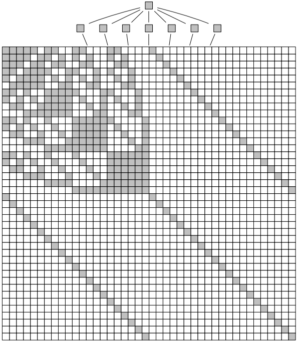
Motzkin monoids



Micky East

York Semigroup

University of York, 5 Aug, 2016

Joint work with Igor Dolinka and Bob Gray

Joint work with Igor Dolinka and Bob Gray

Joint work with Igor Dolinka and Bob Gray

Any questions?

Given a semigroup S:

- What is $\text{rank}(S) = \min\{|A| : A \subseteq S,\ S = \langle A \rangle\}$?
- Is S idempotent-generated?
- If so, what is $\text{idrank}(S) = \min\{|A| : A \subseteq E(S),\ S = \langle A \rangle\}$?

  — Does $\text{idrank}(S) = \text{rank}(S)$?
- If not, what is $\mathbb{E}(S) = \langle E(S) \rangle$?

  — What is $\text{rank}(\mathbb{E}(S))$? $\text{idrank}(\mathbb{E}(S))$? Are these equal?
- Can we enumerate minimal (idempotent) generating sets?
- Ditto for the ideals of S?

Transformation semigroups

Full transformation semigroup

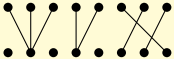
- Let $\mathcal{T}_n$ be the set of all transformations of $\mathbf{n} = \{1, \ldots, n\}$.
- $\mathcal{T}_n$ is the full transformation semigroup of degree n.
- $\mathcal{T}_n$ contains the symmetric group $\mathcal{S}_n$.
- Any finite group G embeds in $\mathcal{S}_{|G|}$.
- Any finite semigroup S embeds in $\mathcal{T}_{|S|+1}$.
- For $f \in \mathcal{T}_n$ and $x \in \mathbf{n}$, write xf instead of $f(x)$.
- For $f, g \in \mathcal{T}_n$, write $fg = f \circ g$ (do f first).

Transformation semigroups

Anatomy of a transformation

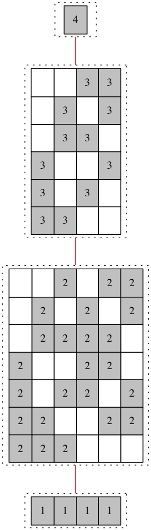
For $f \in \mathcal{T}_n$, define

- $\text{im}(f) = \{xf : x \in \mathbf{n}\}$ — the image of f,
- $\ker(f) = \{(x,y) : xf = yf\}$ — the kernel of f,
- $\text{rank}(f) = |\,\text{im}(f)| = |\mathbf{n}/\ker(f)|$ — the rank of f.

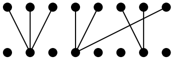
For $f =$  $\in \mathcal{T}_8$:

- $\text{im}(f) = \{2,4,6,7,8\}$,
- $\ker(f) = (\ 1,2,3 \mid 4,5 \mid 6 \mid 7 \mid 8\ )$,
- $\text{rank}(f) = 5$.

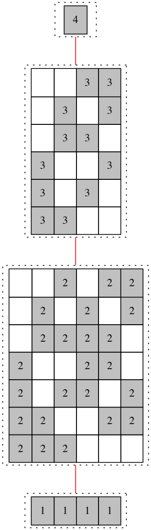
Transformation semigroups



$\mathcal{T}_4$

- Same box $\Leftrightarrow$ equal rank.
- Same column $\Leftrightarrow$ equal image.
- Same row $\Leftrightarrow$ equal kernel.
- Same cell $\Leftrightarrow$ equal image and kernel.
- A cell with a rank r idempotent ($f = f^2$) is a subgroup isomorphic to $\mathcal{S}_r$.
- $f \in \mathcal{T}_n$ is idempotent $\Leftrightarrow$ $f|_{\text{im}(f)} = \text{id}_{\text{im}(f)}$.
- $f =$  $\in \mathcal{T}_8$ is idempotent.
- $E(\mathcal{T}_n) = \{f \in \mathcal{T}_n : f = f^2\}$.
- $|E(\mathcal{T}_n)| = \sum_{r=1}^{n} \binom{n}{r} r^{n-r}$.

Transformation semigroups



$\mathcal{T}_4$

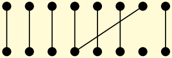
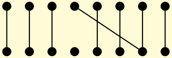
- $D_r = D_r(\mathcal{T}_n) = \{f \in \mathcal{T}_n : \text{rank}(f) = r\}$.
- $I_r = I_r(\mathcal{T}_n) = \{f \in \mathcal{T}_n : \text{rank}(f) \leq r\}$.
- $D_n = \mathcal{S}_n$.
- $I_n = \mathcal{T}_n$.
- $I_{n-1} = \mathcal{T}_n \setminus \mathcal{S}_n$.

Theorem (Howie and others, 1966–)

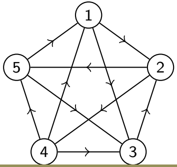
- $\mathcal{T}_n \setminus \mathcal{S}_n$ is idempotent-generated.
- $\text{rank}(\mathcal{T}_n \setminus \mathcal{S}_n) = \binom{n}{2} = \frac{n(n-1)}{2} = \text{idrank}(\mathcal{T}_n \setminus \mathcal{S}_n)$.
- $I_1 \subset I_2 \subset \cdots \subset I_n$ are the ideals of $\mathcal{T}_n$.
- $I_r = \langle D_r \rangle = \langle E(D_r) \rangle$ for $1 \leq r < n$.
- $\text{rank}(I_r) = S(n, r) = \text{idrank}(I_r)$ for $1 < r < n$.

Transformation semigroups

Theorem (Howie and McFadden, 1978)

- $\mathcal{T}_n \setminus \mathcal{S}_n = \langle E(D_{n-1}) \rangle$.
- $E(D_{n-1}) = \{e_{ij}, e_{ji} : 1 \leq i < j \leq n\}$.
- $e_{47} =$  $e_{74} =$ 
- {minimal idempotent generating sets of $\mathcal{T}_n \setminus \mathcal{S}_n$}

  $\leftrightarrow$ {strongly connected tournaments on $\mathbf{n}$}.



$$\leftrightarrow \qquad \mathcal{T}_5 \setminus \mathcal{S}_5 = \left\langle \begin{matrix} e_{12}, e_{13}, e_{41}, e_{51}, e_{32}, \\ e_{24}, e_{25}, e_{43}, e_{53}, e_{45} \end{matrix} \right\rangle$$

Theorem (Wright, 1970)

Full linear monoids

Theorem (Gray, 2008)

- $\mathcal{M}_n(\mathbb{F}_q) =$ semigroup of all $n \times n$ matrices over $\mathbb{F}_q$.
- $D_r = D_r(\mathcal{M}_n(\mathbb{F}_q)) = \{A \in \mathcal{M}_n(\mathbb{F}_q) : \text{rank}(A) = r\}$.
- $I_r = I_r(\mathcal{M}_n(\mathbb{F}_q)) = \{A \in \mathcal{M}_n(\mathbb{F}_q) : \text{rank}(A) \leq r\}$.
- $I_r = \langle D_r \rangle = \langle E(D_r) \rangle$ for $0 \leq r < n$.
- $\text{rank}(I_r) = \text{idrank}(I_r) = \left[{n \atop r} \right]_q$ for $0 \leq r < n$.
- $\text{rank}(\mathcal{M}_n(\mathbb{F}_q) \setminus \mathcal{G}_n(\mathbb{F}_q)) = \text{idrank}(\mathcal{M}_n(\mathbb{F}_q) \setminus \mathcal{G}_n(\mathbb{F}_q)) = \frac{q^n - 1}{q - 1}$

— Erdos, 1967; Dawlings, 1982.

Green's relations

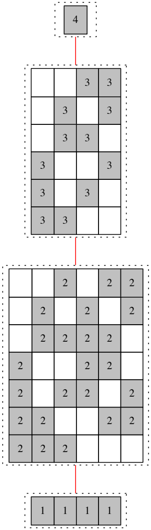
For a semigroup S:

- $x \mathscr{R} y \Leftrightarrow xS^1 = yS^1$
- $x \mathscr{L} y \Leftrightarrow S^1x = S^1y$
- $x \mathscr{J} y \Leftrightarrow S^1xS^1 = S^1yS^1$
- $\mathscr{H} = \mathscr{R} \wedge \mathscr{L}$
- $\mathscr{D} = \mathscr{R} \vee \mathscr{L} = \mathscr{J}$ if S finite

Eggbox diagram:

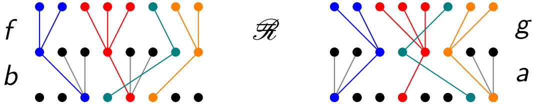
- $\mathscr{D}$-related elements in same box
- $\mathscr{R}$-related elements in same row
- $\mathscr{L}$-related elements in same column
- $\mathscr{H}$-related elements in same cell

Green's relations — $\mathcal{T}_n$



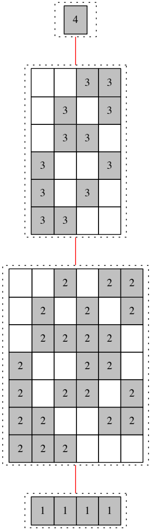
$\mathcal{T}_4$

- $f \mathscr{R} g \Leftrightarrow \ker(f) = \ker(g)$



$f = ga$ $\qquad$ $g = fb$

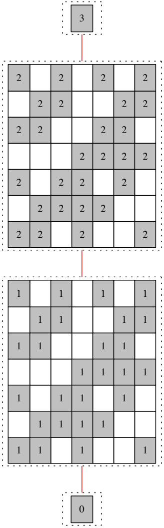
Green's relations — $\mathcal{T}_n$



$\mathcal{T}_4$

- $f \mathscr{R} g \Leftrightarrow \ker(f) = \ker(g)$
- $f \mathscr{L} g \Leftrightarrow \mathrm{im}(f) = \mathrm{im}(g)$
- $f \mathscr{D} g \Leftrightarrow \mathrm{rank}(f) = \mathrm{rank}(g)$
- group $\mathscr{H}$-classes are isomorphic to $\mathcal{S}_r$
- $\mathscr{D}$-classes are $D_r = \{f \in \mathcal{T}_n : \mathrm{rank}(f) = r\}$
- D_r contains $\binom{n}{r}$ $\mathscr{L}$-classes
- D_r contains $S(n,r)$ $\mathscr{R}$-classes
- $|D_r| = \binom{n}{r} S(n,r) r!$
- $|\mathcal{T}_n| = \sum_{r=1}^{n} |D_r|$
- $n^n = \sum_{r=1}^{n} \binom{n}{r} S(n,r) r!$

Green's relations — $\mathcal{M}_n(\mathbb{F}_q)$



$\mathcal{M}_3(\mathbb{F}_2)$

- $A \mathscr{R} B \Leftrightarrow \mathrm{Col}(A) = \mathrm{Col}(B)$
- $A \mathscr{L} B \Leftrightarrow \mathrm{Row}(A) = \mathrm{Row}(B)$
- $A \mathscr{D} B \Leftrightarrow \mathrm{rank}(A) = \mathrm{rank}(B)$
- group $\mathscr{H}$-classes are isomorphic to $\mathrm{GL}(r, \mathbb{F}_q)$
- $\mathscr{D}$-classes are $D_r = \{A \in \mathcal{M}_n(\mathbb{F}) : \mathrm{rank}(A) = r\}$
- D_r contains $\left[{n \atop r} \right]_q$ $\mathscr{L}$-classes (and $\mathscr{R}$-classes)
- $q^{n^2} = \sum_{r=0}^{n} \left[{n \atop r} \right]_q^2 q^{\binom{r}{2}} (q-1)^r [r]_q!$

Note

For $\mathcal{T}_n$ and $\mathcal{M}_n(\mathbb{F}_q)$, $\mathrm{rank}(I_r) = \mathrm{idrank}(I_r)$ is equal to the maximum of the number of $\mathscr{R}$- or $\mathscr{L}$-classes in D_r.

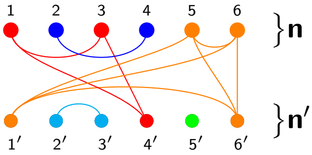
Partition monoids

- Let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- The *partition monoid* on $\mathbf{n}$ is

  $\mathcal{P}_n = \big\{\text{set partitions of } \mathbf{n} \cup \mathbf{n}'\big\}$

  $\equiv \big\{\text{(equiv. classes of) graphs on vertex set } \mathbf{n} \cup \mathbf{n}'\big\}$.

- Eg: $\alpha =$
  $\Big\{\{1,3,4'\}\{1,3,4'\},\{2,4\}\{2,4\},\{5,6,1',6'\}\{5,6,1',6'\},\{2',3'\}\{2',3$
  $\mathcal{P}_6$



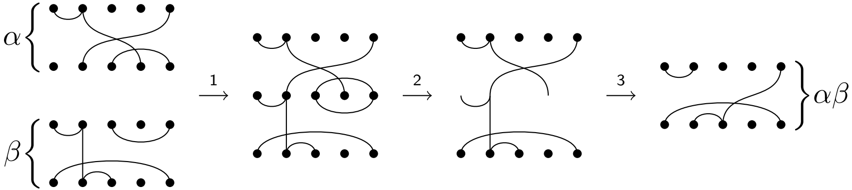
Partition monoids — product in $\mathcal{P}_n$

Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha\beta$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

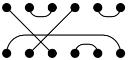
(3) smooth out resulting graph to obtain $\alpha\beta$.

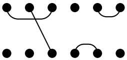


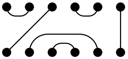
The operation is associative, so $\mathcal{P}_n$ is a semigroup (monoid, etc).

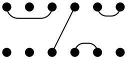
Partition monoids — submonoids of $\mathcal{P}_n$

- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{blocks of } \alpha \text{ have size } 2\}$ — Brauer monoid
- $\mathcal{TL}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is planar}\}$ — Temperley-Lieb monoid
- $\mathcal{PB}_n = \{\alpha \in \mathcal{P}_n : \text{blocks of } \alpha \text{ have size } \leq 2\}$ — partial Brauer monoid
- $\mathcal{M}_n = \{\alpha \in \mathcal{PB}_n : \alpha \text{ is planar}\}$ — Motzkin monoid

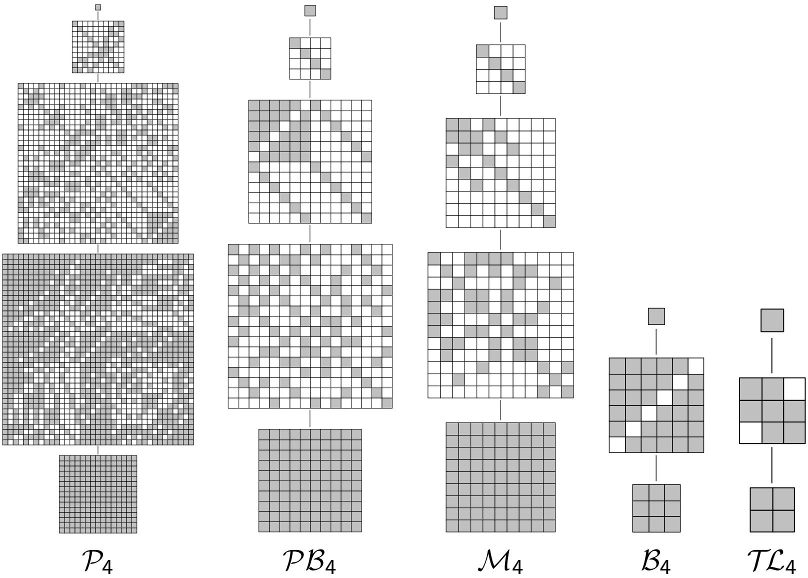

$\in \mathcal{B}_6$


$\in \mathcal{PB}_6$


$\in \mathcal{TL}_6$


$\in \mathcal{M}_6$

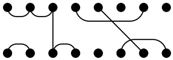
Green's relations — diagram monoids



Green's relations — diagram monoids

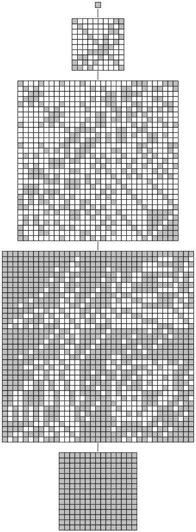
For $\alpha \in \mathcal{P}_n$:

- $\text{dom}(\alpha) = \{i \in \mathbf{n} : \text{the } \alpha\text{-block of } i \text{ intersects } \mathbf{n}'\}$
- $\ker(\alpha) = \{(i,j) : i \text{ and } j \text{ belong to the same } \alpha\text{-block}\}$
- $\text{codom}(\alpha) = \{i \in \mathbf{n} : \text{the } \alpha\text{-block of } i' \text{ intersects } \mathbf{n}\}$
- $\text{coker}(\alpha) = \{(i,j) : i' \text{ and } j' \text{ belong to the same } \alpha\text{-block}\}$
- $\text{rank}(\alpha) = \text{number of transversal } \alpha\text{-blocks}$

For $\alpha =$  $\in \mathcal{P}_8$:

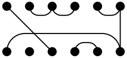
- $\text{dom}(\alpha) = \{1,2,3,5\}$
- $\ker(\alpha) = (\ 1,2,3 \mid 4,7\ )$
- $\text{rank}(\alpha) = 2$
- $\text{codom}(\alpha) = \{3,4,7\}$
- $\text{coker}(\alpha) = (\ 1,2 \mid 3,4 \mid 6,8\ )$

Green's relations — $\mathcal{P}_n$

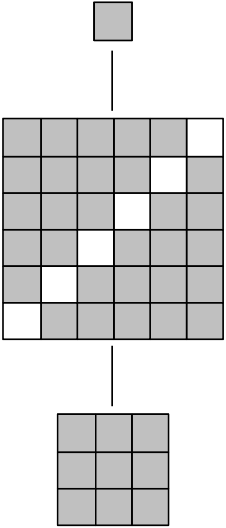


$\mathcal{P}_4$

- $\alpha \mathscr{D} \beta \Leftrightarrow \text{rank}(\alpha) = \text{rank}(\beta)$
- $\alpha \mathscr{R} \beta \Leftrightarrow \begin{cases} \text{dom}(\alpha) = \text{dom}(\beta) \quad \text{and} \\ \ker(\alpha) = \ker(\beta) \end{cases}$
- $\alpha \mathscr{L} \beta \Leftrightarrow \begin{cases} \text{codom}(\alpha) = \text{codom}(\beta) \quad \text{and} \\ \text{coker}(\alpha) = \text{coker}(\beta) \end{cases}$
- Group $\mathscr{H}$-classes are isomorphic to $\mathcal{S}_r$

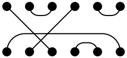
 $\in \mathcal{P}_6$

Green's relations — $\mathcal{B}_n$



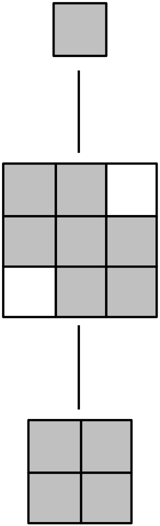
$\mathcal{B}_4$

- $\alpha \mathscr{D} \beta \Leftrightarrow \text{rank}(\alpha) = \text{rank}(\beta)$
- $\alpha \mathscr{R} \beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$
- $\alpha \mathscr{L} \beta \Leftrightarrow \text{coker}(\alpha) = \text{coker}(\beta)$
- Group $\mathscr{H}$-classes are isomorphic to $\mathcal{S}_r$
- Ranks are restricted to $n, n-2, n-4, \ldots$



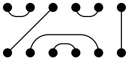
$\in \mathcal{B}_6$

Green's relations — $\mathcal{TL}_n$

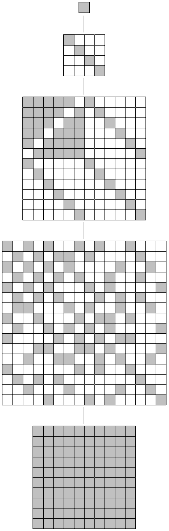


$\mathcal{TL}_4$

- $\alpha \mathscr{D} \beta \Leftrightarrow \text{rank}(\alpha) = \text{rank}(\beta)$
- $\alpha \mathscr{R} \beta \Leftrightarrow \ker(\alpha) = \ker(\beta)$
- $\alpha \mathscr{L} \beta \Leftrightarrow \text{coker}(\alpha) = \text{coker}(\beta)$
- Group $\mathscr{H}$-classes are trivial
- Ranks are restricted to $n, n-2, n-4, \ldots$

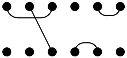
 $\in \mathcal{TL}_6$

Green's relations — $\mathcal{PB}_n$



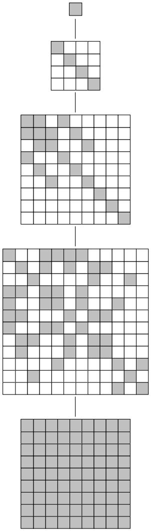
$\mathcal{PB}_4$

- $\alpha \mathscr{D} \beta \Leftrightarrow \text{rank}(\alpha) = \text{rank}(\beta)$
- $\alpha \mathscr{R} \beta \Leftrightarrow \begin{cases} \text{dom}(\alpha) = \text{dom}(\beta) \quad \text{and} \\ \text{ker}(\alpha) = \text{ker}(\beta) \end{cases}$
- $\alpha \mathscr{L} \beta \Leftrightarrow \begin{cases} \text{codom}(\alpha) = \text{codom}(\beta) \quad \text{and} \\ \text{coker}(\alpha) = \text{coker}(\beta) \end{cases}$
- Group $\mathscr{H}$-classes are isomorphic to $\mathcal{S}_r$



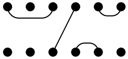
$\in \mathcal{PB}_6$

Green's relations — $\mathcal{M}_n$

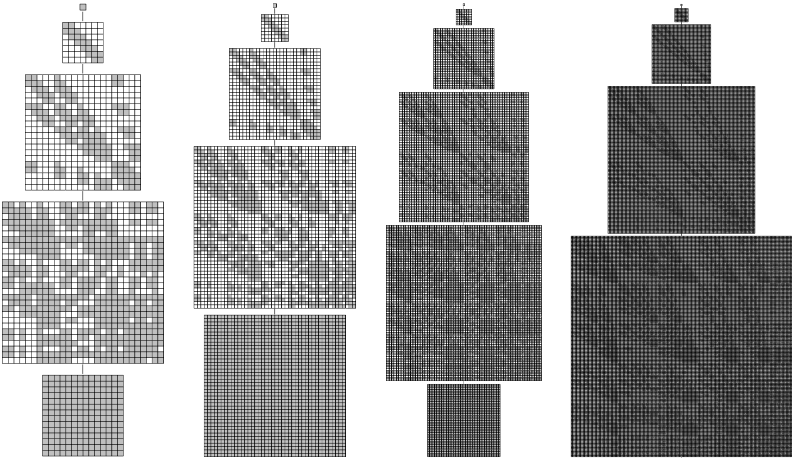


$\mathcal{M}_4$

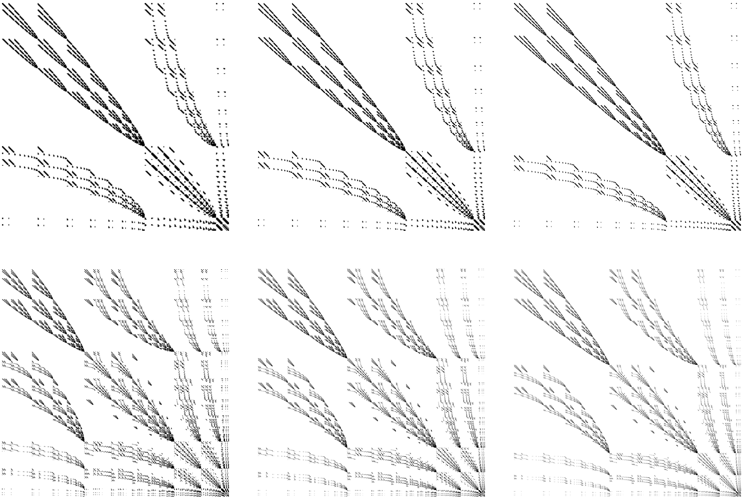
- $\alpha \mathscr{D} \beta \Leftrightarrow \text{rank}(\alpha) = \text{rank}(\beta)$
- $\alpha \mathscr{R} \beta \Leftrightarrow \begin{cases} \text{dom}(\alpha) = \text{dom}(\beta) & \text{and} \\ \ker(\alpha) = \ker(\beta) \end{cases}$
- $\alpha \mathscr{L} \beta \Leftrightarrow \begin{cases} \text{codom}(\alpha) = \text{codom}(\beta) & \text{and} \\ \text{coker}(\alpha) = \text{coker}(\beta) \end{cases}$
- Group $\mathscr{H}$-classes are trivial

 $\in \mathcal{M}_6$

Green's relations — $\mathcal{TL}_8$–$\mathcal{TL}_{11}$

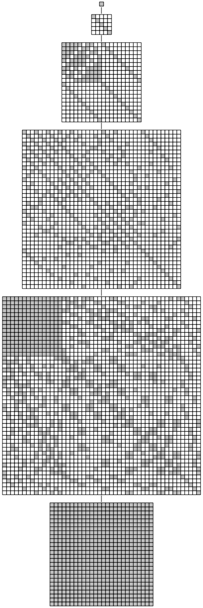


Green's relations — $\mathcal{TL}_{15}$–$\mathcal{TL}_{17}$

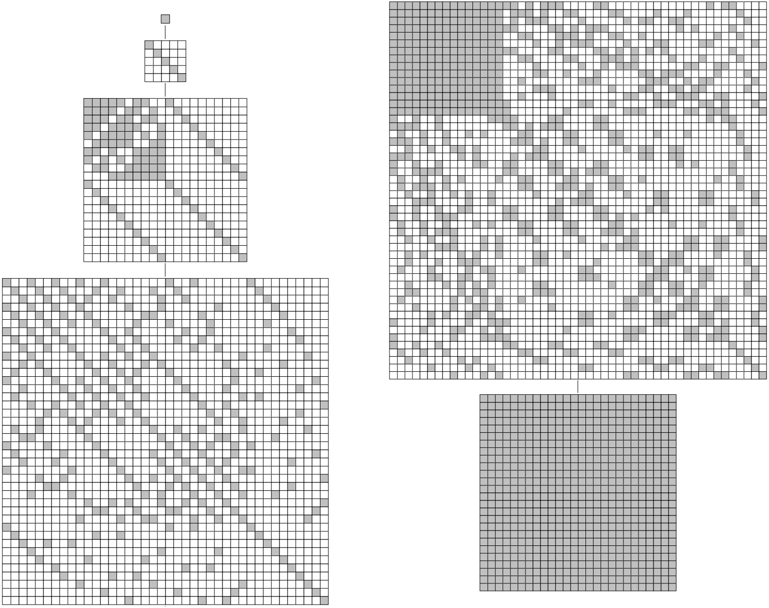


Thanks to Attila Egri-Nagy for these ...

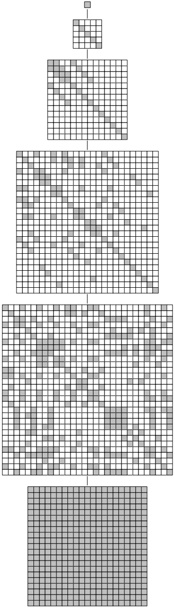
Green's relations — $\mathcal{PB}_5$



Green's relations — $\mathcal{PB}_5$



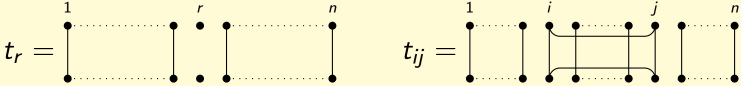
Green's relations — $\mathcal{M}_5$



Idempotent generators — $\mathcal{P}_n$

Theorem (E, 2011)

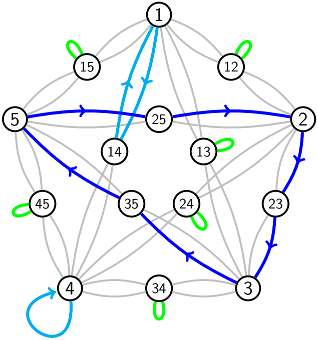
- $\mathcal{P}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{P}_n \setminus \mathcal{S}_n = \langle t_r, t_{ij} : 1 \leq r \leq n,\ 1 \leq i < j \leq n \rangle$.



- $\text{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \binom{n+1}{2} = \frac{n(n+1)}{2}$.
- Defining relations also given.

Idempotent generators — $\mathcal{P}_n \setminus \mathcal{S}_n$

- Minimal (idempotent) generating sets (E and Gray, 2014).



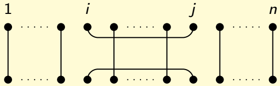
- $\mathcal{P}_5 \setminus \mathcal{S}_5 = \langle t_{12}, t_{13}, t_{15}, t_{24}, t_{34}, t_{45}, t_4,$

$e_{41}, f_{14}, e_{23}, f_{23}, e_{35}, f_{35}, e_{52}, f_{52} \rangle.$

Idempotent generators — $\mathcal{B}_n$

Theorem (Maltcev and Mazorchuk, 2007)

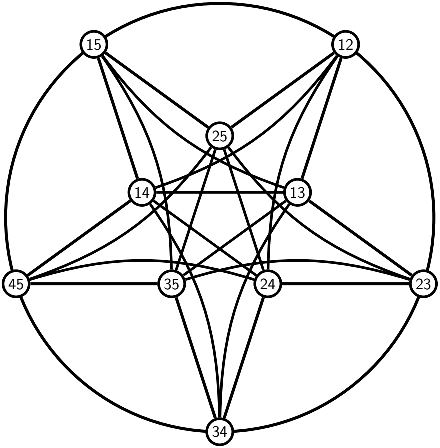
- $\mathcal{B}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{B}_n \setminus \mathcal{S}_n = \langle u_{ij} : 1 \leq i < j \leq n \rangle$.

$u_{ij} =$ 

- $\text{rank}(\mathcal{B}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{B}_n \setminus \mathcal{S}_n) = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Defining relations also given.

Idempotent generators — $\mathcal{B}_n \setminus \mathcal{S}_n$

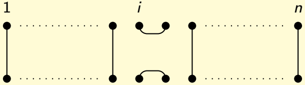
- Minimal (idempotent) generating sets (E and Gray, 2014).



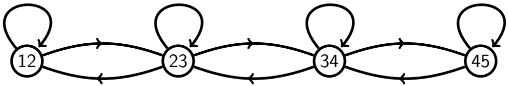
Idempotent generators — $\mathcal{TL}_n$

Theorem (Borisavljević, Došen, Petrić, 2002, etc)

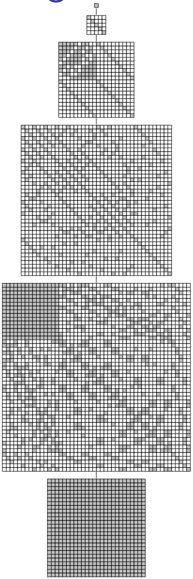
- $\mathcal{TL}_n$ is idempotent generated.
- $\mathcal{TL}_n = \langle u_1, \ldots, u_{n-1} \rangle$.

$u_i =$ 

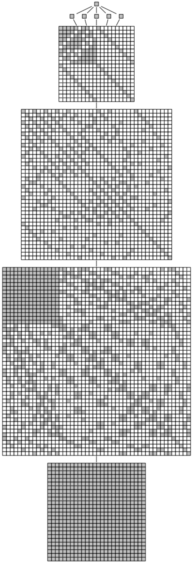
- $\text{rank}(\mathcal{TL}_n) = \text{idrank}(\mathcal{TL}_n) = n - 1$.



Idempotent generators — $\mathbb{E}(\mathcal{PB}_n) = \langle E(\mathcal{PB}_n) \rangle$

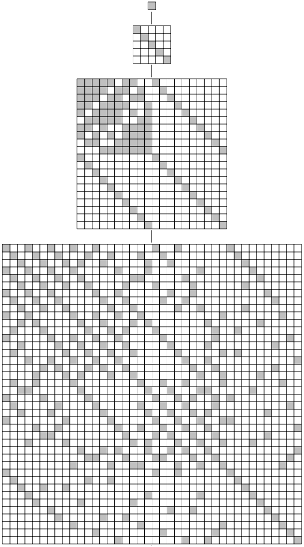


$\mathcal{PB}_5$

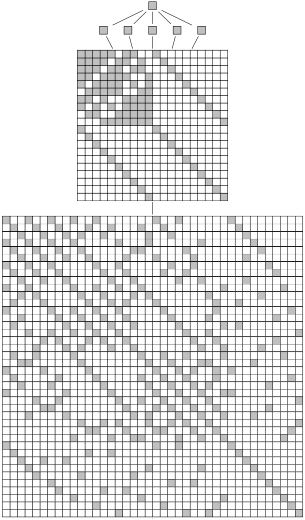


$\mathbb{E}(\mathcal{PB}_5) \neq \mathcal{PB}_5 \setminus \mathcal{S}_5$!

Idempotent generators — $\mathbb{E}(\mathcal{PB}_n) = \langle E(\mathcal{PB}_n) \rangle$

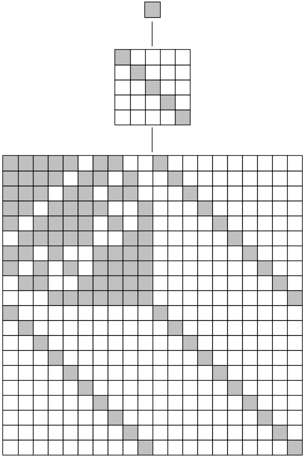


$\mathcal{PB}_5$

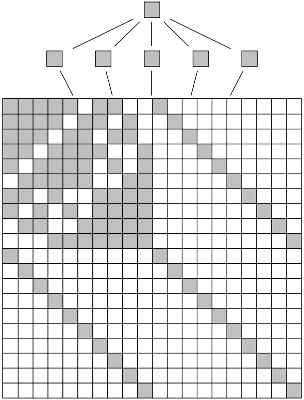


$\mathbb{E}(\mathcal{PB}_5) \neq \mathcal{PB}_5 \setminus \mathcal{S}_5$!

Idempotent generators — $\mathbb{E}(\mathcal{PB}_n) = \langle E(\mathcal{PB}_n) \rangle$

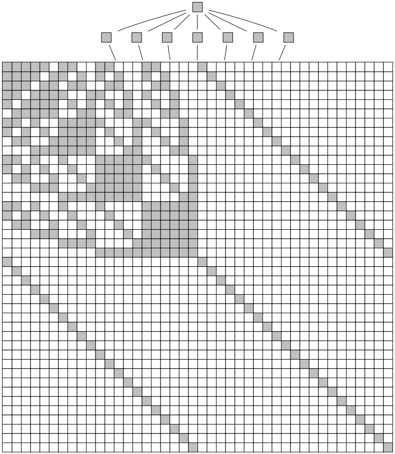


$\mathcal{PB}_5$



$\mathbb{E}(\mathcal{PB}_5) \neq \mathcal{PB}_5 \setminus \mathcal{S}_5$!

Idempotent generators — $\mathbb{E}(\mathcal{PB}_n) = \langle E(\mathcal{PB}_n) \rangle$

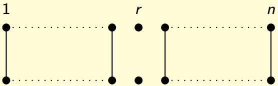


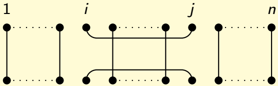
$\mathbb{E}(\mathcal{PB}_7)$

Idempotent generators — $\mathbb{E}(\mathcal{PB}_n) = \langle E(\mathcal{PB}_n) \rangle$

Theorem (Dolinka, E, Gray, 2015)

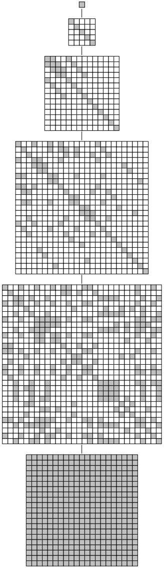
- $\mathbb{E}(\mathcal{PB}_n) = \{1\} \cup \{t_r : 1 \leq r \leq n\} \cup I_{r-2}(\mathcal{PB}_n)$
- $\mathbb{E}(\mathcal{PB}_n) = \langle t_r, u_{ij} : 1 \leq r \leq n,\ 1 \leq i < j \leq n \rangle$.

$t_r =$ 

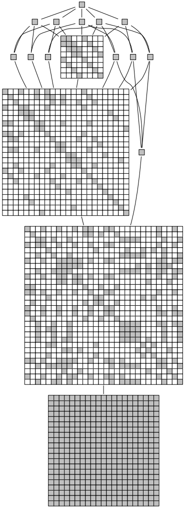
$u_{ij} =$ 

- $\text{rank}(\mathbb{E}(\mathcal{PB}_n)) = \text{idrank}(\mathbb{E}(\mathcal{PB}_n)) = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

Idempotent generators — $\mathbb{E}(\mathcal{M}_n) = \langle E(\mathcal{M}_n) \rangle$

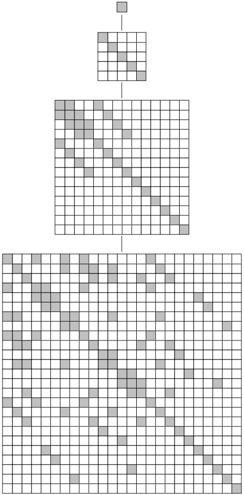


$\mathcal{M}_5$

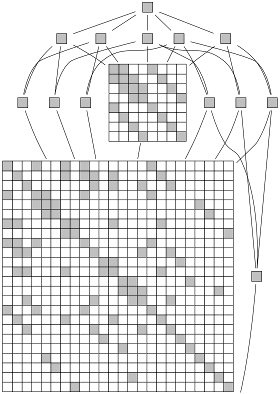


$\mathbb{E}(\mathcal{M}_5)$

Idempotent generators — $\mathbb{E}(\mathcal{M}_n) = \langle E(\mathcal{M}_n) \rangle$



$\mathcal{M}_5$



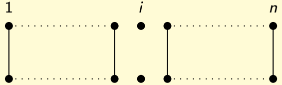
$\mathbb{E}(\mathcal{M}_5)$

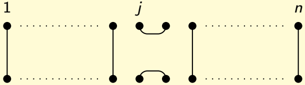
Idempotent generators — $\mathbb{E}(\mathcal{M}_n) = \langle E(\mathcal{M}_n)\rangle$

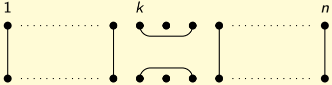
- $A \subseteq \mathbf{n}$ is *cosparse* if: $i \in \mathbf{n} \setminus A \ \Rightarrow\ i+1 \in A$.

Theorem (Dolinka, E, Gray, 2015)

- $\mathbb{E}(\mathcal{M}_n) = \{1\} \cup \{\mathrm{id}_A : A \subseteq \mathbf{n} \text{ cosparse}\}$
  $\cup\{\alpha \in \mathcal{M}_n : \mathrm{dom}(\alpha) \text{ and } \mathrm{codom}(\alpha) \text{ non-cosparse}\}$
- $\mathbb{E}(\mathcal{M}_n) = \langle t_1, \ldots, t_n, u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-2}\rangle$.

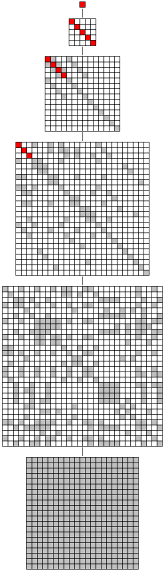
$t_i =$ 

$u_j =$ 

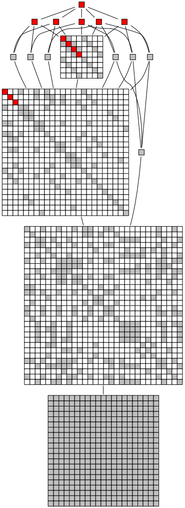
$v_k =$ 

- $\mathrm{rank}(\mathbb{E}(\mathcal{M}_n)) = \mathrm{idrank}(\mathbb{E}(\mathcal{M}_n)) = 3n-3$.

Idempotent generators — $\mathbb{E}(\mathcal{M}_n) = \langle E(\mathcal{M}_n) \rangle$

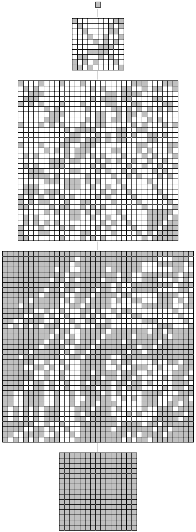


$\mathcal{M}_5$



$\mathbb{E}(\mathcal{M}_5)$

Ideals — $\mathcal{P}_n$

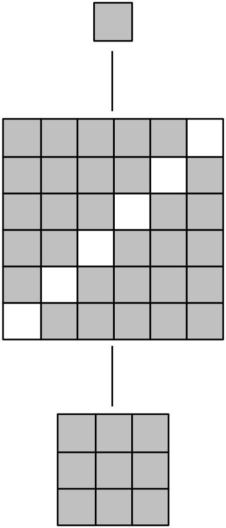


$\mathcal{P}_4$

Theorem (E and Gray, 2014)

- The ideals of $\mathcal{P}_n$ are
$$I_r = I_r(\mathcal{P}_n) = \{\alpha \in \mathcal{P}_n : \text{rank}(\alpha) \leq r\}$$
for $0 \leq r \leq n$.
- $I_r = \langle D_r \rangle = \langle E(D_r) \rangle$ for $0 \leq r < n$,
$$\text{where } D_r = \{\alpha \in \mathcal{P}_n : \text{rank}(\alpha) = r\}.$$
- $\text{rank}(I_r) = \text{idrank}(I_r) = \sum_{j=r}^{n} S(n,j)\binom{j}{r}$.

Ideals — $\mathcal{B}_n$



$\mathcal{B}_4$

Theorem (E and Gray, 2014)

- The ideals of $\mathcal{B}_n$ are

$$I_r = I_r(\mathcal{B}_n) = \{\alpha \in \mathcal{B}_n : \mathrm{rank}(\alpha) \leq r\}$$

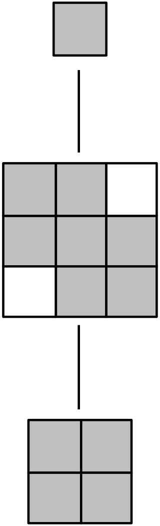
for $0 \leq r = n - 2k \leq n$.

- $I_r = \langle D_r \rangle = \langle E(D_r) \rangle$ for $0 \leq r < n$,

$$\text{where } D_r = \{\alpha \in \mathcal{B}_n : \mathrm{rank}(\alpha) = r\}.$$

- $\mathrm{rank}(I_r) = \mathrm{idrank}(I_r) = \frac{n!}{2^k k! r!}$.

Ideals — $\mathcal{TL}_n$



Theorem (E and Gray, 2014)

- The ideals of $\mathcal{TL}_n$ are

$$I_r = I_r(\mathcal{TL}_n) = \{\alpha \in \mathcal{TL}_n : \mathrm{rank}(\alpha) \leq r\}$$

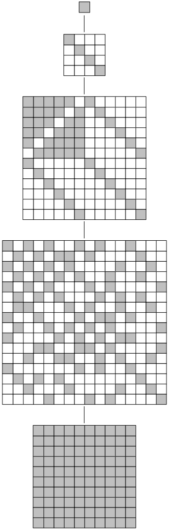
  for $0 \leq r = n - 2k \leq n$.

- $I_r = \langle D_r \rangle = \langle E(D_r) \rangle$ for $0 \leq r < n$,

  where $D_r = \{\alpha \in \mathcal{TL}_n : \mathrm{rank}(\alpha) = r\}$.

- $\mathrm{rank}(I_r) = \mathrm{idrank}(I_r) = \frac{r+1}{n+1}\binom{n+1}{k}$.

Ideals — $\mathcal{PB}_n$



$\mathcal{PB}_4$

Theorem (Dolinka, E, Gray, 2015)

- The ideals of $\mathcal{PB}_n$ are

$$I_r = I_r(\mathcal{PB}_n) = \{\alpha \in \mathcal{PB}_n : \text{rank}(\alpha) \leq r\}$$

  for $0 \leq r \leq n$.

- $I_r = \langle D_r \cup D_{r-1} \rangle$ for $0 \leq r < n$

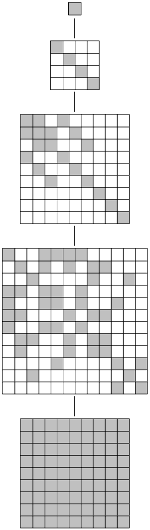
$$= \langle D_r \rangle \text{ iff } r = 0 \text{ or } r \equiv n \pmod 2$$

- $\text{rank}(I_r) = \binom{n}{r-1} \cdot (n-r)!! + \binom{n}{r} \cdot a(n-r)$,

  where $a(k)$ = number of involutions of $\mathbf{k}$.

- I_r is idempotent-generated iff $r \leq n-2$.
- $\text{idrank}(I_r) = \text{rank}(I_r)$ where appropriate.

Ideals — $\mathcal{M}_n$



$\mathcal{M}_4$

Theorem (Dolinka, E, Gray, 2015)

- The ideals of $\mathcal{M}_n$ are
$$I_r = I_r(\mathcal{M}_n) = \{\alpha \in \mathcal{M}_n : \text{rank}(\alpha) \leq r\}$$
for $0 \leq r \leq n$.
- $I_r = \langle D_r \cup D_{r-1} \rangle$ for $0 \leq r < n$.
- $\text{rank}(I_r) = m(n,r) + m'(n,r)$,

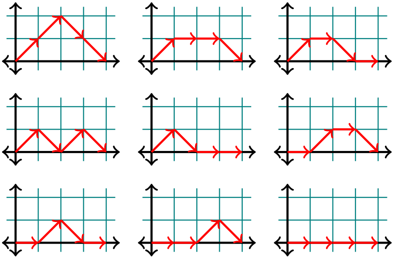
  — Motzkin and Riordan numbers.
- I_r is idempotent-generated iff $r < \lfloor \frac{n}{2} \rfloor$.
- $\text{rank}(\mathbb{E}(I_r)) = \text{idrank}(\mathbb{E}(I_r)) = \text{rank}(I_r)$ for $0 \leq r \leq n-2$.

Motzkin triangle — A064189

- $m(n, r)$ = number of Motzkin paths from $(0, 0)$ to (n, r)

  — stay in 1st quadrant, use steps $U(1, 1)$, $D(1, -1)$, $F(1, 0)$

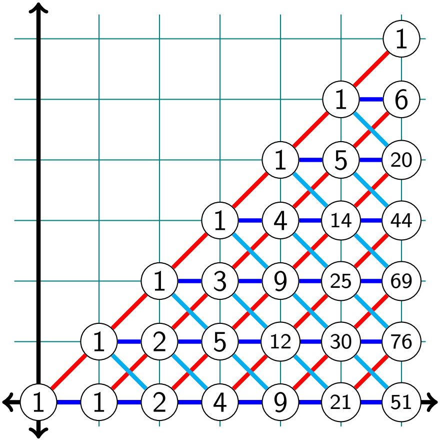
- $m(4, 0) = 9$:



- $m(0, 0) = 1$, $\quad m(n, r) = 0$ if (n, r) out-of-bounds

- $m(n + 1, r) = m(n, r - 1) + m(n, r) + m(n, r + 1)$

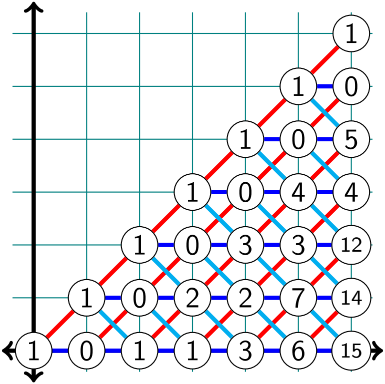
Motzkin triangle — A064189



- $m(0,0) = 1, \quad m(n,r) = 0$ if (n,r) out-of-bounds
- $m(n+1,r) = m(n,r-1) + m(n,r) + m(n,r+1)$

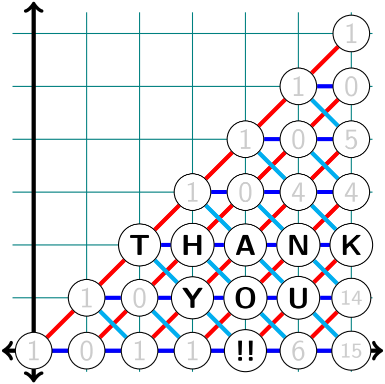
Riordan triangle — A097609

- $m'(n, r)$ = number of Motzkin paths from $(0, 0)$ to $(n, 0)$ with r flats at level 0.
- $m'(0, 0) = 1, \quad m'(n, r) = 0$ if (n, r) out-of-bounds
- $m'(n + 1, r) = m'(n, r - 1) + m'(n, r + 1) + \cdots + m'(n, n)$



Riordan triangle — A097609

- $m'(n,r)$ = number of Motzkin paths from $(0,0)$ to $(n,0)$ with r flats at level 0.
- $m'(0,0) = 1, \quad m'(n,r) = 0$ if (n,r) out-of-bounds
- $m'(n+1,r) = m'(n,r-1) + m'(n,r+1) + \cdots + m'(n,n)$



Special thanks to the York Maths Dept!