. If $n > 3$ this may be done in more than one way. Let $B \subseteq A_n, |B| = 1$. Then $C(B) = B$ and $C(B)\overline{\alpha} = B\overline{\alpha} = C(B\overline{\alpha})$. If $|B| \geq 2$ then $|B\overline{\alpha}| \geq 2$ and $C(B)\overline{\alpha} = A_n\overline{\alpha} = A_n = C(B\overline{\alpha})$. Thus in either case, α extends to a C-morphism. Hence $U_{2,n}$ is a weak independence structure but not an independence structure for $n > 3$. In fact it is easy to show that $U_{2,3}$ is an independence structure.

Example We show that $U_{3,4}$ is an independence structure.

For any partial map $\alpha : A_4 \to A_4$ and any non-empty subset B of A_4 with $|B| \leq 2$, we have $|B\alpha| \leq 2$ and so $C(B)\alpha = B\alpha = C(B\alpha)$. In particular, if $|B| \leq 2$ and $\alpha : B \to A_4$, then α is a C-morphism. Suppose now that $|B| = 3$, say $B = \{x, y, z\}$ where $A_4 = \{x, y, z, t\}$ and $\alpha : B \to A_4$ is a map.

Case (i) $x\alpha = y\alpha = z\alpha = u$. For any C-morphism $\overline{\alpha}$ extending α we must have $A_4\overline{\alpha} = C(B)\overline{\alpha} = C(B\overline{\alpha}) = C(B\alpha) = C(\{u\}) = \{u\}$. So put $t\overline{\alpha} = u$. Then $\overline{\alpha}$ is a C-morphism.

Case (ii) $x\alpha = y\alpha = u, z\alpha = v$ where $u, v \in A$ and $u \neq v$. Again, if $\overline{\alpha}$ is a C-morphism extending α then $A_4\overline{\alpha} = C(B\alpha) = \{u, v\}$. Suppose that $t\overline{\alpha} = u$. Then $C(\{x, y, t\})\overline{\alpha} = A_4\overline{\alpha} = \{u, v\}$ but $C(\{x\overline{\alpha}, y\overline{\alpha}, t\overline{\alpha}\}) = C(\{u\}) = \{u\}$, a contradiction. So put $t\overline{\alpha} = v$. If $D \subseteq A_4$ with $|D| \geq 3$, then $D\overline{\alpha} = \{u, v\}$. So $C(D\overline{\alpha}) = C(\{u, v\}) = \{u, v\} = A_4\overline{\alpha} = C(D)\overline{\alpha}$. Thus $\overline{\alpha}$ is a C-morphism.

Case (iii) α is one-one. Any C-morphism extending α must satisfy $C(B)\overline{\alpha} = C(B\alpha)$, that is, $A_4\overline{\alpha} = A_4$. So complete α to a bijection $\overline{\alpha}$ of A_4; clearly this can be done in a unique way and yields a C-morphism.

This completes the proof that $U_{3,4}$ is an independence structure.

Our last example of this section gives a method of constructing independence structures from rather special independence algebras.

Example Let C be any algebraic closure operator on a set A. A standard construction gives an algebra A_C such that $C = Sg^{A_C}$. The operations of A_C are defined as follows. Put

$$T = \{\overline{a} : \overline{a} = (a_0, a_1, ..., a_n), n \in \mathbf{N}, a_n \in C(\{a_0, ..., a_{n-1}\})\}.$$

For each $\overline{a} \in T$ define an operation $f_{\overline{a}}$ by

$$f_{\overline{a}}(x_0, ..., x_{n-1}) = \begin{cases} a_n & \text{if } x_i = a_i, i < n \\ x_0 & \text{else.} \end{cases}$$

Then $A_C = (A; \{f_{\overline{a}} : \overline{a} \in T\})$ is an algebra such that $Sg^{A_C} = C$.

Let $\alpha \in$ End A_C; we claim that α is idempotent. For if $a \in A$ and $a \neq a\alpha$ put $\overline{a} = (a\alpha, a, a)$. Now $f_{\overline{a}}(a\alpha, a) = a$ so that $a\alpha = f_{\overline{a}}(a\alpha, a)\alpha = f_{\overline{a}}(a\alpha^2, a\alpha)$. But $a \neq a\alpha$ so that $f_{\overline{a}}(a\alpha^2, a\alpha) = a\alpha^2$. Hence α is idempotent.

Suppose now that A is an independence algebra such that every Sg^A-morphism from a subalgebra to A is a homomorphism; put $C = Sg^A$. An example of such an A is a free left G-set over Z_2. The algebra A_C can be constructed as above and $Sg^{A_C} = C$. Thus the Sg^B-morphisms of subalgebras of B to B for the algebras $B = A_C$ and $B = A$ coincide with the homomorphisms from subalgebras of A to A. Further, $X \subseteq A$ is independent in the algebra A if and only if it is independent in A_C. Then if $X \subseteq A$ is independent and $\alpha : X \to A$, α can be uniquely extended to a homomorphism $C(X) \to A$, that is, α has a unique extension to an Sg^{A_C}-morphism. However, if X is a basis of A then, unless α is idempotent, α is not an endomorphism of A_C. Certainly free left Z_2-sets can have non-idempotent endomorphisms.

Finally in this section we consider the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ for a semigroup S. The relation $a\mathcal{L}^*b$ is defined by the rule that for any $a, b \in S$, $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually.

Proposition 2.2 *[F1] For any semigroup S and $a, b \in S$, $a\mathcal{L}^*b$ if and only if for all $x, y \in S^1$,*

$$ax = ay \text{ if and only if } bx = by.$$

The dual proposition holds for $\mathcal{R}^*$.

Clearly $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. It is not difficult to show that if S is regular then these inclusions are equalities. The semigroup S is *right (left) abundant* if every $\mathcal{L}^*$-class (every $\mathcal{R}^*$-class) contains an idempotent. Certainly regular semigroups are left and right abundant; the converse is not true in general.

3 Weak independence structures

We begin with a remark concerning arbitrary subsemigroups of the full transformation monoid $\mathcal{T}_A$ on a set A.

Lemma 3.1 *Let S be a subsemigroup of $\mathcal{T}_A$. Then for any $\alpha, \beta \in S$,*

$$Im\alpha = Im\beta \text{ implies } \alpha\mathcal{L}^*\beta$$

$$Ker\alpha = Ker\beta \text{ implies } \alpha\mathcal{R}^*\beta.$$

Proof In $\mathcal{T}_A$, $Im\alpha = Im\beta$ if and only if $\alpha\mathcal{L}\beta$ so certainly $Im\alpha = Im\beta$ implies that $\alpha\mathcal{L}^*\beta$ in $\mathcal{T}_A$ and hence $\alpha\mathcal{L}^*\beta$ in S. A similar argument gives the result for kernels.

Let A be an independence space and $\alpha : B \to A$ a C-morphism for some closed subset B of A. The *rank* of α is $\rho(Im\alpha)$. Note that as α is a C-morphism, $Im\alpha = B\alpha = C(B)\alpha = C(B\alpha)$ so that $Im\alpha$ is closed. For $n \in \mathbf{N}$ we define a subset T_n of A by

$$T_n = \{\alpha \in \ \mathrm{Mor}_C(A) : \rho(\alpha) \leq n\}.$$

The following result shows that T_n is an ideal of $\mathrm{Mor}_C(A)$ for each $n \in \mathbf{N}$ (where we allow $\emptyset$ to be an ideal).

Lemma 3.2 *Let A be an independence space and $\alpha, \beta \in Mor_C(A)$. Then*

$$\rho(\alpha\beta) \leq min\{\rho(\alpha), \rho(\beta)\}.$$

Proof Since $Im\alpha\beta \subseteq Im\beta$ it is clear that $\rho(\alpha\beta) \leq \rho(\beta)$. Let X be a basis of $Im\alpha$ so that $Im\alpha = C(X)$. Then $Im\alpha\beta = C(X)\beta = C(X\beta)$ and $\rho(\alpha\beta) = \rho(Im\alpha\beta) = \rho(C(X\beta)) = \rho(X\beta) \leq |X\beta| \leq |X| = \rho(\alpha)$.

Thus each T_n is an ideal of A and if $n \geq 1$ and $T_n \setminus T_{n-1} \neq \emptyset$ we can form the Rees quotient T_n/T_{n-1}.

We recall that a semigroup is *primitive* if all its non-zero idempotents are primitive. That is, if $e, f \in E(S) \setminus \{0\}$ and $ef = e = fe$, then $e = f$. A semigroup S with zero is *categorical at* 0 if given $a, b, c \in S$, $ab \neq 0$ and $bc \neq 0$ imply that $abc \neq 0$. We say that 0 *is a prime ideal of* S if for any $a, b \in S \setminus \{0\}$ there exists $c \in S$ such that $acb \neq 0$. *Completely 0-simple semigroups* can be characterised as primitive regular semigroups in which 0 is a prime ideal [H]; the structure of such semigroups may be described using *Rees matrix semigroups.* A fundamental property of a completely 0-simple semigroup is that it is categorical at zero.

Proposition 3.3 *Let A be an independence space. Then if $T_0 \neq \emptyset$, T_0 is primitive . For $n \in \mathbf{N}$, $n \geq 1$, if $T_n \setminus T_{n-1} \neq \emptyset$, the Rees quotient T_n/T_{n-1} is primitive and categorical at 0.*

Proof Let ϵ, ν be non-zero idempotents in T_n/T_{n-1}, or idempotents in T_0. We may think of ϵ, ν as idempotents of the same rank. Suppose that $\epsilon\nu = \epsilon = \nu\epsilon$. Then $Im\epsilon \subseteq Im\nu$; $Im\epsilon$ and $Im\nu$ are closed subsets of the same rank, whence $Im\epsilon = Im\nu$ and from Lemma 3.1, $\epsilon\mathcal{L}^*\nu$ in T_n. Then $\epsilon\epsilon = \epsilon 1$ gives $\nu\epsilon = \nu 1 = \nu$. But $\nu\epsilon = \epsilon$ so that $\nu = \epsilon$ and T_n/T_{n-1} (T_0) is primitive.

Suppose now that $\alpha, \beta, \gamma \in T_n/T_{n-1}$ and $\alpha\beta$, $\beta\gamma$ are non-zero, that is, $\rho(\alpha\beta) = \rho(\beta\gamma) = n$. Then $Im\alpha\beta = Im\beta$ and $Im\alpha\beta\gamma = Im\gamma|_{Im\alpha\beta} = Im\gamma|_{Im\beta} = Im\beta\gamma$ so that $\rho(\alpha\beta\gamma) = \rho(\beta\gamma) = n$ and $\alpha\beta\gamma$ is non-zero in T_n/T_{n-1}. Thus T_n/T_{n-1} is categorical at zero.

Lemma 3.4 *Let A be an independence space and α a C-morphism from a closed subspace of A to A. If α is one-one on a set X and $X\alpha$ is independent, then X is independent.*

Proof Suppose that $x \in X$ and $x \in C(X \setminus \{x\})$. Since α is a C-morphism we obtain $x\alpha \in C(X \setminus \{x\})\alpha = C(X\alpha \setminus \{x\alpha\})$, a contradiction. Thus X is independent.

If A is a weak independence structure we can say more about the factors T_n/T_{n-1}. Before proceeding we introduce the concept of a preimage basis. Let A be an independence space and let $\alpha : B \to A$ be a C-morphism for some closed subset B of A. Then $Im\alpha$ has a basis X where $|X| = \rho(\alpha)$ and $Im\alpha = C(X)$. For each $x \in X$ choose $y_x \in B$ with $y_x\alpha = x$. Put $Y = \{y_x : x \in X\}$. We say that Y is a *preimage basis* of α. Note that α is one-one on Y so that $|Y| = |X|$ and by Lemma 2.4, Y is independent.

For wider application at a later stage we work with Rees factors of subsemigroups of $\mathrm{Mor}_C(A)$. Let A be a weak independence structure. We say that a subsemigroup S of $\mathrm{Mor}_C(A)$ is *extensive* if for any basis X of A and map $\alpha : X \to A, \alpha$ can be extended to a C-morphism in S. Then for $n \in \mathbf{N}$ we define

$$T_n^S = T_n \cap S$$

so that each T_n^S is an ideal of S.

Proposition 3.5 *Let A be a weak independence structure and let S be an extensive subsemigroup of $Mor_C(A)$. Then $T_0^S \neq \emptyset$ if and only if $C(\emptyset) \neq \emptyset$.*

Proof Suppose that $T_0^S \neq \emptyset$. Then there is some $\alpha \in S$ with $\rho(\alpha) = 0$. Thus $\emptyset$ is a basis for $Im\alpha$ and $\emptyset \neq Im\alpha = C(\emptyset)$.

Conversely, if $C(\emptyset) \neq \emptyset$, let X be a basis for A and $c \in C(\emptyset)$. Define $\alpha : X \to A$ by $x\alpha = c$ for all $x \in X$. Extend α to an element of S. Then $Im\alpha = A\alpha = C(X)\alpha = C(X\alpha) = C(c) = C(\emptyset)$ so that $\rho(\alpha) = 0$ and $T_0^S \neq \emptyset$.

Proposition 3.6 *Let A be a weak independence structure and let S be an extensive subsemigroup of $Mor_C(A)$. Let $n \in \mathbf{N}$, where $1 \leq n \leq \rho(A)$. Then $T_n^S \setminus T_{n-1}^S \neq \emptyset$ and 0 is a prime ideal of T_n^S/T_{n-1}^S.*

Proof Let $X \subseteq A$ be an independent set such that $\rho(X) = n$. Extend X to a basis $X \cup Y$ of A. Define $\alpha : X \cup Y \to A$ by

$$\begin{aligned} x\alpha &= x \quad \text{for all } x \in X \\ y\alpha &= x_0 \quad \text{for all } y \in Y \end{aligned}$$

where $x_0 \in X$ is fixed. Then α extends to an element of S, which we refer to also as α. Now $Im\alpha = A\alpha = C(X \cup Y)\alpha = C((X \cup Y)\alpha) = C(X)$ so that $\rho(\alpha) = \rho(Im\alpha) = \rho(C(X)) = \rho(X) = n$.

To see that 0 is a prime ideal of T_n^S/T_{n-1}^S, let $\alpha, \beta \in T_n^S \setminus T_{n-1}^S$ and choose a basis X of $Im\alpha$ and a preimage basis Y of β. Then $|X| = |Y| = n$ and X, Y are independent. Extend X to a basis $X \cup Z$ of A and define $\gamma : X \cup Z \to A$ by

$$\begin{aligned} x\gamma &= y_x \quad \text{for all } x \in X \\ z\gamma &= y_0 \quad \text{for all } z \in Z \end{aligned}$$

where $x \to y_x$ is a bijection $X \to Y$, and $y_0 \in Y$ is fixed. Extend γ to an element of S. Then $\rho(\gamma) = \rho(Im\alpha) = \rho(A\alpha) = \rho(C(X \cup Z)\gamma) = \rho(C((X \cup Z)\gamma)) = \rho(C(Y)) = \rho(Y) = n$ so that $\gamma \in T_n^S \setminus T_{n-1}^S$. Further, $Im\alpha\gamma\beta = Im\gamma\beta|_{Im\alpha} = C(X)\gamma\beta = c(X\gamma\beta) = C(Y\beta) = Im\beta$ so that $\rho(\alpha\gamma\beta) = \rho(\beta) = n$ and $\alpha\gamma\beta \in T_n^S \setminus T_{n-1}^S$. This shows that T_n^S/T_{n-1}^S has 0 as a prime ideal.

The structure of primitive right (left) abundant semigroups is known [FG]. If A is an independence algebra, then in $\mathrm{Mor}_{Sg^A}(A), T_n/T_{n-1}$ is primitive right abundant, as we now show.

Proposition 3.7 *Let A be an independence algebra. Then for $1 \leq n \leq \rho(A)$ where $n \in \mathbf{N}$ the Rees quotient T_n/T_{n-1} (in the semigroup $Mor_{Sg^A}(A)$) is primitive right abundant. Further, if $T_0 \neq \emptyset$ then T_0 is primitive right abundant.*

Proof Let α be an Sg^A-morphism with $\rho(\alpha) = n$. Then $Im\alpha$ is a subuniverse and has a basis X with $Sg^A(X) = Im\alpha$ and $|X| = n$. It follows from Theorem 4.12 of [G] that there is an idempotent endomorhpism β of A with $Im\beta = Sg^A(X)$ and by Lemma 3.1, $\alpha\mathcal{L}^*\beta$ in $\mathrm{Mor}_C(A)$. For any γ in T_n/T_{n-1}, $\alpha\gamma = 0$ if and only if $\gamma = 0$ or $\gamma \neq 0$ and $\rho(\alpha\gamma) \leq n-1$. Note that $\rho(\alpha\gamma) = \rho(\gamma|_{Im\alpha}) = \rho(\gamma|_{Im\beta}) = \rho(\beta\gamma)$ so that $\alpha\gamma = 0$ if and only if $\beta\gamma = 0$. It follows that $\alpha\mathcal{L}^*\beta$ in T_n/T_{n-1}.

An easy adjustment yields the result for T_0.

Let A be an independence space and α a C-morphism with preimage basis X. Then α is one-one on X. To show that α is $\mathcal{R}^*$-related to an idempotent we need the stronger assumption that α is one-one on $C(X)$.

Lemma 3.8 *Let A be an independence space and let α be a C-morphism of A with preimage basis X. Then $Ker\alpha = Ker\beta$ for some idempotent C-morphism β with $Im\beta = C(X)$ if and only if α is one-one on $C(X)$.*

Proof Suppose that α is one-one on $C(X)$. Define $\beta : A \to A$ by $p\beta = q$ where $p\alpha = q\alpha$, $q \in C(X)$. By assumption, β is well defined.

For any $D \subseteq A$,

$$C(D)\beta = \{q : p\alpha = q\alpha, p \in C(D), q \in C(X)\}$$

and

$$C(D\beta) = C\{q : p\alpha = q\alpha, p \in D, q \in C(X)\}$$

so that $C(D)\beta, C(D\beta)$ are subsets of $C(X)$ and

$$C(D)\beta\alpha = \{q\alpha : p\alpha = q\alpha, p \in C(D), q \in C(X)\}$$

$$= \{p\alpha : p \in C(D)\} = C(D)\alpha$$

and

$$C(D\beta)\alpha = C(D\beta\alpha) = C\{q\alpha : p\alpha = q\alpha, p \in D, q \in C(X)\} = C\{p\alpha : p \in D\} = C(D\alpha) = C(D)\alpha.$$

Since α is one-one on $C(X)$ it follows that $C(D)\beta = C(D\beta)$. Clearly β is idempotent with image $C(X)$ and $\beta\alpha = \alpha$. Thus $Ker\beta \subseteq Ker\alpha$. If $p, p' \in A$ and $p\alpha = p'\alpha$ then $p\beta = q, p'\beta = q'$, for some $q, q' \in C(X)$ with $p\alpha = q\alpha, p'\alpha = q'\alpha$ so that $q\alpha = q'\alpha$ and $q = q'$. Hence $(p, p') \in Ker\beta$ and so $Ker\alpha = Ker\beta$.

The converse is straightforward.

Corollary 3.9 *Let A be an independence space such that any C-morphism α has a preimage basis X with α one-one on $C(X)$. Then $Mor_C(A)$ is left abundant. If $1 \leq n \leq \rho(A)$ where $n \in \mathbf{N}$ and $T_n \setminus T_{n-1} \neq \emptyset$ then T_n/T_{n-1} is primitive left abundant. Further, if $T_0 \neq \emptyset$, then T_0 is primitive left abundant.*

Proof Note that in the proof of Lemma 3.8, $\rho(\alpha) = \rho(\beta)$. Lemma 3.8 together with Lemma 3.1 and Proposition 3.3 then give that $\mathrm{Mor}_C(A)$ is left abundant and if $T_0 \neq \emptyset$, T_0 is primitive left abundant.

Now let $1 \leq n \leq \rho(A)$ where $n \in \mathbf{N}$ and suppose that $T_n \setminus T_{n-1} \neq \emptyset$. For any $\alpha \in \mathrm{Mor}_C(A)$ with rank n there is an idempotent $\beta \in \mathrm{Mor}_C(A)$ with rank n such that $Ker\alpha = Ker\beta$. Let $\gamma \in \mathrm{Mor}_C(A)$ have rank n and suppose that $\gamma\alpha = 0$ in T_n/T_{n-1}, that is, $\rho(\alpha\gamma) < n$. Writing $Im\gamma = C(\{u_1, ..., u_n\})$ we have that $\rho(Im\gamma\alpha) = \rho(\alpha|_{Im\gamma}) = \rho(C(\{u_1, ..., u_n\})\alpha) = \rho(C(\{u_1\alpha, ..., u_n\alpha\})) = \rho(\{u_1\alpha, ..., u_n\alpha\})$ so that for some $i \in \{1, ..., n\}$, say $i = 1$, $u_1\alpha \in C(\{u_2\alpha, ..., u_n\alpha\}) = C(\{u_2, ..., u_n\})\alpha$. Hence $u_1\beta \in C(\{u_2, ..., u_n\})\beta$ and it follows that $\rho(Im\gamma\beta) < n$ and $\gamma\beta = 0$ in T_n/T_{n-1}. It is now straightforward to see that $\beta\mathcal{R}^*\alpha$ in T_n/T_{n-1}.

4 The regular case

Let A be a weak independence structure. If S is a regular extensive subsemigroup of $\mathrm{Mor}_C(A)$, then each ideal T_n^S $(n \leq \rho(A))$ and each Rees quotient T_n^S/T_{n-1}^S $(1 \leq n \leq \rho(A))$ is also regular. Hence

Lemma 4.1 *Let A be a weak independence structure and let S be a regular extensive subsemigroup of $Mor_C(A)$. Then for each $n \in \mathbf{N}$ with $1 \leq n \leq \rho(A)$, T_n^S/T_{n-1}^S is completely 0-simple. Further, if $T_0^S \neq \emptyset$, then T_0^S is completely simple.*

Proof This follows from Propositions 3.3 and 3.6 and Theorem III 3.5 of [H].

Lemma 4.2 *Let A be a weak independence structure and let S be a regular subsemigroup of $Mor_C(A)$. Then for all $\alpha, \beta \in S$,*

$$Im\alpha = Im\beta \text{ if and only if } \alpha\mathcal{L}\beta$$

$$Ker\alpha = Ker\beta \text{ if and only if } \alpha\mathcal{R}\beta.$$

Proof If $Im\alpha = Im\beta$ then Lemma 3.1 gives that $\alpha\mathcal{L}^*\beta$ in S. But S is regular so that $\mathcal{L}^* = \mathcal{L}$ in S. Conversely, it is easy to see that if $\alpha\mathcal{L}\beta$ then $Im\alpha = Im\beta$.

The proof for Ker and $\mathcal{R}$ is dual.

Example Let V be a left vector space over a field F and let G be a subgroup of Aut F. A map $\alpha : V \to V$ is a *g-map* where $g \in G$ if there exists a basis X of V such that α is given by

$$(\sum_i a_i x_i)\alpha = \sum_i (a_i g)(x_i \alpha).$$

It is not difficult to show that if β is a g-map defined using a basis Y of V such that $x\beta = x\alpha$ for all $x \in X$, then $\beta = \alpha$. Further, for any $v_1, ..., v_n \in V$ and $c_1, ..., c_n \in F$,

$$(\sum_i c_i v_i)\alpha = \sum_i (c_i g)(v_i \alpha)$$

so that α may be defined using any basis of V.

Let W be a subset of V. Then

$$Sg^V(W)\alpha = \{\sum a_i w_i : a_i \in F, w_i \in W\}\alpha$$

where the sums are all finite. So

$$Sg^V(W)\alpha = \{(\sum a_i w_i)\alpha : a_i \in F, w_i \in W\}$$

$$= \{\sum (a_i g)(w_i\alpha) : a_i \in F, w_i \in W\}$$

$$= \{\sum a_i(w_i\alpha) : a_i \in F, w_i \in W\} = Sg^V(W\alpha)$$

giving that α is an Sg^V-morphism. If $g \neq 1_F$, it is clear that α is not a homomorphism. We claim that

$$S = \{\alpha \in \text{ Mor}_{Sg^V}(V) : \alpha \text{ is a } g\text{-map}, g \in G\}$$

is a regular subsemigroup of $\text{Mor}_{Sg^V}(V)$. Clearly it is extensive, since it contains all the linear maps.

It is easy to see that if α is a g-map and β is an h-map where $g, h \in G$, then $\alpha\beta$ is a gh-map. To show that S is regular, let $\alpha \in S$ be a g-map and let X be a basis of $Im\alpha$. For each $x \in X$ pick $y_x \in V$ with $y_x\alpha = x$. Extend X to a basis $X \cup W$ of V. Let β be the g^{-1}-map defined by

$$\begin{array}{ll} x\beta = y_x & \text{for all } x \in X \\ w\beta = w & \text{for all } w \in W. \end{array}$$

Then $\beta \in S$ since G is a subgroup. For any $v = \sum a_i x_i \in Im\alpha$ we have

$$v\beta\alpha = (\sum a_i x_i)\beta\alpha = (\sum (a_i g^{-1}) y_{x_i})\alpha$$

$$= \sum (a_i g^{-1} g)(y_{x_i}\alpha) = \sum a_i x_i$$

so that $\beta\alpha|_{Im\alpha} = 1|_{Im\alpha}$ and $\alpha\beta\alpha = \alpha$.

Lemma 4.2 describes Green's relations $\mathcal{L}$ and $\mathcal{R}$ on S. One can also show that for S, $\mathcal{D} = \mathcal{J}$ and elements α, β of S are $\mathcal{D}$-related if and only if $\rho(\alpha) = \rho(\beta)$.

5 Independence structures

In this final section we show that $\text{Mor}_C(A)$ is regular, where A is an independence structure, so that by Lemma 4.1, the factors T_n/T_{n-1} are completely 0-simple. In Theorem 5.12 we describe a Rees matrix semigroup isomorphic to T_n/T_{n-1}. The proof is analogous to that of Theorem 4.12 of [G], which relies on Corollary 3.11 and Proposition 3.12 of that paper and a description of the relations $\mathcal{L}$ and $\mathcal{R}$. In the present context, $\mathcal{L}$ and $\mathcal{R}$ are described in Lemma 4.2. We prove here the analogues of Corollary 3.11 and Proposition 3.12 of [G] and show how these allow us to construct a Rees matrix semigroup isomorphic to T_n/T_{n-1}. We omit those arguments which may easily be lifted from those of [G]. A corollary of our results is a description of the relations $\mathcal{D}$ and $\mathcal{J}$ between elements of $\text{Mor}_C(A)$ of finite rank.

To widen the scope of application, we consider certain regular subsemigroups of $\text{Mor}_C(A)$, where A is a weak independence structure. In fact, we begin by looking at semigroups of continuous maps of A, and show that under the conditions we impose they must all be C-morphisms. We recall that if A is an independence space, then a map $\alpha : B \to A$, where $B \subseteq A$ is closed, is *continuous* if $C(X)\alpha \subseteq C(X\alpha)$ for all $X \subseteq B$. Clearly the continuous maps on A form a submonoid of $\mathcal{T}_A$, and the restriction of a continuous map on A to a closed subset is continuous.

For the remainder of the paper, let A be a weak independence structure and let (P) be a property defined for continuous maps from closed subsets of A to A. Put

$$S^{(P)} = \{\alpha : A \to A : \alpha \text{ is continuous }, \alpha \text{ has (P)}\}$$

and suppose that $S^{(P)}$ is a submonoid of $\mathcal{T}_A$ such that conditions (L) and (R) hold :

(L) if $X \subseteq A$ is independent and $\alpha : X \to A$ is a map, then α can be extended to a unique continuous map $C(X) \to A$ satisfying (P);

(R) if α is in $S^{(P)}$ then $\alpha|_B$ has (P) for all closed subsets B of A.

Lemma 5.1 *If B is a closed subset of A and $\alpha : B \to A$ is a continuous map with (P), then α can be extended to an element of $S^{(P)}$.*

Proof Let X be a basis for B and extend X to a basis $X \cup Y$ of A. Define $\beta : X \cup Y \to A$ by

$$\begin{aligned} x\beta &= x\alpha \quad \text{for all } x \in X \\ y\beta &= x_0\alpha \quad \text{for all } y \in Y \end{aligned}$$

where $x_0 \in X$ is fixed. Then β has a unique extension to a map $\gamma \in S^{(P)}$. Now $\gamma|_X = \beta|_X = \alpha|_X$ and by (R) $\gamma|_B$ has (P). Hence by (L), $\gamma|B = \alpha$ so that γ extends α.

At this point we note that if U, V are closed sets and $\alpha : U \to A, \beta : V \to A$ are continuous maps with (P) such that $Im\alpha \subseteq V$, then $\alpha\beta$ has (P). For α, β extend to elements $\overline{\alpha}, \overline{\beta}$ of $S^{(P)}$, respectively. Now for $u \in U$, $u\overline{\alpha}\overline{\beta} = u\alpha\overline{\beta} = u\alpha\beta$ so that $\overline{\alpha}\overline{\beta}|_U = \alpha\beta$. Condition (R) gives that $\alpha\beta$ has (P).

Lemma 5.2 *The monoid $S^{(P)}$ is a submonoid of $Mor_C(A)$.*

Proof Let $\alpha \in S^{(P)}$ and let B be a subset of A. Choose a basis U of $B\alpha$ and for each $u \in U$, pick $b_u \in B$ with $b_u\alpha = u$. Put $V = \{b_u : u \in U\}$. Then $\alpha : V \to U$ is one-one and as in Lemma 2.4, V is independent. Define $\beta : U \to V$ by $u\beta = b_u$. Then β extends to a unique continuous map $C(U) \to A$ with (P) and by Lemma 5.1, this map can be extended to an element of $S^{(P)}$, which we refer to also as β. Then $\alpha\beta, \beta\alpha \in S^{(P)}, \beta\alpha|_U = 1|_U, \alpha\beta|_V = 1|_V$ and $\beta\alpha|_{C(U)}, 1|_{C(U)}, \alpha\beta|_{C(V)}, 1|_{C(V)}$ are all continuous with (P). By uniqueness in (L) we have $\beta\alpha|_{C(U)} = 1_{C(U)}$ and $\alpha\beta|_{C(V)} = 1|_{C(V)}$. Then

$$C(V) = C(V)\alpha\beta \subseteq C(V\alpha)\beta \subseteq C(V\alpha\beta) = C(V)$$

so that $C(V) = C(U)\beta$ and $C(V)\alpha = C(U)$. Now

$$C(B\alpha) = C(U) = C(V)\alpha \subseteq C(B)\alpha \subseteq C(B\alpha)$$

giving $C(B\alpha) = C(B)\alpha$ and $\alpha \in \mathrm{Mor}_C(A)$.

Corollary 5.3 *Let $\alpha : B \to A$ be continuous with (P), where B is a closed subset of A. Then α is a C-morphism.*

Proof By Lemma 5.1, α can be extended to an element $\overline{\alpha}$ in $S^{(P)}$. Lemma 5.2 says that $\overline{\alpha}$ is a C-morphism. Hence α, being the restriction of a C-morphism, is also a C-morphism.

Lemma 5.4 *Let B be a closed subset of A and $\alpha : B \to A$ a C-morphism with (P). If X is a preimage basis for α then $\alpha|_{C(X)} : C(X) \to Im\alpha$ is a bijection.*

Proof First note that $C(X)\alpha = C(X\alpha) = Im\alpha$ so that $\alpha|_{C(X)} : C(X) \to Im\alpha$ is onto. Define $\beta : X\alpha \to X$ by $(x\alpha)\beta = x$ for all $x\alpha \in X\alpha$, and extend β to a C-morphism with (P) from $C(X\alpha) = Im\alpha$ to A. Then $Im\beta = C(X\alpha)\beta = C(X\alpha\beta) = C(X)$ and so by a previous remark, $\alpha\beta, \beta\alpha$ have (P). Further, $\alpha\beta|_X = 1|_X, \beta\alpha|_{X\alpha} = 1|_{X\alpha}$ and $\alpha\beta|_{C(X)}, 1|_{C(X)}, \beta\alpha|_{C(X\alpha)}, 1|_{C(X\alpha)}$ are all continuous maps with (P) so that $\alpha\beta|_{C(X)} = 1_{C(X)}, \beta\alpha = 1_{C(X\alpha)}$. Putting $\gamma = \alpha|_{C(X)}$ we have $\gamma\beta = 1_{C(X)}, \beta\gamma = 1_{C(X\alpha)}$ so that β is the inverse of γ and γ is one-one.

We remark that in the previous result, if α is one-one then $C(X) = B$ and α has an inverse $\beta : Im\alpha \to B$ with (P). For if α is one-one and $b \in B$, then $b\alpha \in Im\alpha = C(X)\alpha$ so that $b\alpha = a\alpha$ for some $a \in C(X)$. Since α is one-one we have $b = a \in C(X)$.

Corollary 5.5 *The monoid $S^{(P)}$ is regular.*

Proof Let $\alpha \in S^{(P)}$ have preimage basis X and define β as in Lemma 5.4. Extend β to an element $\overline{\beta}$ of $S^{(P)}$. Then $\overline{\beta}\alpha$ is the identity on $Im\alpha$ so that $\alpha\overline{\beta}\alpha = \alpha$ and $S^{(P)}$ is regular.

The next corollary follows immediately from Lemma 4.1 and Corollary 5.5.

Corollary 5.6 *For $n \in \mathbf{N}$ where $1 \leq n \leq \rho(A)$, the Rees factor $T_n^{S^{(P)}}/T_{n-1}^{S^{(P)}}$ is completely 0-simple. If $T_0^{S^{(P)}} \neq \emptyset$, then $T_0^{S^{(P)}}$ is completely simple.*

We are now in a position to prove the analogue of Proposition 3.12 of [G].

Proposition 5.7 *Let B, D be closed subsets of A with $\rho(B) = \rho(D) = n \in \mathbf{N}$. The following conditions are equivalent for a C-morphism $\alpha : B \to D$ with (P) :*

(i) $\rho(\alpha) = n$;

(ii) α is onto;

(iii) α is one-one;

(iv) α is a bijection.

Proof Let X be a preimage basis for α so that by Lemma 5.4, $\alpha|_{C(X)} : C(X) \to Im\alpha$ is a bijection.

Now $Im\alpha$ is a closed subset of D, so that $\rho(Im\alpha) = \rho(D)$ if and only if $Im\alpha = D$. That is, $\rho(\alpha) = n$ if and only if α is onto.

Suppose that α is onto; then $n = |X\alpha| = |X|$ and so $C(X) = B$ and α is certainly one-one. On the other hand, if α is one-one then by the remark following Lemma 5.4, $C(X) = B$. Thus $|X| = n$, so that $|X\alpha| = n$ and $Im\alpha = C(X\alpha) = C$.

The analogue of Proposition 3.10 of [G] is

Proposition 5.8 *Let $\alpha, \beta \in S^{(P)}$ with $Ker\alpha \subseteq Ker\beta$. Let X be a preimage basis for β. Then X can be extended to a preimage basis for α. Moreover, if $Ker\alpha = Ker\beta$, then X is a preimage basis for α.*

Proof The proof of Lemma 3.8 of [G] carries over to $S^{(P)}$. The result is then proved in the same manner as Proposition 3.10 of [G].

Corollary 5.9 *Let $\alpha, \beta \in S^{(P)}$ with $Ker\alpha = Ker\beta$ and $\alpha|_X = \beta|_X$ where X is a preimage basis for α. Then $\alpha = \beta$.*

Proof As for Corollary 3.11 of [G].

Corollary 5.10 *Let $\alpha, \beta \in S^{(P)}$ with $Ker\alpha = Ker\beta$. Then $\rho(\alpha) = \rho(\beta)$.*

Proof Let X be a common preimage basis for α and β. Then $\rho(\alpha) = |X\alpha| = |X| = |X\beta| = \rho(\beta)$.

Corollary 5.11 *For elements $\alpha, \beta \in S^{(P)}$ of finite rank, $\mathcal{D} = \mathcal{J}$ and $\alpha\mathcal{D}\beta$ if and only if $\rho(\alpha) = \rho(\beta)$.*

Proof If $\alpha, \beta \in S^{(P)}$ and $\alpha\mathcal{D}\beta$ then there exists $\gamma \in S^{(P)}$ with $\alpha\mathcal{R}\gamma\mathcal{L}\beta$. Hence $Ker\alpha = Ker\gamma$ and $Im\gamma = Im\beta$ so that by Corollary 5.10, $\rho(\alpha) = \rho(\gamma)$, giving $\rho(\alpha) = \rho(\beta)$.

Conversely, if $\rho(\alpha) = \rho(\beta) = n \in \mathbf{N}$ then as $T_n^{S^{(P)}}/T_{n-1}^{S^{(P)}}$ $(T_0^{S^{(P)}})$ is completely 0-simple (completely simple) we have $\alpha\mathcal{D}\beta$ in $T_n^{S^{(P)}}/T_{n-1}^{S^{(P)}}$ $(T_0^{S^{(P)}})$, hence $\alpha\mathcal{D}\beta$ in $S^{(P)}$.

From Lemma 3.2, if $\alpha\mathcal{J}\beta$ then $\rho(\alpha) = \rho(\beta)$ and from the above, $\alpha\mathcal{D}\beta$.

We remark that Corollary 5.11 is also true without the restriction on the ranks of α and β.

We are now in a position where we can describe the Rees matrix semigroups isomorphic to $T_n^{S^{(P)}}/T_{n-1}^{S^{(P)}}$ $(1 \leq n \leq \rho(A), n \in \mathbf{N})$ and $T_0^{S^{(P)}}$ (where $T_0^{S^{(P)}} \neq \emptyset$). Since $T_n^{S^{(P)}}/T_{n-1}^{S^{(P)}}$ and $T_0^{S^{(P)}}$ must be regular, so are the corresponding Rees matrix semigroups.

Let $n \in \mathbf{N}$ be less than the rank of A. Put $Q = T_n^{S^{(P)}}/T_{n-1}^{S^{(P)}}$; we also allow $Q = T_0^{S^{(P)}}$ in the case where $T_0^{S^{(P)}} \neq \emptyset$. Let

$$I = \{i : i \text{ is the kernel of } \alpha \in S^{(P)}, \rho(\alpha) = n\}$$

and

$$\Lambda = \{\lambda : \lambda \subseteq A, C(\lambda) = \lambda, \rho(\lambda) = n\}.$$

For each $i \in I$, choose and fix a preimage basis $\{u_1^i, ..., u_n^i\}$ of α where $Ker\alpha = i$. By Proposition 5.8, if $Ker\alpha = Ker\beta$ for $\beta \in S^{(P)}$, then $\{u_1^i, ..., u_n^i\}$ is also a preimage basis for β. Further, $\rho(\beta) = n$. Denote by B_i the closed subset $C(\{u_1^i, ..., u_n^i\})(i \in I)$.

For each $\lambda \in \Lambda$ choose and fix a basis $\{v_1^\lambda, ..., v_n^\lambda\}$ of λ. Now fix $\lambda_0 \in \Lambda$ and label $v_j^{\lambda_0}$ as w_j for $j \in \{1, ..., n\}$. Put $B = C(\{w_1, ..., w_n\}) = \lambda_0$ and let

$$G = \{\alpha : B \to B : \alpha \text{ is a } C\text{-morphism}, \alpha \text{ has (P)}, \alpha \text{ is a bijection}\}.$$

By the remark following Lemma 5.4, G is a group.

Let $\lambda \in \Lambda, i \in I$ and choose $\alpha, \beta \in S^{(P)}$ with $Im\alpha = \lambda, Ker\beta = i$. If $\rho(\alpha\beta) < n$ put $p_{\lambda i} = 0$. In view of Lemma 4.2 it is easy to use an argument similar to that in [G] to show that the property $\rho(\alpha\beta) < n$ is independent of the choice of α and β. If $\rho(\alpha\beta) = n > 0$ then $p_{\lambda i}$ is defined as follows.

First note that by Lemma 5.4 $v_j^\lambda\beta = \overline{u}_j^i\beta$ for some unique $\overline{u}_j^i \in B_i (1 \leq j \leq n)$. Define for each $i \in I$ and $\lambda \in \Lambda$ $\mu_i : B \to B_i$ and $\tau_\lambda : \lambda \to B$ by

$$w_j\mu_i = u_j^i, v_j^\lambda\tau_\lambda = w_j (1 \leq j \leq n)$$

where μ_i, τ_λ are C-morphisms with P. By Proposition 5.7, μ_i, τ_λ are bijections and by the remark following Lemma 5.4, $\mu_i^{-1}, \tau_\lambda^{-1}$ are C-morphisms with (P). The element $p_{\lambda i}$ of G is then defined by

$$w_j p_{\lambda i} = \overline{u}_j^i \mu_i^{-1}.$$

That $p_{\lambda i}$ is well defined and is indeed an element of G, follows as in [G], using Proposition 5.7.

Put $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda, P)$. In the case where $n = 0$, $\Lambda = \{C(\emptyset)\}, B = C(\emptyset)$ and $G = \{e\}$ is the trivial group; let $p_{C(\emptyset)i} = e$ for all $i \in I$ and put $\mathcal{M} = \mathcal{M}(G; I, \Lambda; P)$. We claim that $\mathcal{M}^0$ (or $\mathcal{M}$) is isomorphic to Q.

Let $\alpha \in S^{(P)}$ where $\rho(\alpha) = n \in \mathbf{N}$. Let $i = Ker\alpha$ and $\lambda = Im\alpha$. By Lemma 5.4, $\alpha' = \alpha|_{B_i} : B_i \to \lambda$ is a bijective C-morphism with (P). Hence $\mu_i\alpha'\tau_\lambda$ is in G. Define $\psi : Q \to \mathcal{M}^0(\mathcal{M})$ by $0\psi = 0$ and

$$\alpha\psi = (i, \mu_i\alpha'\tau_\lambda, \lambda)$$

where $i = Ker\alpha$ and $\lambda = Im\alpha$.

Theorem 5.12 *With $Q, \mathcal{M}^0(\mathcal{M})$ and ψ defined as above, ψ is an isomorphism.*

Proof The proof that ψ is one-one and a homomorphism is similar to that in [G].

To see that ψ is onto, consider $(i, \theta, \lambda) \in \mathcal{M}^0($ or $\mathcal{M})$ and choose $\alpha \in S^{(P)}$ with $Ker\alpha = i$. Suppose that $Im\alpha = \kappa$ so that $\gamma : B_i \to \kappa$ is a bijective C-morphism with (P), where $\gamma = \alpha|_{B_i}$. Further, γ^{-1} is a bijective C-morphism with (P). Define $\beta \in S^{(P)}$ by

$$\beta = \alpha\gamma^{-1}\mu_i^{-1}\theta\tau_\lambda^{-1}$$

so that $Im\beta = \lambda$. All the maps but α are one-one, giving that $Ker\beta = Ker\alpha = i$. Finally,

$$\mu_i\beta'\tau_\lambda = \mu_i\alpha\gamma^{-1}\mu_i^{-1}\theta\tau_\lambda^{-1}\tau_\lambda = \mu_i\gamma\gamma^{-1}\mu_i^{-1}\theta\tau_\lambda^{-1}\tau_\lambda = \theta$$

giving $\beta\psi = (i, \theta, \lambda)$ and ψ is onto.

We end the paper with two corollaries to Theorem 5.12.

Corollary 5.13 *Let A be an independence algebra. Then in End A the principal factors of finite rank are completely 0-simple and are isomorphic to a Rees matrix semigroup as described in [G].*

Proof This follows from Theorem 5.12, where we choose condition (P) to be the condition that a map is a homomorphism of a subalgebra of A to A.

Corollary 5.14 *Let A be an independence structure. Then for $n \in \mathbf{N}$ where $n \leq \rho(A)$ the principal factor T_n/T_{n-1} (T_0) is completely 0-simple (completely simple) and is isomorphic to a Rees matrix semigroup as described in Theorem 5.12.*

Proof In Theorem 5.12 take condition (P) to be the condition that a map is a C-morphism.

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