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Restriction semigroups and λ-Zappa-Szép products

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Abstract The aim of this paper is to study λ-semidirect and λ-Zappa-Szép products of restriction semigroups. The former concept was introduced for inverse semigroups by Billhardt, and has been extended to some classes of left restriction semigroups. The latter was introduced, again in the inverse case, by Gilbert and Wazzan. We unify these concepts by considering what we name the *scaffold* of a Zappa-Szép product $S \bowtie T$ where S and T are restriction. Under certain conditions this scaffold becomes a category. If one action is trivial, or if S is a semilattice and T a monoid, the scaffold may be ordered so that it becomes an inductive category. A standard technique, developed by Lawson and based on the Ehresmann-Schein-Nambooripad result for inverse semigroups, allows us to define a product on our category. We thus obtain restriction semigroups that are λ-*semidirect products* and λ-*Zappa-Szép products*, extending the work of Billhardt and of Gilbert and Wazzan. Finally, we explicate the internal structure of λ-semidirect products.

Keywords Restriction semigroups · Semidirect products · Zappa-Szép products · Categories

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1 Introduction

The techniques of decomposing semigroups into direct, semidirect or Zappa-Szép products, are now well established. McAlister [17, 18] demonstrated the utility of semidirect products in understanding inverse semigroups, results subsequently extended by a number of authors

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to the broader context of (left) restriction semigroups. Zappa-Szép products were introduced by Zappa [22] and after being widely developed in the context of groups (see for example Szép [20]) were applied to more general structures by Kunze [12] and Brin [4], who used the term Zappa-Szép product. As shown in [12] and explicated in [6], Zappa-Szép products are closely related to the action of Mealy machines (automata with output depending on input and current state). The concept being so natural, a number of other names have been used, in particular that of *general* product [13]. Such constructions may be approached from two directions: internal and external. We focus largely on the second in this article, but the two are closely related, as is well established.

For the convenience of the reader, we now give the relevant definitions, before outlining the direction of this article. A semigroup T is said to *act* on the left of a set S if there is a map $T \times S \to S$ where $(t, s) \mapsto t \cdot s$, such that for all $s \in S, t, t' \in T$ we have $t \cdot (t' \cdot s) = (tt') \cdot s$. Clearly the action of T induces a morphism from T to $\mathcal{T}_S^*$, the (dual of) the full transformation monoid on S. If S is a semigroup and for all $s, s' \in S$ and $t \in T$ we have $t \cdot (ss') = (t \cdot s)(t \cdot s')$, that is, the morphism $T \to \mathcal{T}_S^*$ is into the endomorphism monoid End S of S, then we naturally say that T *acts by endomorphisms*.

Definition 1.1 Let T be a semigroup acting on the left of a semigroup S by endomorphisms. Then $S \rtimes T$ is the semigroup on $S \times T$ with binary operation given by

$$(s, t)(s', t') = (s(t \cdot s'), tt')$$

and is called an *(external) semidirect product* of S by T.

If T is a monoid with identity 1_T acting on S then we normally insist that $1_T \cdot s = s$ for all $s \in S$; where necessary we will emphasise this by saying that T acts *monoidally*.

Definition 1.2 Let S and T be semigroups and suppose that we have maps

$$T \times S \to S,\ (t, s) \mapsto t \cdot s \quad \text{and} \quad T \times S \to T,\ (t, s) \mapsto t^s$$

such that for all $s, s' \in S, t, t' \in T$:

$$\begin{array}{ll} \text{(ZS1)}\ tt' \cdot s = t \cdot (t' \cdot s); & \text{(ZS3)}\ (t^s)^{s'} = t^{ss'}; \\ \text{(ZS2)}\ t \cdot (ss') = (t \cdot s)(t^s \cdot s'); & \text{(ZS4)}\ (tt')^s = t^{t' \cdot s} t'^s. \end{array}$$

Define a binary operation on $S \times T$ by

$$(s, t)(s', t') = (s(t \cdot s'), t^{s'} t').$$

Then $S \times T$ is a semigroup, referred to as the (external) *Zappa-Szép product of S and T* and denoted by $S \bowtie T$.

In Definition 1.2 (ZS1) and (ZS3) are simply saying that S and T act on each other. Notice that a semidirect product $S \rtimes T$ (respectively, reverse semidirect product $S \ltimes T$) is merely a Zappa-Szép product $S \bowtie T$ in which the action of S on T (respectively, of T on S) is trivial, that is, $t \mapsto t^s = t$ (respectively, $s \mapsto t \cdot s = s$) for all $s \in S, t \in T$. If both actions are trivial then we obtain the (external) direct product semigroup $S \times T$.

If S and T are monoids in Definition 1.2, and if the following four axioms also hold:

$$\begin{array}{ll} \text{(ZS5)}\ t \cdot 1_S = 1_S; & \text{(ZS7)}\ 1_T \cdot s = s; \\ \text{(ZS6)}\ t^{1_S} = t; & \text{(ZS8)}\ 1_T^s = 1_T, \end{array}$$

then $S \bowtie T$ is a monoid with identity $(1_S, 1_T)$.

Restriction semigroups are a variety of biunary semigroups, naturally extending the class of inverse semigroups. Our broad aim is to study restriction semigroups using semidirect and Zappa-Szép products. It is known that the former works reasonably well if one component is a semilattice [5]. However, difficulties arise in the general case even for inverse semigroups: the semidirect product of two inverse semigroups is not inverse in general. Billhardt showed how to modify the definition of a semidirect product of two inverse semigroups to obtain what he termed as a λ-*semidirect product* [1]. The λ-semidirect product of two inverse semigroups is again inverse. This result was generalised to the left ample case, again by Billhardt, where one component is a semilattice [2] (with an indication that the same should work for any two left ample semigroups). The case where S is a semilattice and T is weakly left ample (a left restriction semigroup with $E_T = E(T)$) was considered in [7] and a subsequent extension of the techniques of [7] appear in [3].

In view of the above, it is clear that the Zappa-Szép product of two inverse semigroups is not inverse in general. Gilbert and Wazzan generalised Billhardt's concept of λ-semidirect product to what they named as λ-*Zappa-Szép products* [6,21]. Their approach is to pick out a subset of a Zappa-Szép product $S \bowtie T$ of inverse semigroups S and T and to show that with the restriction of the binary operation in $S \bowtie T$ the given subset is a groupoid. We refer to it here as the *scaffold* of $S \bowtie T$. In the special case where $S = E$ is a semilattice and $T = G$ is a group, $E \bowtie G$ may be ordered such that it becomes an inductive groupoid. An application of the Ehresmann–Schein–Nambooripad Theorem [15, Theorem 4.1.8] results in an inverse semigroup.

Our article puts the work of Billhardt on λ-semidirect products, and that of Gilbert and Wazzan on λ-Zappa-Szép products, into the broader context of restriction semigroups. Note that even in the case for semidirect products with one component a semilattice, the arguments for *two-sided* restriction semigroups differ from those in the one-sided case. We also outline a number of examples, in Sects. 4 and 6, of Zappa-Szép products, that we believe to be of some independent interest.

In Sect. 2 we give the necessary preliminaries, including the result of Lawson [14] generalising the Ehresmann–Schein–Nambooripad Theorem, showing that the category of *restriction semigroups* and appropriate morphisms is isomorphic to a certain category of *inductive categories*. In Sect. 3 we consider restriction semigroups S and T and a Zappa-Szép product $S \bowtie T$ satisfying a number of extra conditions; in particular, S and T must act on each other on *both* sides in a way we name a *double* action. Note that all of these conditions are satisfied automatically if S and T are inverse. We define a subset $V_{\bowtie}(S \bowtie T)$, the *scaffold* of $S \bowtie T$, and show that under the restriction of the binary operation of $S \bowtie T$ we have a category.

Sections 5 and 6 use the notion of scaffold in two different ways. In Sect. 5 we consider the case where one action is trivial. In this case, we can order $V_{\bowtie}(S \bowtie T)$ so that it becomes and inductive category. According to Lawson's result, we then obtain a restriction semigroup. We show that the multiplication coincides with that in the previous case of a λ-semidirect product. In Sect. 6 we again show that $V_{\bowtie}(S \bowtie T)$ is inductive, this time for a Zappa-Szép product where S is a semilattice and T a monoid. The resulting restriction semigroup has a multiplication corresponding to that in [6] in case T is a group. We therefore refer to it as a λ-*Zappa-Szép product* of S and T. In our final section we begin the investigation of the structure of restriction semigroups obtained as λ-semidirect products; they contain a 'core', that is, they have a restriction subsemigroup that is a strong semilattice of restriction semigroups that are isomorphic to restriction subsemigroups of S. On this core T acts in a way reflecting the original action on S.

2 Restriction semigroups and categories

Restriction semigroups and their one-sided versions arise from two rather distinct viewpoints, as algebraic models of partial mappings and as the setting for generalised Green's relations.

A *left restriction semigroup* S is a unary semigroup $(S, \cdot, ^+)$, that is, a semigroup equipped with an additional unary operation $s \mapsto s^+$, such that the following identities hold:

$$x^+x = x, \quad x^+y^+ = y^+x^+, \quad (x^+y)^+ = x^+y^+ \quad \text{and} \quad xy^+ = (xy)^+x.$$

Putting $E_S = \{a^+ : a \in S\}$, it is easy to see that E_S is a semilattice, the *semilattice of projections* of S.

For any set X the partial transformation semigroup $\mathcal{PT}_X$ becomes left restriction under the map $\alpha \mapsto \alpha^+$, where α^+ is the local identity on dom α. The semilattice of projections of $\mathcal{PT}_X$ consists of the local identity maps of X, and thus is isomorphic to the semilattice of subsets of X under intersection. Moreover, any left restriction semigroup S embeds into $\mathcal{PT}_S$ [10, Theorem 3.9]. If S is inverse, then defining $a^+ = aa^{-1}$, it is easy to see that S is left restriction. It follows from the the Wagner-Preston representation theorem that the embedding of S into $\mathcal{PT}_S$ has image contained in the symmetric inverse semigroup $\mathcal{I}_S$. An example of a left restriction semigroup of quite another ilk is obtained from any monoid M: put $a^+ = 1$ for all $a \in M$. In this case the semilattice of projections is trivial; any left restriction semigroup with trivial semilattice of projections is said to be *reduced*.

Dually, a right restriction semigroup is a unary semigroup $(S, \cdot, ^*)$ satisfying the dual of the above identities. A *restriction semigroup* $S = (S, \cdot, ^+, ^*)$ is a biunary semigroup (that is, a semigroup equipped with two unary operations) such that $(S, \cdot, ^+)$ is left restriction, $(S, \cdot, ^*)$ is right restriction and the linking identities

$$(x^+)^* = x^+ \quad \text{and} \quad (x^*)^+ = x^*$$

hold. It follows that the semilattices of projections of S regarded as a left or as a right restriction semigroup coincide. As above, it is clear that inverse semigroups become restriction under the operations $a \mapsto a^+ = aa^{-1}$ and $a \mapsto a^* = a^{-1}a$. Moreover, reduced restriction semigroups, that is, restriction semigroups with trivial semilattices of projections, directly correspond to monoids.

A partial order on a left restriction semigroup S is defined by the rule for any $a, b \in S$

$$a \leq b \quad \text{if and only if } a = eb \text{ for some } e \in E_S.$$

Clearly, $\leq$ restricts to the standard order on the semilattice E_S. It is easy to see that $a \leq b$ if and only if $a = a^+b$. In the case that S is restriction, this definition is left right dual, that is, $a \leq b$ if and only if $a = bf$ for some $f \in E_S$. Moreover, $\leq$ is compatible with all the basic operations of S. We therefore refer to $\leq$ as the *natural partial order* on S.

Suppose now that T is a semigroup acting on the left of a semigroup S. If S is left restriction and the morphism $T \mapsto \mathcal{T}_S^*$ is into the semigroup of endomorphisms of S *as a unary semigroup*, then we may emphasise this by saying that T acts on S by *left restriction* endomorphisms or by *restriction* endomorphisms if the context is clear. The corresponding definition holds if S is restriction.

Lemma 2.1 *Let T be a semigroup acting on the left of a left restriction semigroup S by restriction endomorphisms. Then the action of T preserves $\leq$ on S (and hence on E_S).*

Proof Let $a, b \in S$ and $t \in T$ and let $a \leq b$. Then $a = a^+b$, so that

$$t \cdot a = t \cdot (a^+b) = (t \cdot a^+)(t \cdot b) = (t \cdot a)^+(t \cdot b),$$

yielding $t \cdot a \leq t \cdot b$. □

As remarked in the Introduction, restriction semigroups may be obtained from inductive categories. We explain this connection by first giving the definition of a category convenient for our purpose.

Let C be a set and let $\cdot$ be a partial binary operation on C. For $x, y \in C$, whenever we write '$\exists\, x \cdot y$', we mean that the product $x \cdot y$ is defined, so that when we will write '$\exists\,(x \cdot y) \cdot z$', it will be understood that we mean $\exists\, x \cdot y$ and $\exists\,(x \cdot y) \cdot z$. We say that an element $e \in C$ is *idempotent* if $\exists\, e \cdot e$ and $e \cdot e = e$. If e is an idempotent of C which satisfies:

$$\exists\, e \cdot x \Rightarrow e \cdot x = x \quad \text{and} \quad \exists\, x \cdot e \Rightarrow x \cdot e = x,$$

then we say that e is an *identity* for C and we call this identity a *local identity* for C.

Definition 2.2 Let $\mathbf{C} = (C, \cdot, \mathbf{d}, r)$, where $\cdot$ is a partial binary operation on C and $\mathbf{d}$, $\mathbf{r} : C \to C$ such that

(C1) $\exists\, x \cdot y$ if and only if $\mathbf{r}(x) = \mathbf{d}(y)$ and then $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$ and $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$;
(C2) $\exists\, x \cdot (y \cdot z)$ (so that from (C1), $\exists\,(x \cdot y) \cdot z$), implies $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(C3) $\exists\, \mathbf{d}(x) \cdot x$ and $\mathbf{d}(x) \cdot x = x$ and $\exists\, x \cdot \mathbf{r}(x)$ and $x \cdot \mathbf{r}(x) = x$.

Let $E_C = \{\mathbf{d}(x) : x \in C\}$. It follows from the axioms that $E_C = \{\mathbf{r}(x) : x \in C\}$ and $\mathbf{C}$ is a small category in the standard sense with set of local identities E_C and set of objects identified with E_C. Further, $\mathbf{d}(x)$ is the domain of x and $\mathbf{r}(x)$ is the range of x.

Small categories can be seen as generalisations of monoids as if $\mathbf{C}$ has precisely one object, then all products are necessarily defined and thus $\mathbf{C}$ is essentially a monoid. Correspondingly, if we define a *groupoid* to be a category $\mathbf{C}$ such that for every $a \in C$ there exists a $b \in C$ with $a \cdot b = \mathbf{d}(a)$ and $b \cdot a = \mathbf{r}(a)$, then a small groupoid with one object is a group.

We make it clear that whenever we write $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ for a category $\mathbf{C}$, we mean that $\mathbf{C}$ is a small category, described as in Definition 2.2.

We now define what we mean by an *ordering* on our category $\mathbf{C}$.

Definition 2.3 Let $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ be a category with set of local identities E_C. Let $\leq$ be a partial order on C such that for all $e \in E_C$, $x, y \in C$:

(IC1) if $x \leq y$ then $\mathbf{r}(x) \leq \mathbf{r}(y)$ and $\mathbf{d}(x) \leq \mathbf{d}(y)$;
(IC2) if $x \leq y$ and $x' \leq y'$, $\exists\, x \cdot x'$ and $\exists\, y \cdot y'$, then $x \cdot x' \leq y \cdot y'$;
(IC3) if $e \leq \mathbf{d}(x)$ then $\exists$ unique ${}_e|x \in C$ such that ${}_e|x \leq x$ and $\mathbf{d}({}_e|x) = e$;
(IC4) if $e \leq \mathbf{r}(x)$ then $\exists$ unique $x|_e \in C$ such that $x|_e \leq x$ and $\mathbf{r}(x|_e) = e$.

We then say that $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an *ordered category*.

The element ${}_e|x$ of Condition (IC3) is called the *restriction* of x to e and the element $x|_e$ of Condition (IC4) is called the *co-restriction* of x to e.

If $\mathbf{C}$ is an ordered category with set of local identities E_C, then for $e, f \in E_C$, we denote the greatest lower bound (meet) of e and f, where it exists, by $e \wedge f$.

Definition 2.4 Let $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ be an ordered category. Then $\mathbf{C}$ is an *inductive category* if (IC5) holds:

(IC5) $(E_C, \leq)$ is a meet semilattice.

Inductive categories correspond to restriction semigroups in a manner analogous to the correspondence, given by the Ehresmann–Schein–Nambooripad Theorem, between inverse semigroups and inductive groupoids. The latter result was generalised to restriction semigroups by Lawson [14] (though he used different terminology). Lawson's approach was made explicit for restriction semigroups in Hollings [9].

Theorem 2.5 [9, Theorems 7.2.4 and 7.2.6] *Let $(S, \cdot, ^+, ^*)$ be a restriction semigroup with natural partial order $\leq$. Then $(S, \odot, ^+, ^*, \leq)$ is an inductive category with set of local identities E_S, where $\odot$ is the restriction of $\cdot$. The restriction and co-restriction operations are defined by:*

$$_e|a = ea \quad \text{and} \quad b|_f = bf,$$

where $e \leq a^+ = \mathbf{d}(a)$ and $f \leq b^* = \mathbf{r}(b)$.

Conversely, let $(C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ be an inductive category. Then $(C, \otimes, \mathbf{d}, \mathbf{r})$ is a restriction semigroup where $\otimes$ is given by

$$a \otimes b = (a|_{\mathbf{r}(a)\wedge\mathbf{d}(b)})(_{\mathbf{r}(a)\wedge\mathbf{d}(b)}|b).$$

Moreover, the ordering $\leq$ from the category coincides with the natural partial order of the restriction semigroup and $\otimes$ coincides with $\cdot$ whenever it is defined.

The operation $\otimes$ in Theorem 2.5 is called a *pseudoproduct*.

3 The scaffold of a Zappa-Szép product of restriction semigroups

As commented earlier, a Zappa-Szép product $S \bowtie T$ of inverse semigroups S and T need not be inverse. Inspired by the work of Billhardt for semidirect products, Gilbert and Wazzan pick out a partial subalgebra of $S \bowtie T$, and show that it is a groupoid. We refer to this groupoid as the *scaffold* of $S \bowtie T$. We outline their approach, and then explain how to put it into the wider context of restriction semigroups.

Theorem 3.1 [21, Theorem 4.5.6] *Let $Z = S \bowtie T$ be a Zappa-Szép product of inverse semigroups S and T. Put*

$$B_{\bowtie}(Z) = \{(a, t) \in S \times T : tt^{-1} \cdot a^{-1} = a^{-1},\ tt^{-1} \cdot a^{-1}a = a^{-1}a,$$
$$(t^{-1})^{a^{-1}a} = t^{-1},\ (tt^{-1})^{a^{-1}a} = tt^{-1}\}.$$

For $(a, t) \in B_{\bowtie}(Z)$ put

$$\mathbf{d}(a, t) = (aa^{-1}, (tt^{-1})^{a^{-1}}) \quad \text{and} \quad \mathbf{r}(a, t) = (t^{-1} \cdot (a^{-1}a), t^{-1}t).$$

Then

$$(B_{\bowtie}(Z), \cdot, \mathbf{d}, \mathbf{r})$$

is a groupoid under the restriction of the binary operation in Z. The set of local identities is

$$E_{B_{\bowtie}(Z)} = \{(e, f) \in E(S) \times E(T) : f \cdot e = e,\ f^e = f\},$$

and the inverse of (a, t) is given by $(a, t)^{-1} = (t^{-1} \cdot a^{-1}, (t^{-1})^{a^{-1}})$.

Gilbert and Wazzan note that, in general, $E_{B_{\bowtie}(Z)}$ does not form a semilattice under the partial order of idempotents inherited from the multiplication in Z. They proceed to specialise to the case where $S = E$ is a semilattice and $T = G$ is a group.

Theorem 3.2 [21, Proposition 4.5.19] *Let $Z = E \bowtie G$ where E is a semilattice and G is a group. Then $B_{\bowtie}(Z) = \{(e, g) \in E \times G : (g^{-1})^e = g^{-1}\}$ and has set of local identities*

$$E_{B_{\bowtie}(Z)} = \{(e, 1) : e \in E\}.$$

Moreover, $(B_{\bowtie}(Z), \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive groupoid where

$$(e, g) \leq (f, h) \Leftrightarrow e \leq f \quad \textit{and} \quad g = h^{h^{-1} \cdot e}.$$

Restrictions and corestrictions are given by

$$_{(f,1)}|(e, g) = (f, g^{g^{-1} \cdot f}) \quad \textit{and} \quad (e, g)|_{(f,1)} = (e(g \cdot f), g^f).$$

The inverse of (e, g) simplifies to $(g^{-1} \cdot e, g^{-1})$.

Our aim is to extend the above ideas to Zappa-Szép products of restriction semigroups. To this end, we need a notion of *double action*, already apparent from the study of semidirect products of restriction semigroups [5].

Definition 3.3 Let S and T be restriction semigroups and suppose that S acts on the right of T and T acts on the left of S. We say that S and T *act doubly* on each other if in addition S acts on the left of T and T acts on the right of S via

$$S \times T \to T,\ (s, t) \mapsto {}^{s}t \quad \text{and} \quad S \times T \to S,\ (s, t) \mapsto s \circ t$$

such that the actions satisfy the following compatibility conditions:

$$\begin{aligned} (t \cdot s) \circ t &= s \circ t^* = t^* \cdot s \\ t \cdot (s \circ t) &= s \circ t^+ = t^+ \cdot s \end{aligned} \tag{CP1}$$

and

$$\begin{aligned} ({}^{s}t)^s &= t^{s^*} = {}^{s^*}t \\ {}^{s}(t^s) &= t^{s^+} = {}^{s^+}t. \end{aligned} \tag{CP2}$$

Consider a Zappa-Szép product $Z = S \bowtie T$ of two restriction semigroups S and T where S and T act doubly on each other. Let

$$V_{\bowtie}(Z) = \{(a, t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)^{a^*} = t^+, {}^{a}t^+ \cdot a = a, t^{a^* \circ t} = t\}.$$

We first prove that if S and T are inverse semigroups and are regarded as restriction semigroups in the usual way, then with a canonical approach to the secondary actions, we have $B_{\bowtie}(Z) = V_{\bowtie}(Z)$.

Lemma 3.4 *Let S and T be inverse semigroups regarded as restriction semigroups. Suppose $Z = S \bowtie T$ is a Zappa-Szép product of S and T. Defining two extra maps*

$$S \times T \to S,\ (s, t) \mapsto s \circ t \quad \textit{and} \quad S \times T \to T,\ (s, t) \mapsto {}^{s}t$$

by

$$s \circ t = t^{-1} \cdot s \quad \textit{and} \quad {}^{s}t = t^{s^{-1}}$$

we have that S and T act doubly on each other. Moreover, $B_{\bowtie}(Z) = V_{\bowtie}(Z)$.

Proof It is clear from the fact that $a \mapsto a^{-1}$ is an involution for any inverse semigroup that the given maps are actions. Moreover, it is easy to check that (CP1) and (CP2) are satisfied, so that S and T indeed act doubly on each other.

By definition

$$V_{\bowtie}(Z) = \{(a,t) \in S \times T : tt^{-1} \cdot a^{-1}a = a^{-1}a,\ (tt^{-1})^{a^{-1}a} = tt^{-1},\\ (tt^{-1})^{a^{-1}} \cdot a = a,\ t^{t^{-1} \cdot a^{-1}a} = t\}.$$

Let $(a,t) \in B_{\bowtie}(Z)$. From [21, Lemma 4.5.3], we know that $t^{t^{-1} \cdot a^{-1}a} = t$ and $(tt^{-1})^{a^{-1}} \cdot a = a$. Thus $(a,t) \in V_{\bowtie}(Z)$, so that $B_{\bowtie}(Z) \subseteq V_{\bowtie}(Z)$.

Conversely, let $(a,t) \in V_{\bowtie}(Z)$. We want to show that $tt^{-1} \cdot a^{-1} = a^{-1}$ and $(t^{-1})^{a^{-1}a} = t^{-1}$. We see that

$$\begin{aligned} a &= aa^{-1}a \\ &= a(tt^{-1} \cdot a^{-1}a) && \text{as } tt^{-1} \cdot a^{-1}a = a^{-1}a \\ &= a(tt^{-1} \cdot a^{-1})((tt^{-1})^{a^{-1}} \cdot a) && \text{using (ZS2)} \\ &= a(tt^{-1} \cdot a^{-1})a && \text{because } (tt^{-1})^{a^{-1}} \cdot a = a, \end{aligned}$$

and

$$\begin{aligned} (tt^{-1} \cdot a^{-1})a(tt^{-1} \cdot a^{-1}) &= (tt^{-1} \cdot a^{-1})((tt^{-1})^{a^{-1}} \cdot a)(tt^{-1} \cdot a^{-1}) && \text{because } (tt^{-1})^{a^{-1}} \cdot a = a \\ &= (tt^{-1} \cdot a^{-1}a)(tt^{-1} \cdot a^{-1}) && \text{using (ZS2)} \\ &= (tt^{-1} \cdot a^{-1}a)((tt^{-1})^{a^{-1}a} \cdot a^{-1}) && \text{because } tt^{-1} = (tt^{-1})^{a^{-1}a} \\ &= tt^{-1} \cdot a^{-1}aa^{-1} && \text{again using (ZS2)} \\ &= tt^{-1} \cdot a^{-1}. \end{aligned}$$

Hence $tt^{-1} \cdot a^{-1} = a^{-1}$. By left-right duality we must also have $(t^{-1})^{a^{-1}a} = t^{-1}$. It follows that $(a,t) \in B_{\bowtie}(Z)$, so that $V_{\bowtie}(Z) \subseteq B_{\bowtie}(Z)$ and equality follows. □

The following lemma was an important tool in the proof that $B_{\bowtie}(Z)$ is a groupoid.

Lemma 3.5 [21, Lemma 4.5.2] *Let S and T be inverse semigroups and $Z = S \bowtie T$ be a Zappa-Szép product of S and T.*

(i) If $t^{bb^{-1}} = t$, then $(t \cdot b)^{-1} = t^b \cdot b^{-1}$;
(ii) If $t^{-1}t \cdot b = b$, then $(t^b)^{-1} = (t^{-1})^{t \cdot b}$.

Let $Z = S \bowtie T$ be a Zappa-Szép product of two restriction semigroups S and T where S and T act doubly on each other. We define:

$$t^{b^+} = t \Rightarrow \begin{cases} (t \cdot b)^* = t^b \cdot b^* \\ (t \cdot b)^+ = t \cdot b^+ \end{cases} \tag{D1}$$

$$t^* \cdot b = b \Rightarrow \begin{cases} (t^b)^* = (t^*)^b \\ (t^b)^+ = (t^+)^{t \cdot b} \end{cases} \tag{D2}$$

$$(a,t) \in B_{\bowtie}(Z) \Rightarrow a^* \circ t \in E_S \tag{D3}$$

and

$$(a,t) \in B_{\bowtie}(Z) \Rightarrow {}^{a}t^+ \in E_T. \tag{D4}$$

The result of the following lemma was used frequently in [21] to prove that $B_{\bowtie}(Z)$ is a groupoid. As it was not stated explicitly, we prove it now for completeness. The key is Lemma 3.5.

Lemma 3.6 *Let S and T be inverse semigroups and $Z = S \bowtie T$ be a Zappa-Szép product of S and T. Then (D1) and (D2) are satisfied, that is:*

$$t^{bb^{-1}} = t \Rightarrow \begin{cases} (t \cdot b)^{-1}(t \cdot b) = t^b \cdot b^{-1}b \\ (t \cdot b)(t \cdot b)^{-1} = t \cdot bb^{-1} \end{cases} \quad \textit{and} \quad t^{-1}t \cdot b = b \Rightarrow \begin{cases} (t^b)^{-1}t^b = (t^{-1}t)^b \\ t^b(t^b)^{-1} = (tt^{-1})^{t \cdot b}. \end{cases}$$

Further, (D3) and (D4) are satisfied, that is:

$$(a, t) \in B_{\bowtie}(Z) \Rightarrow t^{-1} \cdot a^{-1}a \in E_S \quad \textit{and} \quad (tt^{-1})^{a^{-1}} \in E_T.$$

Proof Let $t^{bb^{-1}} = t$. From Lemma 3.5 we know that $(t \cdot b)^{-1} = t^b \cdot b^{-1}$. Thus

$$\begin{aligned} (t \cdot b)^{-1}(t \cdot b) &= (t^b \cdot b^{-1})(t \cdot b) && \text{because} \\ &= (t^b \cdot b^{-1})(t^{bb^{-1}} \cdot b) && \text{because } t^{bb^{-1}} = t \\ &= t^b \cdot b^{-1}b && \text{using (ZS2),} \end{aligned}$$

and

$$\begin{aligned} (t \cdot b)(t \cdot b)^{-1} &= (t \cdot b)(t^b \cdot b^{-1}) && \text{because } (t \cdot b)^{-1} = t^b \cdot b^{-1} \\ &= t \cdot bb^{-1} && \text{using (ZS2).} \end{aligned}$$

Thus (D1) and dually, (D2) hold.

If $(a, t) \in B_{\bowtie}(Z)$, then as $(t^{-1})^{a^{-1}a} = t^{-1}$, we have

$$t^{-1} \cdot a^{-1}a = (t^{-1} \cdot a^{-1}a)((t^{-1})^{a^{-1}a} \cdot a^{-1}a) = (t^{-1} \cdot a^{-1}a)^2,$$

so that (D3) and dually, (D4), hold. □

We have not been able to show Conditions (D1)–(D4) hold for arbitrary Zappa-Szép products of restriction semigroups acting doubly on each other. We will see, however, that they do so in a variety of cases upon which we will expand later in the article. For the moment we must impose them to proceed to find a suitable scaffold below.

Theorem 3.7 *Let S and T be restriction semigroups and suppose that $Z = S \bowtie T$ is the Zappa-Szép product of S and T where S and T act doubly on each other satisfying (D1)–(D4). Let*

$$V = V_{\bowtie}(Z) = \{(a, t) \in S \times T : t^+ \cdot a^* = a^*, (t^+)^{a^*} = t^+, {}^{a}t^+ \cdot a = a, t^{a^* \circ t} = t\}.$$

Then V is a category under the restriction of the binary operation in Z where

$$\mathbf{d}(a, t) = (a^+, {}^{a}t^+) \quad \textit{and} \quad \mathbf{r}(a, t) = (a^* \circ t, t^*).$$

The set of local identities is given by

$$E_V = \{(e, f) \in E_S \times E_T : f \cdot e = e, f^e = f\}.$$

Proof We first note that for any $(b, u) \in V$

$$u^+ \cdot b^* = b^*, (u^+)^{b^*} = u^+, {}^{b}u^+ \cdot b = b, u^{b^* \circ u} = u.$$

Now we prove some preliminary results needed to show that V is a category.

Lemma 3.8 *Let $(a,t),(b,u)\in V$ be such that $(a,t)\cdot(b,u)$ is defined. Then*

(1) $t^{b^+}=t$ *and* $t^*\cdot b=b$;
(2) $a^*=t\cdot b^+$ *and* $(t^*)^b=u^+$;
(3) $(t^bu)^+=(t^b)^+$;
(4) ${}^{a(t\cdot b)}(t^bu)^+={}^a t^+$;
(5) $(a(t\cdot b))^+=a^+$;
(6) $(a(t\cdot b))^*=(t\cdot b)^*$;
(7) $(a(t\cdot b))^*\circ t^bu=b^*\circ u$;
(8) $(t^bu)^*=u^*$.

Proof As $(a,t)\cdot(b,u)$ is defined, $\mathbf{r}(a,t)=\mathbf{d}(b,u)$, so that $(a^*\circ t,t^*)=(b^+,{}^bu^+)$, giving

$$a^*\circ t=b^+\quad\text{and}\quad t^*={}^bu^+.$$

(1) We have $t^{b^+}=t^{a^*\circ t}=t$ and $t^*\cdot b={}^bu^+\cdot b=b$, because $(a,t),(b,u)\in V$.
(2) From $a^*\circ t=b^+$, we have $t\cdot(a^*\circ t)=t\cdot b^+$ and so using (CP1), $a^*=t^+\cdot a^*=t\cdot b^+$. Dually, $(t^*)^b=u^+$.
(3) From (1) and (2) proved above , we have that $t^*\cdot b=b$ and $(t^*)^b=u^+$. Hence

$$(t^bu)^+=(t^bu^+)^+=\big(t^{t^*\cdot b}(t^*)^b\big)^+=\big((tt^*)^b\big)^+=(t^b)^+.$$

(4) We see that

$$\begin{aligned}
{}^{a(t\cdot b)}(t^bu)^+ &= {}^{a(t\cdot b)}(t^b)^+ && \text{by (3) proved above}\\
&= {}^{a(t\cdot b)}\big((t^+)^{t\cdot b}\big) && \text{using (D2)}\\
&= {}^{a}\big({}^{t\cdot b}((t^+)^{t\cdot b})\big) && \\
&= {}^{a}((t^+)^{(t\cdot b)^+}) && \text{using (CP2)}\\
&= {}^{a}((t^+)^{t\cdot b^+}) && \text{using (D1)}\\
&= {}^{a}((t^+)^{a^*}) && \text{from (2) proved above}\\
&= {}^{a}({}^{a^*}t^+) && \text{using (CP2)}\\
&= {}^{aa^*}t^+ && \\
&= {}^{a}t^+. &&
\end{aligned}$$

(5) From (D1), we know that $(t\cdot b)^+=t\cdot b^+$ and from (2) proved above, we have that $t\cdot b^+=a^*$. Hence

$$(a(t\cdot b))^+=(a(t\cdot b)^+)^+=(a(t\cdot b^+))^+=(aa^*)^+=a^+.$$

Conditions (6), (7) and (8) are dual of (3), (4) and (5), respectively. □

Now we continue with the proof of Theorem 3.7.

We first check that $\cdot$ is a closed partial binary operation. For this let $(a,t),(b,u)\in V$ such that $(a,t)\cdot(b,u)$ is defined. To show that $(a,t)\cdot(b,u)=(a(t\cdot b),t^bu)\in V$, we need to check that

$$\begin{aligned}
&(t^bu)^+\cdot(a(t\cdot b))^*=(a(t\cdot b))^*, && ((t^bu)^+)^{(a(t\cdot b))^*}=(t^bu)^+,\\
&{}^{a(t\cdot b)}(t^bu)^+\cdot(a(t\cdot b))=a(t\cdot b)\quad\text{and} && (t^bu)^{(a(t\cdot b))^*\circ t^bu}=t^bu.
\end{aligned}$$

We calculate

$$\begin{aligned}(t^b u)^+ \cdot (a(t\cdot b))^* &= (t^b)^+ \cdot (t\cdot b)^* && \text{from Lemma 3.8 ((3) and (6))}\\ &= (t^b)^+ \cdot (t^b \cdot b^*) && \text{by (D1)}\\ &= (t^b)^+ t^b \cdot b^* && \text{using (ZS1)}\\ &= t^b \cdot b^* \\ &= (t\cdot b)^* && \text{by (D1)}\\ &= (a(t\cdot b))^* && \text{from Lemma 3.8 (6).}\end{aligned}$$

Also,

$$\begin{aligned}{}^{a(t\cdot b)}(t^b u)^+ \cdot (a(t\cdot b)) &= {}^a t^+ \cdot (a(t\cdot b)) && \text{from Lemma 3.8 (4)}\\ &= ({}^a t^+ \cdot a)(({}^a t^+)^a \cdot (t\cdot b)) && \text{using (ZS2)}\\ &= a((t^+)^{a^*} \cdot (t\cdot b)) && \text{as } (a,t)\in V \text{ and using (CP2)}\\ &= a(t^+ \cdot (t\cdot b)) && \text{as } (a,t)\in V\\ &= a(t\cdot b).\end{aligned}$$

Together with the dual arguments, we have shown $(a(t\cdot b), t^b u) \in V$.

Next for $e \in E_S$ and $f \in E_T$, we show that

$$(e, f) \in V \Leftrightarrow f \cdot e = e,\ f^e = f.$$

Clearly if $(e, f) \in V$, then $f \cdot e = e$ and $f^e = f$. Conversely, if these conditions hold, then ${}^e f \cdot e = f^e \cdot e = f \cdot e = e$ and $f^{e\circ f} = f^{f\cdot e} = f^e = f$. Thus $(e, f) \in V$.

Now let $(a, t) \in V$. By definition, $\mathbf{d}(a, t) = (a^+, {}^a t^+)$ and by our assumption, ${}^a t^+ \in E_T$. Thus to show that $\mathbf{d}(a, t) \in V$, we need only to show that $({}^a t^+)^{a^+} = {}^a t^+$ and ${}^a t^+ \cdot a^+ = a^+$. We see that

$$({}^a t^+)^{a^+} = {}^{a^+}({}^a t^+) = {}^{a^+a}(t^+) = {}^a t^+$$

so that ${}^a t^+ \cdot a^+ = ({}^a t^+ \cdot a)^+ = a^+$, using (D1).

Dually, $\mathbf{r}(a, t) = (a^* \circ t, t^*) \in V$. Next we see that

$$\mathbf{d}(e, f) = (e, {}^e f) = (e, f^e) = (e, f)$$

and

$$\mathbf{r}(e, f) = (e \circ f, f) = (f \cdot e, f) = (e, f).$$

Thus $\mathbf{d}(e, f) = (e, f) = \mathbf{r}(e, f)$ where $(e, f) \in E_V$, $e \in E_S$ and $f \in E_T$. We show below that V is a category, and it will follow that E_V is as given in the statement of the theorem.

(C1) Let $(a, t), (b, u) \in V$ and suppose $\exists (a, t) \cdot (b, u)$, so that $(a^* \circ t, t^*) = (b^+, {}^b u^+)$. Thus

$$\begin{aligned}\mathbf{d}\big((a,t)\cdot(b,u)\big) &= \mathbf{d}\big(a(t\cdot b), t^b u\big)\\ &= \big((a(t\cdot b))^+, {}^{a(t\cdot b)}(t^b u)^+\big)\\ &= (a^+, {}^a t^+) && \text{from Lemma 3.8 ((5) and (4))}\\ &= \mathbf{d}(a,t).\end{aligned}$$

Dually, $\mathbf{r}\big((a, t) \cdot (b, u)\big) = \mathbf{r}(b, u)$. Hence (C1) is satisfied.

(C2) This directly follows from (C1) and the fact that multiplication is associative in a Zappa-Szép product.

(C3) Let $(a, t) \in V$. Then $\mathbf{r}(\mathbf{d}(a, t)) = \mathbf{d}(a, t)$ because $\mathbf{d}(e, f) = (e, f) = \mathbf{r}(e, f)$ for all $(e, f) \in E_V$.

Hence there exists $\mathbf{d}(a,t)\cdot(a,t)$ and then

$$\begin{aligned}\mathbf{d}(a,t)\cdot(a,t) &= (a^{+},\, {}^{a}t^{+})\cdot(a,t)\\ &= (a^{+}({}^{a}t^{+}\cdot a),\, ({}^{a}t^{+})^{a}t)\\ &= (a^{+}a,\, (t^{+})^{a^{*}}t) && \text{using (CP2)}\\ &= (a,\, t^{+}t) && \text{because } (a,t)\in V\\ &= (a,t).\end{aligned}$$

Dually, $\mathbf{d}(\mathbf{r}(a,t)) = \mathbf{r}(a,t)$, so $(a,t)\cdot\mathbf{r}(a,t)$ exists and equals (a,t). Hence $V = V_{\bowtie}(Z)$ is a category. $\square$

4 Examples

This section presents a number of examples of Zappa-Szép products. The details of the verifications are left to the reader.

Example 4.1 [23, Chap. 4] This example is inspired by the corresponding example for inverse semigroups given in [6]. Let S be a left restriction semigroup with semilattice of projections E. It is easily checked that we have actions

$$S\times E\to E,\,(s,e)\mapsto s\cdot e=(se)^{+}$$

and

$$S\times E\to E,\,(s,e)\mapsto s^{e}=se$$

satisfying (SZ1)–(ZS4). The semigroup S embeds into $E\bowtie S$ under $s\mapsto (s^{+},s)$; we denote by $\overline{E}$ the image of E under this embedding, so that $\overline{E}=\{(e,e): e\in E\}$. Then $(e,s)\mapsto es$ is a morphism of $E\bowtie S$ onto S, separating the elements of $\overline{E}$.

Example 4.2 A link is known between Zappa-Szép products and self-similar group actions [8] which provides some of the motivation for the approach of [16]. This idea may be extended from groups to monoids, as we now briefly explain.

Let $\mathbf{T}$ be a k-regular rooted tree. The vertices of $\mathbf{T}$ may be labelled in a natural way by the elements of a free monoid X^{*} on k generators, with the root labelled by ϵ. The elements of the monoid $\mathcal{T}_{\mathbf{T}}$ of endomorphisms of $\mathbf{T}$ (where here we compose right-to-left) may be identified with certain length-preserving elements of the full transformation monoid $\mathcal{T}_{X^{*}}$.

For any $w\in X^{*}$, the right ideal wX^{*} labels the vertices of a subtree of $\mathbf{T}$ with root w. Moreover, the map $X^{*}\to X^{*}$, where $v\mapsto wv$, corresponds to an isomorphism of $\mathbf{T}$ onto the subtree with vertices in wX^{*}. It follows that if $\varphi\in\mathcal{T}_{\mathbf{T}}$ and $x\in X$, then $\varphi^{x}\in\mathcal{T}_{\mathbf{T}}$, where for all $v\in X^{*}$, $\varphi^{x}(v)=(\varphi(x))^{-1}\varphi(xv)$, the multiplication being taken in the free group on X. This gives, in a natural way, an action of the free monoid X^{*} on the right of $\mathcal{T}_{\mathbf{T}}$. On the other hand, $\mathcal{T}_{\mathbf{T}}$ acts on the left of X^{*} by mappings, i.e. $\varphi\cdot w=\varphi(w)$. Using an inductive argument, one can show that (ZS2) and (ZS4) hold and further, (ZS5)–(ZS8) also hold. Thus we may form the monoidal Zappa-Szép product $X^{*}\bowtie\mathcal{T}_{\mathbf{T}}$, which contains as a submonoid the Zappa-Szép product $X^{*}\bowtie\mathcal{S}_{\mathbf{T}}$, where $\mathcal{S}_{\mathbf{T}}$ is the automorphism group of $\mathbf{T}$. Lawson explains in [16] that Perrot's work [19] characterises the structure of Zappa-Szép products of X^{*} with a group as what he calls *left Rees monoids*.

Example 4.3 Let X be a set and let $S=T=\mathcal{P}(X)$ be the power set of X, under the operation of union. Then $S=T$ is a semilattice and so certainly inverse. For convenience we keep the separate labels S and T. We define two actions

$$T \times S \to S, (t, s) \mapsto t \cdot s = t \cup s$$

and

$$T \times S \to S, (t, s) \mapsto t^s = t \setminus s.$$

One can check that (ZS1)–(ZS4) hold and thus $Z = S \bowtie T$ is a Zappa-Szép product. Since S and T are inverse, Lemmas 3.4 and 3.6 give that all the conditions for Theorem 3.7 hold.

Example 4.4 Let X be a non empty set. For notational convenience we let $T = \mathcal{PT}_X$ be the partial transformation monoid on X (with composition right-to-left) and let $\mathcal{E} = \mathrm{FSL}(X)$ be the free semilattice monoid on X, that is, $\mathcal{E}$ is the set of finite subsets of X under union. We define actions

$$T \times \mathcal{E} \to \mathcal{E}, (\alpha, E) \mapsto \alpha \cdot E = \{\alpha(y) : y \in E \cap \operatorname{dom} \alpha\}$$

and

$$T \times \mathcal{E} \to T, (\alpha, E) \mapsto \alpha^E = \alpha|_{[\alpha^{-1}\alpha(E)]^c}$$

where $E^c = X \setminus E$. Again, one can check that (ZS1)–(ZS4) hold. Thus $Z = \mathcal{E} \bowtie T$ is a Zappa-Szép product. However, regarding T as a reduced restriction monoid, in order for T to act doubly on $\mathcal{E}$, we would need that the action be via bijections, which it is clearly not.

We may amend this example slightly by replacing $\mathcal{PT}_X$ with the symmetric inverse monoid $\mathcal{I}_X$ on X. We define

$$\mathcal{I}_X \times \mathcal{E} \to \mathcal{E}, (\alpha, E) \mapsto \alpha \cdot E = \{\alpha(y) : y \in E \cap \operatorname{dom} \alpha\}$$

and

$$\mathcal{I}_X \times \mathcal{E} \to \mathcal{I}_X, (\alpha, E) \mapsto \alpha^E = \alpha|_{E^c}$$

where $E^c = X \setminus E$. Noticing that the action of $\mathcal{I}_X$ on $\mathcal{E}$ is the restriction to $\mathcal{I}_X$ of that of $\mathcal{PT}_X$, it is clear that the Zappa-Szép axioms hold. Since $\mathcal{E}$ and T are inverse, the same remarks apply as in Example 4.3.

5 λ-semidirect product of restriction semigroups

As mentioned in the Introduction, a semidirect product of inverse semigroups need not be inverse. To counter this, Billhardt [1] introduced the notion of a λ-semidirect product $S \rtimes^\lambda T$ of inverse semigroups S and T, which has subsequently been extended as far as the case for left restriction semigroups in which one component is a semilattice. For completeness, we first give the following result, concerning the case where both components are left restriction, with no further conditions applied. Since we are primarily interested in the two-sided case in this article, we leave some details to the reader.

Proposition 5.1 *Let S and T be left restriction semigroups such that T acts on S on the left by restriction endomorphisms. Then*

$$S \rtimes^\lambda T = \{(a, t) \in S \times T : t^+ \cdot a = a\}$$

is a left restriction semigroup where

$$(a, t)(b, u) = (((tu)^+ \cdot a)(t \cdot b), tu)$$

and $(a,t)^+ = (a^+, t^+)$.

Proof Let $(a,t),\ (b,u) \in S \rtimes^\lambda T$. To show that $(a,t)(b,u) \in S \rtimes^\lambda T$, we see that

$$\begin{aligned}(tu)^+ \cdot (((tu)^+ \cdot a)(t\cdot b)) &= ((tu)^+ \cdot ((tu)^+ \cdot a))((tu)^+ \cdot (t \cdot b))\\ &= ((tu)^+(tu)^+ \cdot a\big)((tu)^+ t \cdot b)\\ &= ((tu)^+ \cdot a)(tu^+ \cdot b) && \text{using } (xy)^+x = xy^+\\ &= ((tu)^+ \cdot a)(t \cdot (u^+ \cdot b))\\ &= ((tu)^+ \cdot a)(t \cdot b).\end{aligned}$$

Thus multiplication is closed.

Now to check associativity, let $(a,t), (b,u), (c,v) \in S \rtimes^\lambda T$. Then, again using the above identity, and the fact that the identity $x^+(xy)^+ = (xy)^+$ holds in any left restriction semigroup, we have:

$$\begin{aligned}\big((a,t)(b,u)\big)(c,v) &= \big(((tu)^+ \cdot a)(t\cdot b), tu\big)(c,v)\\ &= \big(((tuv)^+ \cdot (((tu)^+ \cdot a)(t \cdot b))\big)(tu \cdot c), (tu)v\big)\\ &= \big(((tuv)^+ \cdot ((tu)^+ \cdot a)))((tuv)^+ \cdot (t \cdot b))(tu \cdot c), t(uv)\big)\\ &= \big(((tuv)^+(tu)^+ \cdot a)((t(uv)^+)^+ t \cdot b)(tu \cdot c), t(uv)\big)\\ &= \big(((tuv)^+ \cdot a)(t(uv)^+ \cdot b)(tu \cdot c), t(uv)\big)\\ &= \big(((t(uv))^+ \cdot a)(t \cdot (((uv)^+ \cdot b)(u \cdot c)), t(uv)\big)\\ &= (a,t)\big(((uv)^+ \cdot b)(u \cdot c), uv\big)\\ &= (a,t)\big((b,u)(c,v)\big).\end{aligned}$$

Thus the associative law holds and hence $S \rtimes^\lambda T$ is a semigroup.

It is a straightforward exercise to show that with $^+$ as defined, $S \rtimes^\lambda T$ is left restriction. □

The semigroup $S \rtimes^\lambda T$ is called the λ-*semidirect product* of S and T. Recall that we may regard a monoid as a reduced left restriction semigroup. Notice that if T is a monoid, and acting as such (that is, $1_T \cdot u = u$ for all $u \in S$), then the λ-*semidirect product* of S and T is simply the semidirect product.

Suppose now that we begin with restriction semigroups S and T. Even in the case where T is a monoid, the ($\lambda-$)semidirect product $S \rtimes T$ need not be restriction. It was this that prompted the introduction in [5] of the notion of a double action, in order to determine the structure of free ample monoids.

Definition 5.2 Let S and T be restriction semigroups. Then T acts *doubly* on S if T acts on the left and on the right of S satisfying (CP1).

The above definition is consistent with Definition 3.3, since if S acts trivially on T, then clearly (CP2) holds. The following is easy to check.

Lemma 5.3 *Let S and T be restriction semigroups such that T acts doubly on S via restriction endomorphisms. With the trivial action of S on the left and right of T, conditions (D1)–(D4) hold.*

Let T be a restriction semigroup and let $S = E_T = E$. Suppose T acts on the left and right of S by $t \cdot e = (te)^+$ and $e \circ t = (et)^*$. It is shown in [5] that T acts on E by morphisms. We check that (CP1) holds. For this let $t \in T$ and $e \in E$, then $(t \cdot e) \circ t = (te)^+ \circ t = ((te)^+ t)^* = (te)^*$. Also $e \circ t^* = (et^*)^* = (t^*e)^* = (te)^*$ and $t^* \cdot e = (t^*e)^+ = ((te)^*)^+ = (te)^*$. Hence we see that $(t \cdot e) \circ t = e \circ t^* = t^* \cdot e$. Dually, it is easy to check that $t \cdot (e \circ t) = e \circ t^+ = t^+ \cdot e$ and hence (CP1) holds.

Theorem 5.4 *Let S and T be restriction semigroups such that T acts doubly on S via restriction endomorphisms. Let $Z = S \rtimes T$ and put*

$$C = S \rtimes^{\lambda} T = \{(a, t) \in S \times T : t^+ \cdot a = a\}.$$

Then $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ is a category under the restriction of the binary operation in Z where

$$\mathbf{d}(a, t) = (a^+, t^+) \quad \textit{and} \quad \mathbf{r}(a, t) = (a^* \circ t, t^*).$$

The set of local identities is

$$E_C = \{(f \cdot e, f) : f \in E_T, e \in E_S\}.$$

Proof Since the action of T on S preserves *, we have in particular that $t^+ \cdot a^* = (t^+ \cdot a)^*$, for all $t \in T$ and $a \in S$. It follows that

$$S \rtimes^{\lambda} T = V_{\bowtie}(Z)$$

as in Theorem 3.7. The result now follows from that theorem. □

In this case, where the action of S on T is trivial, we can order our category $\mathbf{C}$, so that it becomes inductive.

Theorem 5.5 *Let S and T be restriction semigroups such that T acts doubly on S via restriction endomorphisms. Let*

$$C = S \rtimes^{\lambda} T = \{(a, t) \in S \times T : t^+ \cdot a = a\}.$$

A partial order $\leq$ on C is defined by

$$(a, t) \leq (b, u) \quad \textit{if and only if } a \leq t^+ \cdot b \textit{ and } t \leq u.$$

Under this partial order,

$$\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$$

is an inductive category under the restriction of the binary operation in $S \rtimes T$ where

$$\mathbf{d}(a, t) = (a^+, t^+) \quad \textit{and} \quad \mathbf{r}(a, t) = (a^* \circ t, t^*).$$

The set of local identities is

$$E_C = \{(f \cdot e, f) : f \in E_T, e \in E_S\}$$

and restriction and co-restriction are given by

$$_{(f \cdot e, f)}|(a, t) = ((f \cdot e)(f \cdot a), ft) = (f \cdot ea, ft)$$

where $(f \cdot e, f) \leq \mathbf{d}(a, t)$ and

$$(a, t)|_{(f \cdot e, f)} = \big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big)$$

where $(f \cdot e, f) \leq \mathbf{r}(a, t)$.

Proof From Theorem 5.4, we know that $\mathbf{C}$ is a category. We now show that $\mathbf{C}$ is inductive under the given relation.

We first check that $\leq$ is a partial order. It is clear that $\leq$ is reflexive.

Let $(a, t), (b, u) \in C$ and suppose that $(a, t) \leq (b, u)$ and $(b, u) \leq (a, t)$. Then $a \leq t^+ \cdot b$, $t \leq u$ and $b \leq u^+ \cdot a$, $u \leq t$. By antisymmetry of the natural partial order we have $t = u$, and so $a \leq t^+ \cdot b = u^+ \cdot b = b$. By symmetry $b \leq a$ and so $a = b$. Thus $(a, t) = (b, u)$ and hence $\leq$ is antisymmetric.

Next let $(a, t), (b, u), (c, v) \in C$ be such that $(a, t) \leq (b, u)$ and $(b, u) \leq (c, v)$. Then $a \leq t^+ \cdot b,\ t \leq u$ and $b \leq u^+ \cdot c,\ u \leq v$. Clearly $t \leq v$ and using Lemma 2.1

$$a \leq t^+ \cdot b \leq t^+ \cdot (u^+ \cdot c) = (t^+ u^+) \cdot c = t^+ \cdot c.$$

Thus $(a, t) \leq (c, v)$ and hence $\leq$ is a partial order on C.

We now check that the conditions hold for $\mathbf{C}$ to be an ordered category.

(IC1) Let $(a, t), (b, u) \in C$ and let $(a, t) \leq (b, u)$. Then $a \leq t^+ \cdot b$ and $t \leq u$.

To show that $\mathbf{d}(a, t) \leq \mathbf{d}(b, u)$, we see that $a^+ \leq (t^+ \cdot b)^+ = t^+ \cdot b^+$ and also $t^+ \leq u^+$. Thus $\mathbf{d}(a, t) \leq \mathbf{d}(b, u)$. Next to show that $\mathbf{r}(a, t) \leq \mathbf{r}(b, u)$, we need to check that $a^* \circ t \leq t^* \cdot (b^* \circ u)$ and $t^* \leq u^*$. We have

$$\begin{aligned}
a^* \circ t &\leq (t^+ \cdot b)^* \circ t && \text{using Lemma 2.1}\\
&= (t^+ \cdot b^*) \circ t\\
&= (b^* \circ t^+) \circ t && \text{using (CP1)}\\
&= b^* \circ t^+ t = b^* \circ t\\
&= b^* \circ u t^* = (b^* \circ u) \circ t^*\\
&= t^* \cdot (b^* \circ u).
\end{aligned}$$

Hence $\mathbf{r}(a, t) \leq \mathbf{r}(b, u)$.

(IC2) Let $(a, t), (b, u), (c, v), (d, p) \in C$ be such that $(a, t) \leq (b, u),\ (c, v) \leq (d, p)$, so that

$$a \leq t^+ \cdot b\ ,\ t \leq u \quad \text{and} \quad c \leq v^+ \cdot d\ ,\ v \leq p.$$

Suppose also $\exists\ (a, t) \cdot (c, v)$ and $\exists\ (b, u) \cdot (d, p)$ in $\mathbf{C}$. Then $\mathbf{r}(a, t) = \mathbf{d}(c, v)$ and $\mathbf{r}(b, u) = \mathbf{d}(d, p)$, that is

$$(a^* \circ t, t^*) = (c^+, v^+) \ \text{ and } \ (b^* \circ u, u^*) = (d^+, p^+).$$

Now $t \leq u\ ,\ v \leq p$ implies that $tv \leq up$. Also

$$\begin{aligned}
a(t \cdot c) &= a^+(t^+ \cdot b)\big(t \cdot (c^+(v^+ \cdot d))\big) && \text{because } a \leq t^+ \cdot b \text{ and } c \leq v^+ \cdot d\\
&= a^+(t^+ \cdot b)(t \cdot c^+)(t \cdot (v^+ \cdot d))\\
&= a^+(t^+ \cdot b)(t \cdot c)^+(tv^+ \cdot d)\\
&= a^+\big((t^+ \cdot b)(t \cdot c)^+\big)^+(t^+ \cdot b)((tv)^+ t \cdot d) && \text{using the left ample condition}\\
&= \big(a^+(t^+ \cdot b)(t \cdot c)\big)^+((tt^*)^+ \cdot b)((tv)^+ t^+ u \cdot d) && \text{because } t \leq u\\
&= (a(t \cdot c))^+((tv^+)^+ \cdot b)((tv)^+ \cdot (u \cdot d)) && \text{because } a \leq t^+ \cdot b \text{ and } t^* = v^+\\
&= (a(t \cdot c))^+((tv)^+ \cdot b)((tv)^+ \cdot (u \cdot d))\\
&= (a(t \cdot c))^+((tv)^+ \cdot (b(u \cdot d)))
\end{aligned}$$

and so $a(t \cdot c) \leq (tv)^+ \cdot b(u \cdot d)$. Hence

$$(a, t) \cdot (c, v) = (a(t \cdot c), tv) \leq (b(u \cdot d), up) = (b, u) \cdot (d, p).$$

(IC3) Let $(f \cdot e, f) \in E_C$ and $(a, t) \in C$ be such that $(f \cdot e, f) \leq \mathbf{d}(a, t) = (a^+, t^+)$ so that $f \cdot e \leq f \cdot a^+$ and $f \leq t^+$.

We show that $(f \cdot ea, ft)$ is the unique element of C such that

$$(f \cdot ea, ft) \leq (a, t) \quad \text{and} \quad \mathbf{d}(f \cdot ea, ft) = (f \cdot e, f).$$

We first check that $(f \cdot ea, ft) \in C$. For this we see that

$$(ft)^+ \cdot (f \cdot ea) = ft^+ f \cdot ea = f \cdot ea.$$

Clearly $ft \leq t$ and

$$f \cdot ea = (f \cdot e)(f \cdot a) \leq f \cdot a = f \cdot (t^+ \cdot a) = (ft)^+ \cdot a.$$

Thus $(f \cdot ea, ft) \leq (a, t)$. Also

$$\begin{aligned}\mathbf{d}(f \cdot ea, ft) &= \big((f \cdot ea)^+, (ft)^+\big)\\ &= \big(f \cdot (ea)^+, ft^+\big)\\ &= (f \cdot (ea^+), f)\\ &= ((f \cdot e)(f \cdot a^+), f)\\ &= (f \cdot e, f) \qquad\qquad \text{because } f \cdot e \leq f \cdot a^+.\end{aligned}$$

Next suppose that (c, s) is another element such that $(c, s) \leq (a, t)$ and $\mathbf{d}(c, s) = (f \cdot e, f)$. Then $c \leq s^+ \cdot a$, $s \leq t$, $c^+ = f \cdot e$ and $s^+ = f$. Now $c = c^+(s^+ \cdot a) = (f \cdot e)(f \cdot a) = f \cdot ea$ and $s = s^+ t = ft$. Hence $(f \cdot ea, ft)$ is the unique element such that

$$(f \cdot ea, ft) \leq (a, t) \quad \text{and} \quad \mathbf{d}(f \cdot ea, ft) = (f \cdot e, f).$$

(IC4) Let $(a, t) \in C$, $(f \cdot e, f) \in E_C$ be such that $(f \cdot e, f) \leq \mathbf{r}(a, t) = (a^* \circ t, t^*)$, so that $f \cdot e \leq f \cdot (a^* \circ t)$ and $f \leq t^*$. We will prove that $\big(((tf)^+ \cdot a)(t \cdot (f \cdot e)), tf\big) = \big(((tf)^+ \cdot a)(tf \cdot e), tf\big)$ is the unique element such that

$$\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) \leq (a, t) \quad \text{and} \quad \mathbf{r}\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) = (f \cdot e, f).$$

As T acts by morphisms, we see that $(tf)^+ \cdot ((tf)^+ \cdot a)(tf \cdot e)) = ((tf)^+ \cdot a)(tf \cdot e)$ and so $\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) \in C$. Now $tf \cdot e \in E_S$ and so

$$\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) \leq (a, t).$$

Clearly $(tf)^* = f$ and

$$\begin{aligned}\big((tf)^+ \cdot a)(tf \cdot e)\big)^* \circ tf &= \big((tf)^+ \cdot a^*)(tf \cdot e)\big) \circ tf\\ &= \big((tf \cdot (a^* \circ tf))(tf \cdot e)\big) \circ tf \text{ using (CP1)}\\ &= \big(tf \cdot \big((a^* \circ tf)e\big)\big) \circ tf\\ &= (tf)^* \cdot ((a^* \circ tf)e)\\ &= f \cdot (a^* \circ tf)e\\ &= (f \cdot (a^* \circ tf))(f \cdot e)\\ &= f \cdot e \text{ as } f \cdot e \leq f \cdot (a^* \circ t) = f \cdot (a^* \circ tf).\end{aligned}$$

Thus $\mathbf{r}\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) = (f \cdot e, f)$.

Next, suppose that (c, n) is another element such that $(c, n) \leq (a, t)$ and $\mathbf{r}(c, n) = (f \cdot e, f)$, so that $c \leq n^+ \cdot a$, $n \leq t$, $c^* \circ n = f \cdot e$ and $n^* = f$. Then $n = tf$ and $c \leq n^+ \cdot a$ implies $c = (n^+ \cdot a)c^* = ((tf)^+ \cdot a)c^*$. Now

$$c^* = n^+ \cdot c^* = n \cdot (c^* \circ n) = tf \cdot (f \cdot e) = tf \cdot e.$$

Hence $\big(((tf)^+ \cdot a)(tf \cdot e), tf\big)$ is the unique element such that

$$\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) \leq (a, t) \quad \text{and} \quad \mathbf{r}\big(((tf)^+ \cdot a)(tf \cdot e), tf\big) = (f \cdot e, f).$$

(IC5) To show that $(E_C, \leq)$ is a meet semilattice, let $(f \cdot e, f), (h \cdot g, h) \in E_C$. An easy calculation gives that $(fh \cdot (eg), fh) \in E_C$ and is the greatest lower bound of $(f \cdot e, f)$ and $(h \cdot g, h)$. Thus $(f \cdot e, f) \wedge (h \cdot g, h)$ exists and equals $(fh \cdot (eg), fh)$.

We have now verified all the axioms to yield that $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is an inductive category. $\square$

We now consider the pseudo-product on $\mathbf{C}$ and show that it coincides with that in Billhardt's λ-semidirect product as in the case for *left* restriction semigroups.

Theorem 5.6 *Let S and T be restriction semigroups such that T acts doubly on S via restriction endomorphisms. Let*

$$S \rtimes^{\lambda} T = \{(a, t) \in S \times T : t^{+} \cdot a = a\}.$$

Then $S \rtimes^{\lambda} T$ is a restriction semigroup with multiplication given by

$$(a, t)(b, u) = (((tu)^{+} \cdot a)(t \cdot b), tu).$$

For any $(a, t) \in S \rtimes^{\lambda} T$ we have $(a, t)^{+} = (a^{+}, t^{+})$ and $(a, t)^{*} = (a^{*} \circ t, t^{*})$. The natural partial order is given by

$$(a, t) \leq (b, u) \quad \textit{if and only if} \quad a \leq t^{+} \cdot b \quad \textit{and} \quad t \leq u.$$

Proof By Theorems 2.5 and 5.5, we need only show that the pseudo-product of the category $\mathbf{C}$ of Theorem 5.5 is given by the formula above. To this end, let $(a, t), (b, u) \in S \rtimes^{\lambda} T$. We first compute $\mathbf{r}(a, t) \wedge \mathbf{d}(b, u)$. We have

$$\begin{aligned}
\mathbf{r}(a, t) \wedge \mathbf{d}(b, u) &= (a^{*} \circ t, t^{*}) \wedge (b^{+}, u^{+}) \\
&= \big((t^{*}u^{+} \cdot (a^{*} \circ t))(t^{*} \cdot b^{+}), t^{*}u^{+}\big) \\
&= \big((u^{+} \cdot (t^{*} \cdot (a^{*} \circ t)))(t^{*} \cdot b^{+}), t^{*}u^{+}\big) \\
&= \big((u^{+} \cdot (a^{*} \circ t))(t^{*} \cdot b^{+}), t^{*}u^{+}\big).
\end{aligned}$$

Now

$$\begin{aligned}
((a, t)|_{\mathbf{r}(a,t)\wedge\mathbf{d}(b,u)}) &= ((a, t)|_{((u^{+}\cdot(a^{*}\circ t))(t^{*}\cdot b^{+}), t^{*}u^{+})}) \\
&= \big(((t(t^{*}u^{+}))^{+} \cdot a\big)\big(t \cdot ((u^{+} \cdot (a^{*} \circ t))(t^{*} \cdot b^{+})\big), tt^{*}u^{+}\big) \\
&= \big(((tu)^{+} \cdot a\big)\big(t \cdot (u^{+} \cdot (a^{*} \circ t))\big)\big(t \cdot (t^{*} \cdot b^{+})\big), tu^{+}\big) \\
&= \big(((tu)^{+} \cdot a)(tu^{+} \cdot (a^{*} \circ t))(t \cdot b^{+}), tu^{+}\big).
\end{aligned}$$

Also

$$\begin{aligned}
(\mathbf{r}_{(a,t)\wedge\mathbf{d}(b,u)}|(b, u)) &= (_{((u^{+}\cdot(a^{*}\circ t))(t^{*}\cdot b^{+}), t^{*}u^{+})}|(b, u)) \\
&= \big((u^{+} \cdot (a^{*} \circ t))(t^{*} \cdot b^{+})(t^{*}u^{+} \cdot b), t^{*}u^{+}u\big) \\
&= \big((u^{+} \cdot (a^{*} \circ t))(t^{*} \cdot b), t^{*}u\big).
\end{aligned}$$

Thus

$$\begin{aligned}
(a, t) \otimes (b, u) &= ((a, t)|_{\mathbf{r}(a,t)\wedge\mathbf{d}(b,u)})(_{\mathbf{r}(a,t)\wedge\mathbf{d}(b,u)}|(b, u)) \\
&= \big(((tu)^{+} \cdot a)(tu^{+} \cdot (a^{*} \circ t))(t \cdot b^{+}), tu^{+}\big) \cdot \big((u^{+} \cdot (a^{*} \circ t))(t^{*} \cdot b), t^{*}u\big) \\
&= \big(((tu)^{+} \cdot a)(tu^{+} \cdot (a^{*} \circ t))(t \cdot b^{+})(tu^{+} \cdot (u^{+} \cdot (a^{*} \circ t)))(tu^{+} \cdot (t^{*} \cdot b)), tu\big) \\
&= \big(((tu)^{+} \cdot a)(tu^{+} \cdot (a^{*} \circ t))((t \cdot b^{+})(tu^{+} \cdot (a^{*} \circ t))(tu^{+} \cdot b), tu\big) \\
&= \big(((tu)^{+} \cdot a)((tu)^{+}t \cdot (a^{*} \circ t)))(t \cdot b), tu\big) \text{ as } u^{+} \cdot b = b \\
&= \big(((tu)^{+} \cdot a)((tu)^{+} \cdot (t \cdot (a^{*} \circ t))))(t \cdot b), tu\big) \\
&= \big(((tu)^{+} \cdot a)((tu)^{+} \cdot (t^{+} \cdot a^{*}))(t \cdot b), tu\big) \qquad \text{using (CP1)} \\
&= \big(((tu)^{+} \cdot a)((tu)^{+} \cdot a^{*})(t \cdot b), tu\big) \\
&= \big(((tu)^{+} \cdot a)(t \cdot b), tu\big).
\end{aligned}$$

$\square$

6 λ-Zappa-Szép product of a semilattice and a monoid

Let S and T be restriction semigroups acting doubly on each other, satisfying $(D1)$–$(D4)$. We would like to determine how to put a multiplication on the scaffold $V_{\bowtie}(S \bowtie T)$ of the Zappa-Szép product in Theorem 3.7, thus extending the notion of λ-semidirect product. In the inverse case, Gilbert and Wazzan [6,21] succeeded in doing so where S is a semilattice and T a group. We extend this to the case where S is a semilattice and T is a monoid, regarded as a reduced restriction semigroup. First, we recall the result in the inverse case.

Theorem 6.1 [6, Theorem 4.3, Lemma 4.4] *Let E be a semilattice, G be a group and let $Z = E \bowtie G$ be a Zappa-Szép product such that (ZS7) holds. Then*

$$B_{\bowtie}(Z) = \{(e, g) \in E \times G : (g^{-1})^e = g^{-1}\}$$

is an inverse semigroup with multiplication defined by

$$(e, g) \otimes (f, h) = \big(e(g \cdot f), g^f h^{h^{-1}g^{-1}\cdot e}\big).$$

We use the symbol $\otimes$ for the binary operation above since in [21] Theorem 6.1 is proved by showing that $B_{\bowtie}(Z)$ can be endowed with an order so that it becomes an inductive groupoid; $\otimes$ is the corresponding pseudoproduct.

An inverse semigroup $B_{\bowtie}(Z)$ as in Theorem 6.1 is referred to as a λ-*Zappa-Szép product* of E and G.

Suppose now that $E \bowtie T$ is a Zappa-Szép product of a semilattice E and a monoid T. Note that E and T act doubly on each other if and only if there is an action $E \times T \to E$ where $(e, t) \mapsto e \circ t$ such that

$$e = (t \cdot e) \circ t = t \cdot (e \circ t) \tag{CP3}$$

for all $e \in E$ and $t \in T$; the left action of E on T is perforce the same as the right action. Notice that (CP3) is simply saying that the action of T on the left of E is by bijections. It is convenient, however, to retain the notation for a right action of T. We suppose that (ZS7) holds. If (D2) holds, then for any $e \in E$ we have $1 = (1^e)^+ = 1^{1\cdot e} = 1^e$, so that (ZS8) holds. Conversely, if (ZS8) holds, then (D1)–(D4) are easily seen to be true. We therefore suppose that (ZS7) and (ZS8) hold. Clearly the description of $V = V_{\bowtie}(E \bowtie T)$ simplifies to

$$V = V_{\bowtie}(E \bowtie T) = \{(e, t) \in E \times T : t = t^{e\circ t}\}.$$

We now prove that, in this special case, V can be made into an inductive category if the extra condition

$$e \leq f \Rightarrow t^f \cdot e = t \cdot e \tag{D5}$$

holds, for all $e, f \in E$ and $t \in T$. Notice that from [6, Lemma 3.1], this is automatically true if T is a group.

Theorem 6.2 *Let $E \bowtie T$ be a Zappa-Szép product of a semilattice E and a monoid T such that T acts by bijections and (ZS7), (ZS8) and (D5) hold. Let*

$$V = V_{\bowtie}(E \bowtie T) = \{(e, t) \in E \times T : t = t^{e\circ t}\}.$$

Then $\mathbf{V} = (V, \mathbf{d}, \mathbf{r}, \leq)$ *where*

$\mathbf{d}(e, t) = (e, 1)$, $\mathbf{r}(e, t) = (e \circ t, 1)$ *and* $(e, s) \leq (f, t)$ *if and only if* $e \leq f$ *and* $s = t^{e\circ t}$, *is an inductive category under the restriction of the binary operation in $E \bowtie T$. The set of local identities is $E_V = \{(e, 1) : e \in E_S\}$. For $(e, 1) \in E_V$, $(f, t) \in V$ with $(e, 1) \leq \mathbf{d}(f, t)$, the restriction is given by*

$$_{(e,1)}|(f, t) = (e, t^{e\circ t})$$

and for $(e, 1) \leq \mathbf{r}(f, t)$, the co-restriction is given by

$$(f, t)|_{(e,1)} = (f(t \cdot e), t^e).$$

Proof From Theorem 3.7, V is a category with set of local identities $E_V = \{(e, 1) : e \in E_S\}$ where $\mathbf{d}(e, t) = (e, 1)$ and $\mathbf{r}(e, t) = (e \circ t, 1)$. We notice that for $(e, 1), (f, 1) \in E_V$, $(e, 1) \leq (f, 1) \Leftrightarrow e \leq f$.

Before giving the proof of Theorem 6.2, we record some preliminary computations.

Lemma 6.3 *Let $(e, s), (f, t) \in V$. Then:*

(1) $ef \circ s = (e \circ s)(f \circ s)$;
(2) if $(e, s) \leq (f, t)$, then $e \circ s = e \circ t$ and $e \circ t \leq f \circ t$;
(3) for $e, f \in E$ and $t \in T$, if $e \leq f$, then $t \cdot e \leq t \cdot f$.

Proof (1) We have

$$\begin{aligned}(e \circ s)(f \circ s) &= \big(s \cdot ((e \circ s)(f \circ s))\big) \circ s && \text{using (CP3)}\\ &= \big((s \cdot (e \circ s))(s^{e\circ s} \cdot (f \circ s))\big) \circ s && \text{using (ZS2)}\\ &= ef \circ s && \text{because } s = s^{e\circ s} \text{ and using (CP3).}\end{aligned}$$

(2) Suppose that $(e, s) \leq (f, t)$. Then $e \leq f$ and $s = t^{e\circ t}$. To prove that $e \circ s = e \circ t$, we see that

$$\begin{aligned}(e \circ s)(e \circ t) &= \big(s \cdot ((e \circ s)(e \circ t))\big) \circ s && \text{using (CP3)}\\ &= \big((s \cdot (e \circ s))(s^{e\circ s} \cdot (e \circ t))\big) \circ s && \text{using (ZS2)}\\ &= \big(e(s \cdot (e \circ t))\big) \circ s && \text{using (CP3) and } s = s^{e\circ s}\\ &= \big((t \cdot (e \circ t))(t^{e\circ t} \cdot (e \circ t))\big) \circ s && \text{using (CP3) and } s = t^{e\circ t}\\ &= \big(t \cdot ((e \circ t)(e \circ t))\big) \circ s && \text{using (ZS2)}\\ &= e \circ s && \text{using (CP3)}\end{aligned}$$

and

$$\begin{aligned}(e \circ t)(e \circ s) &= \big(t \cdot ((e \circ t)(e \circ s))\big) \circ t && \text{using (CP3)}\\ &= \big((t \cdot (e \circ t))(t^{e\circ t} \cdot (e \circ s))\big) \circ t && \text{using (ZS2)}\\ &= \big(e(s \cdot (e \circ s))\big) \circ t && \text{because } s = t^{e\circ t} \text{ and using (CP3)}\\ &= e \circ t && \text{using (CP3).}\end{aligned}$$

Thus $e \circ t = e \circ s$. Next we see that $e \circ t \leq f \circ t$, since

$$\begin{aligned}(e \circ t)(f \circ t) &= ef \circ t && \text{from (1) above}\\ &= e \circ t && \text{because } e \leq f.\end{aligned}$$

(3) Suppose $e \leq f$. Then $e = ef$, so that

$$t \cdot e = t \cdot (ef) = t \cdot (fe) = (t \cdot f)(t^f \cdot e) \quad \text{using (ZS2)}$$

and hence $t \cdot e \leq t \cdot f$. ∎

We now continue the proof of Theorem 6.2 by showing that V is an inductive category. First we prove that $\leq$ is a partial order. It is clear that $\leq$ is reflexive. Let $(e, s), (f, t) \in V$ and suppose that $(e, s) \leq (f, t)$ and $(f, t) \leq (e, s)$. Then $e \leq f \leq e$ implies $e = f$ and $s = t^{e\circ t} = t^{f\circ t} = t$. Hence $\leq$ is antisymmetric.

To check transitivity, let $(e, s), (f, t), (g, u) \in V$ and suppose that $(e, s) \leq (f, t)$ and $(f, t) \leq (g, u)$. Then $e \leq f \leq g$, $s = t^{e\circ t}$ and $t = u^{f\circ u}$. Now $e \leq f \leq g$ implies that $e \leq g$ and

$$\begin{aligned}
s &= t^{e\circ t} \\
&= (u^{f\circ u})^{(e\circ u^{f\circ u})} && \text{because } t = u^{f\circ u} \\
&= u^{(f\circ u)(e\circ u^{f\circ u})} \\
&= u^{\left(u\cdot((f\circ u)(e\circ u^{f\circ u}))\right)\circ u} && \text{using (CP3)} \\
&= u^{\left((u\cdot(f\circ u))(u^{f\circ u}\cdot(e\circ u^{f\circ u}))\right)\circ u} && \text{using (ZS2)} \\
&= u^{fe\circ u} && \text{using (CP3)} \\
&= u^{e\circ u}.
\end{aligned}$$

Hence $(e, s) \leq (g, u)$ and thus $\leq$ is a partial order on V.

We now show that V satisfies the axioms of an inductive category.

(IC1) Let $(e, s), (f, t) \in V$ and suppose that $(e, s) \leq (f, t)$. Then $e \leq f$ and $s = t^{e\circ t}$. Clearly $\mathbf{d}(e, s) \leq \mathbf{d}(f, t)$. Now to show that $\mathbf{r}(e, s) \leq \mathbf{r}(f, t)$, we need to check that $e \circ s \leq f \circ t$. From Lemma 6.3, we see that $e \circ s = e \circ t \leq f \circ t$, as required.

(IC2) Let $(e, s), (f, t), (g, u), (h, v) \in V$ and $(e, s) \leq (f, t)$, $(g, u) \leq (h, v)$. Then

$$e \leq f,\ s = t^{e\circ t} \quad \text{and} \quad g \leq h,\ u = v^{g\circ v}.$$

Suppose that $\exists(e, s)\cdot(g, u)$ and $\exists(f, t)\cdot(h, v)$. Then $\mathbf{r}(e, s) = \mathbf{d}(g, u)$ and $\mathbf{r}(f, t) = \mathbf{d}(h, v)$, so that $(e \circ s, 1) = (g, 1)$ and $(f \circ t, 1) = (h, 1)$. To prove that $(e, s)\cdot(g, u) \leq (f, t)\cdot(h, v)$, that is, $(e(s \cdot g), s^g u) \leq (f(t \cdot h), t^h v)$, we need to show

$$e(s \cdot g) \leq f(t \cdot h) \quad \text{and} \quad s^g u = (t^h v)^{e(s\cdot g)\circ t^h v}.$$

We see that $e(s \cdot g) = e(s \cdot (e \circ s)) = ee = e$, and $f(t \cdot h) = f(t \cdot (f \circ t)) = ff = f$, so that $e(s \cdot g) \leq f(t \cdot h)$. Now

$$\begin{aligned}
s^g u &= s^{e\circ s} u \\
&= su && \text{because } s = s^{e\circ s} \\
&= t^{e\circ t} v^{g\circ v} && \text{because } s = t^{e\circ t},\ u = v^{g\circ v} \\
&= t^{v\cdot((e\circ t)\circ v)} v^{(e\circ s)\circ v} && \text{using (CP3)} \quad \text{and} \quad g = e\circ s \\
&= t^{v\cdot((e\circ t)\circ v)} v^{(e\circ t)\circ v} && \text{because } e\circ s = e\circ t \text{from Lemma 6.3 (2)} \\
&= (tv)^{(e\circ t)\circ v} && \text{using (ZS4)} \\
&= (t^{f\circ t} v)^{e\circ t^{f\circ t} v} && \text{because } t = t^{f\circ t} \\
&= (t^h v)^{(e(s\cdot g))\circ t^h v} && \text{because } h = f\circ t \quad \text{and} \quad e = e(s\cdot g).
\end{aligned}$$

Hence $(e, s)\cdot(g, u) \leq (f, t)\cdot(h, v)$.

(IC3) Let $(e, 1) \in E_V$ and $(f, t) \in V$ be such that $(e, 1) \leq \mathbf{d}(f, t)$ so that $e \leq f$. We first note that $(e, t^{e\circ t}) \in V$ as

$$\begin{aligned}
(t^{e\circ t})^{(e\circ t^{e\circ t})} &= t^{(e\circ t)(e\circ t^{e\circ t})} \\
&= t^{\left(t\cdot((e\circ t)(e\circ t^{e\circ t}))\right)\circ t} && \text{using (CP3)} \\
&= t^{\left((t\cdot(e\circ t))(t^{e\circ t}\cdot(e\circ t^{e\circ t}))\right)\circ t} && \text{using (ZS2)} \\
&= t^{e\circ t}.
\end{aligned}$$

Clearly $\mathbf{d}(e, t^{e\circ t}) = (e, 1)$ and $(e, t^{e\circ t}) \leq (f, t)$.

We show that $(e, t^{e\circ t})$ is the unique element with these properties. Let (g, u) be another element such that $(g, u) \leq (f, t)$ and $\mathbf{d}(g, u) = (e, 1)$. Then $g \leq f$, $u = t^{g\circ t}$ and $(g, 1) = (e, 1)$. Thus $g = e$ and $u = t^{g\circ t} = t^{e\circ t}$. Hence $(e, t^{e\circ t})$ has the required properties.

(IC4) Let $(e, 1) \in E$ and $(f, t) \in V$ be such that $(e, 1) \leq \mathbf{r}(f, t)$. Then $(e, 1) \leq (f \circ t, 1)$, so that $e \leq f \circ t$. We want to show that $(f(t \cdot e), t^e)$ is the unique element in V such that

$$(f(t \cdot e), t^e) \leq (f, t) \quad \text{and} \quad \mathbf{r}(f(t \cdot e), t^e) = (e, 1).$$

We first check that $(f(t \cdot e), t^e) \in V$. First we notice that as $e \leq f \circ t$, so $t \cdot e \leq t \cdot (f \circ t) = f$, by Lemma 6.3 (3). Next we see that

$$\begin{aligned} (t^e)^{(f(t\cdot e))\circ t^e} &= (t^e)^{(t\cdot e)\circ t^e} && \text{because } t \cdot e \leq f, \text{ so } f(t \cdot e) = t \cdot e \\ &= (t^e)^{(t^e\cdot e)\circ t^e} && \text{since } t^e \cdot e = t \cdot e \text{ by}(D5) \\ &= (t^e)^e = t^e && \text{using (CP3).} \end{aligned}$$

Hence $(f(t \cdot e), t^e) \in V$. Clearly $f(t \cdot e) \leq f$ and as $t \cdot e \leq f$,

$$t^{(f(t\cdot e))\circ t} = t^{(t\cdot e)\circ t} = t^e.$$

Hence $(f(t \cdot e), t^e) \leq (f, t)$. Also

$$\begin{aligned} \mathbf{r}(f(t \cdot e), t^e) &= ((f(t \cdot e)) \circ t^e, 1) \\ &= ((t \cdot e) \circ t^e, 1) && \text{because } t \cdot e \leq f \\ &= ((t^e \cdot e) \circ t^e, 1) && \text{since } t^e \cdot e = t \cdot e \text{ by (D5)} \\ &= (e, 1). \end{aligned}$$

Next to prove the uniqueness of $(f(t \cdot e), t^e)$, let (g, u) be another element in V such that $(g, u) \leq (f, t)$ and $\mathbf{r}(g, u) = (e, 1)$. Then $g \leq f$, $u = t^{g\circ t}$ and $g \circ u = e$. Now

$$\begin{aligned} f(t \cdot e) &= f(t \cdot (g \circ u)) && \text{because } e = g \circ u \\ &= f(t \cdot (g \circ t)) && \text{because } g \circ u = g \circ t \text{ from Lemma 6.3 (2)} \\ &= fg = g, \end{aligned}$$

and $u = t^{g\circ t} = t^{g\circ u} = t^e$. Hence $(f(t \cdot e), t^e)$ is the unique element with the required properties.

(IC5) It is clear that $E_V = \{(e, 1) : e \in E\}$ is a meet semilattice with $(e, 1) \wedge (f, 1) = (ef, 1)$.

Hence V is an inductive category with restriction and co-restriction as given. $\square$

Next we define a pseudo-product on our inductive category to obtain a restriction semigroup.

Theorem 6.4 *Let $E \bowtie T$ be a Zappa-Szép product of a semilattice E and a monoid T such that T acts by bijections and (ZS7), (ZS8) and (D5) hold. Let*

$$V = V_{\bowtie}(E \bowtie T) = \{(e, t) \in E \times T : t = t^{e\circ t}\}.$$

Then V is a restriction semigroup under

$$(e, s) \otimes (f, t) = \big(e(s \cdot f), s^f t^{e\circ st}\big),$$

such that

$$(e, s)^+ = (e, 1) \quad \textit{and} \quad (e, s)^* = (e \circ s, 1).$$

Proof In view of Theorems 2.5 and 6.2, we need only to show that the pseudo-product on V simplifies to the given formula.

We first see that

$$\mathbf{r}(e,s) \wedge \mathbf{d}(f,t) = (e \circ s, 1) \wedge (f, 1) = ((e \circ s)f, 1).$$

Now

$$\begin{aligned}(e,s)|_{\mathbf{r}(e,s)\wedge\mathbf{d}(f,t)} &= (e,s)|_{((e\circ s)f,1)} \\ &= \big(e(s.((e\circ s)f)), s^{(e\circ s)f}\big) \\ &= \big(e((s\cdot(e\circ s))(s^{e\circ s}\cdot f), s^f\big) \\ &= (e(s\cdot f), s^f)\end{aligned}$$

and

$$\begin{aligned}{}_{\mathbf{r}(e,s)\wedge\mathbf{d}(f,t)}|(f,t) &= {}_{((e\circ s)f,1)}|(f,t) \\ &= ((e\circ s)f, t^{((e\circ s)f)\circ t}).\end{aligned}$$

Thus

$$\begin{aligned}(e,s)\otimes(f,t) &= (e(s\cdot f), s^f)((e\circ s)f, t^{((e\circ s)f)\circ t}) \\ &= \big(e(s\cdot f)(s^f\cdot((e\circ s)f)), (s^f)^{(e\circ s)f}t^{((e\circ s)f)\circ t}\big) \\ &= \big(e(s\cdot(f^2(e\circ s)), s^{(e\circ s)f}t^{((e\circ s)f)\circ t}\big) && \text{using (ZS2)} \\ &= \big(e(s\cdot(f(e\circ s)), s^{(e\circ s)f}t^{((e\circ s)f)\circ t}\big) \\ &= \big(e(s\cdot((e\circ s)f), s^{(e\circ s)f}t^{((e\circ s)f)\circ t}\big) \\ &= \big(e(s\cdot(e\circ s))(s^{e\circ s}\cdot f), s^f t^{((e\circ s)f)\circ t}\big) && \text{using (ZS2) and (CP3)} \\ &= \big(e(s\cdot f), s^f t^{((e\circ s)\circ t)(f\circ t)}\big) && \text{from Lemma 6.3 (1)} \\ &= \big(e(s\cdot f), s^f (t^{f\circ t})^{(e\circ s t)}\big) \\ &= \big(e(s\cdot f), s^f t^{(e\circ s t)}\big) && \text{because } t^{f\circ t} = t.\end{aligned}$$

□

Remark 6.5 If $E \bowtie G$ is a Zappa-Szép product of a group G with a semilattice E, then (ZS7) (in our notation) is easily seen to imply that (ZS8) holds (indeed, we only require the unipotency of G here). It follows that, in view of Lemmas 3.4 and 3.6, Theorem 6.1 is an immediate specialisation of Theorem 6.4 to the inverse case.

We now construct examples in which we show that all the conditions of Theorems 6.2 and 6.4 hold. To construct these examples we utilise our knowledge that if $Z = E \bowtie G$ is a Zappa-Szép product of a semilattice and a group G such that (ZS7) holds, then so do (ZS8) and (D5).

Example 6.6 Let $S \bowtie T$ be a Zappa-Szép product of monoids S and T. Suppose that $U = TV = T \ltimes V$ is a monoid semidirect product of T and a monoid V (certainly such U can be found - one needs only to consider forming direct products). Define actions

$$U \times S \to S,\ (u,s) \mapsto u \star s = t \cdot s \quad \text{and} \quad U \times S \to U,\ (u,s) \mapsto u_s = t^s$$

where $u = tv$, $t \in T$, $v \in V$. It is straightforward to check the Zappa-Szép axioms (ZS1)–(ZS4) hold. Hence $Z = S \bowtie U$ is Zappa-Szép product of S and U.

Suppose now that in the above, $S = E$ is a semilattice and $T = G$ is a group and suppose that (ZS7) holds for the action of G on E. For examples of such, we refer the reader to [6]. Next we suppose that $U = GV$ as above. It is then easy to check that U acts by bijections, and (ZS7), (ZS8) and (D5) hold for $E \bowtie U$. Thus all the conditions of Theorems 6.2 and 6.4 hold.

For details of our final example, we refer the reader to [11].

Example 6.7 Let S be an ample monoid which is *perfect*. Consequently, every σ-class $[s]$ (where σ is the least congruence identifying the projections) has a greatest element m_s and for all $s, t \in S$ we have $m_s m_t = m_{st}$. Suppose in addition that $G = S/\sigma$ is a group: for example, this will happen if S is finite.

Now let E be a semilattice and $E \bowtie G$ a Zappa-Szép product such that (ZS7) holds. We define

$$S \times E \to E,\ (s, e) \mapsto s \star e = [s] \cdot e$$

and

$$S \times E \to S,\ (s, e) \mapsto s_e = m_t \quad \text{where } [s]^e = [t].$$

One may verify that these actions satisfy the Zappa-Szép axioms and that all conditions of Theorem 6.2 hold for this Zappa-Szép product.

7 Internal structure of $S \rtimes^\lambda T$

Let S and T be (left) restriction semigroups such that T acts on S satisfying the conditions required to build the (left) restriction semigroup $S \rtimes^\lambda T$. What characterises the structure of $S \rtimes^\lambda T$? This is known in some special cases (see, for example, [1–3]), where S is a semilattice. The same question could be asked for a restriction semigroup of the form $S \rtimes T$, where S is a semilattice and T a monoid. Even in the case where T is a group, so that the latter semigroup is inverse, the connection between its internal structure and that of the constituent components T and S is a little mysterious. In this section we begin to consider these questions, by examining the structure of $S \rtimes^\lambda T$ in the one- and two-sided cases.

Throughout this section, let S and T be left restriction semigroups such that T acts on S on the left by left restriction endomorphisms. From Theorem 5.1 we know that

$$S \rtimes^\lambda T = \{(a, t) \in S \times T : t^+ \cdot a = a\}$$

is a left restriction semigroup where

$$(a, t)(b, u) = (((tu)^+ \cdot a)(t \cdot b), tu)$$

and $(a, t)^+ = (a^+, t^+)$. If, in addition, S and T are restriction, we suppose that T acts doubly on S via restriction endomorphisms. From Theorem 5.6, $S \rtimes^\lambda T$ is then restriction, where $(a, t)^* = (a^* \circ t, t^*)$.

Let $\theta : S \rtimes^\lambda T \to T$ be defined by $(a, t)\theta = t$.

Lemma 7.1 *The map θ is a (left) restriction onto morphism. For each $e \in E_T$ let*

$$K_e = K_e(S \rtimes^\lambda T) = \{(a, e) : e \cdot a = a\} = e\theta^{-1}$$

and let

$$K = K(S \rtimes^\lambda T) = \bigcup_{e \in E_T} K_e = E_T \theta^{-1}.$$

Then K_e and K are (left) restriction subsemigroups of $S \rtimes^\lambda T$.

Proof In view of the given operations in $S \bowtie^{\lambda} T$, we need only show that θ is onto. Let $t \in T$. Then for any $a \in S$ we have $(t^+ \cdot a, t) \in S \bowtie^{\lambda} T$, and $(t^+ \cdot a, t)\theta = t$. $\square$

The semigroup K plays an important role in what follows. Essentially, if forms a 'core' of $S \bowtie^{\lambda} T$ on which T acts by restriction endomorphisms - on the left in the one-sided case and on both sides and doubly in the two-sided case.

Lemma 7.2 *The left restriction semigroup T acts on the left of $S \bowtie^{\lambda} T$ preserving $^+$ via*

$$t \cdot (a, u) = (t \cdot a, (tu)^+)$$

such that $T \cdot (S \bowtie^{\lambda} T) \subseteq K$, and the restriction of the action of T to K is by left restriction endomorphisms.

If S and T are restriction, then in addition T acts on the right of $S \bowtie^{\lambda} T$ preserving * via

$$(a, u) \circ t = (a \circ ut, (ut)^*).$$

In this case we further have that $(S \bowtie^{\lambda} T) \circ T \subseteq K$, the restriction of the actions of T to K is by restriction endomorphisms, and T acts doubly on K.

Proof To see that the actions are well-defined, let $(a, u) \in S \bowtie^{\lambda} T$ and $t \in T$. As T is left restriction,

$$(tu)^+ \cdot (t \cdot a) = ((tu)^+ t) \cdot a = tu^+ \cdot a = t \cdot (u^+ \cdot a) = t \cdot a.$$

In the two sided case,

$$(ut)^* \cdot (a \circ ut) = (a \circ ut) \circ (ut)^* = a \circ (ut(ut)^*) = a \circ ut.$$

It is easy to check that T acts on $S \bowtie^{\lambda} T$ with image contained in K and that the left and right action of T preserves $^+$ and *, respectively. Straightforward (but long) calculations give that the restriction of these actions to K are by restriction morphisms.

In the two-sided case, we now check that T acts doubly on K. To this end, let $(a, e) \in K$, $t \in T$ and $f \in E_T$. It is easy to see that

$$f \cdot (a, e) = (f \cdot a, fe) = (a \circ ef, ef) = (a, e) \circ f.$$

Further,

$$\begin{aligned} t \cdot ((a, e) \circ t) &= t \cdot (a \circ et, (et)^*) &&= t \cdot (a \circ t, (et)^*) \\ &= (t \cdot (a \circ t), (t(et)^*)^+) &&= (t^+ \cdot a, (et)^+) \\ &= (t^+ \cdot a, et^+) &&= t^+ \cdot (a, e). \end{aligned}$$

On the other hand,

$$\begin{aligned} (t \cdot (a, e)) \circ t &= (t \cdot a, (te)^+) \circ t &&= ((t \cdot a) \circ (te)^+ t, ((te)^+ t)^*) \\ &= ((t \cdot a) \circ te, (te)^*) &&= \big(((t \cdot a) \circ t) \circ e, t^* e\big) \\ &= (a \circ t^* e, t^* e) &&= (t^* \cdot a, t^* e) \\ &= t^* \cdot (a, e). \end{aligned}$$

$\square$

In fact the relation between K and the left action of T is very rich. It is clear that K is a semilattice E_T of the (left) restriction semigroups K_e, $e \in E_T$. Moreover, it is a strong semilattice with connecting morphisms $\psi_{e,f}$ for $e \geqslant f$ given by $(a, e)\psi_{e,f} = f \cdot (a, e) = (f \cdot a, f)$.

We now define $\psi : K \to S$ by $(a, e)\psi = a$ and for any $\alpha \in S \rtimes^{\lambda} T$ we let $\overline{\alpha} \in K$ be given by

$$\overline{\alpha} = (\alpha\theta)^{+} \cdot \alpha.$$

Clearly if $\overline{\alpha}\psi = \overline{\beta}\psi$ and $\alpha\theta = \beta\theta$, then $\alpha = \beta$. The next lemma is easily verified.

Lemma 7.3 *The restriction of ψ to any K_e, $e \in E_T$ is a (left) restriction morphism. Further, for any $\alpha, \beta \in S \rtimes^{\lambda} T$, $\gamma, \delta \in K$ and $t \in T$ we have*:

(S1) $\overline{\alpha}\theta = (\alpha\theta)^{+}$ *and* $(\delta\theta) \cdot \delta = \delta$;
(S2) $\gamma\psi = \delta\psi$ *implies that* $(t \cdot \gamma)\psi = (t \cdot \delta)\psi$ *and* $\alpha\theta = \beta\theta$ *implies that* $(t \cdot \alpha)\theta = (t \cdot \beta)\theta$;
(S3) $\overline{\alpha\beta}\psi = \big(((\alpha\beta)\theta)^{+} \cdot \overline{\alpha}\big)\psi\big((\alpha\theta) \cdot \overline{\beta}\big)\psi$;
(S4) *if* $((\beta\theta)^{+} \cdot \overline{\alpha})\psi = \overline{\alpha}\psi$, *then there exists* $\gamma \in S \rtimes^{\lambda} T$ *such that* $\gamma\theta = \beta\theta$ *and* $\overline{\gamma}\psi = \overline{\alpha}\psi$;
and in the two sided case:
(S5) $\gamma\psi = \delta\psi$ *implies that* $(\gamma \circ t)\psi = (\delta \circ t)\psi$ *and* $\alpha\theta = \beta\theta$ *implies that* $(\alpha \circ t)\theta = (\beta \circ t)\theta$;
(S6) $\overline{\alpha}^{*} = \alpha\theta \cdot \alpha^{*}$.

We now aim to prove that the conditions extracted from $S \rtimes^{\lambda} T$ are enough to determine its structure. For simplicity, we do this only in a special case, where Condition (U) holds:

$$\text{for all } a, b \in S \quad \text{there exists } e \in E_T \quad \text{such that} \quad e \cdot a = a \quad \text{and} \quad e \cdot b = b. \tag{U}$$

Note that (U) always holds if T is a monoid acting monoidally, or if E_T is a join semilattice, and every $a \in S$ is such that $e \cdot a = a$ for some $e \in E_T$. In turn, (U) gives that ψ is a surjection.

We begin with the one-sided case.

Theorem 7.4 *Let $\mathcal{Z}$, S and T be left restriction semigroups and suppose there is a left restriction epimorphism $\theta : \mathcal{Z} \to T$. Let $K = K(\mathcal{Z}) = E_T\theta^{-1}$ and suppose there is a surjection $\psi : K \to S$. Suppose also that T acts on the left of $\mathcal{Z}$ preserving $^{+}$ such that $T \cdot \mathcal{Z} \subseteq K$ and T acts by left restriction endomorphisms on K. Further, we assume the restriction of ψ to any $K_e = K_e(\mathcal{Z}) = e\theta^{-1}$, $e \in E_T$, is a left restriction morphism. For any $\alpha \in \mathcal{Z}$ we let $\overline{\alpha} \in K$ be defined by*

$$\overline{\alpha} = (\alpha\theta)^{+} \cdot \alpha$$

We suppose that (S1)–(S4) hold and if $\overline{\alpha}\psi = \overline{\beta}\psi$ and $\alpha\theta = \beta\theta$, then $\alpha = \beta$. Finally, we assume

(U)′: for any $\alpha, \beta \in K$ there exists $e \in E_T$ and $\alpha', \beta' \in K_e$ with $\alpha\psi = \alpha'\psi$ and $\beta\psi = \beta'\psi$.

Then T acts on S on the left by left restriction endomorphisms, satisfying (U), such that $\mathcal{Z}$ is isomorphic to $S \rtimes^{\lambda} T$ via $\alpha\varphi = (\overline{\alpha}\psi, \alpha\theta)$.

Conversely, if S and T are left restriction semigroups such that T acts on S by (left) restriction endomorphisms satisfying (U), then with $\mathcal{Z} = S \rtimes^{\lambda} T$, we have that θ, ψ and the action of T on $S \rtimes^{\lambda} T$ exists, satisfying the listed conditions.

Proof We have already shown the converse with the exception of the easily seen fact that (U) gives us (U)′.

For the direct part, we assume the given conditions hold for $\mathcal{Z}$, T and S. From (S1) we have $\overline{\alpha} = \alpha$ for any $\alpha \in K$. As the restriction of ψ to each K_e is a left restriction morphism, it follows that ψ preserves $^{+}$.

Define a map $T \times S \to S$ by

$$t \cdot (\alpha\psi) = (t \cdot \alpha)\psi.$$

Since ψ is onto we have that the map is everywhere defined and it is well defined by (S2); as T acts on K, it is easily seen to also be an action. Moreover, for any $t \in T$ and $\alpha \in K$ we have

$$t \cdot (\alpha\psi)^+ = t \cdot (\alpha^+\psi) = (t \cdot \alpha^+)\psi = (t \cdot \alpha)^+\psi = (t \cdot \alpha)\psi^+ = (t \cdot \alpha\psi)^+,$$

so the action preserves $^+$. Suppose now that also $\beta \in K$, then we use (U)$'$ to choose $\alpha', \beta' \in K$ and $e \in E_T$ such that $\alpha\psi = \alpha'\psi$, $\beta\psi = \beta'\psi$ and $\alpha', \beta' \in K_e$. Since $\alpha'\theta = \beta'\theta$, it follows from (S2) that $(t \cdot \alpha')\theta = (t \cdot \beta')\theta = f \in E_T$. Using the fact that $\psi|_{K_e}$ and $\psi|_{K_f}$ are morphisms, we calculate:

$$\begin{aligned} t \cdot (\alpha\psi\, \beta\psi) &= t \cdot (\alpha'\psi\, \beta'\psi) && = t \cdot (\alpha'\beta')\psi \\ &= (t \cdot \alpha'\beta')\psi && = \big((t \cdot \alpha')(t \cdot \beta')\big)\psi \\ &= (t \cdot \alpha')\psi(t \cdot \beta')\psi && = (t \cdot \alpha'\psi)(t \cdot \beta'\psi) \\ &= (t \cdot \alpha\psi)(t \cdot \beta\psi). \end{aligned}$$

Thus we have shown that T acts on S by left restriction endomorphisms.

If $\alpha\psi, \beta\psi \in S$, then from (U)$'$ we can assume $\alpha, \beta \in K_e$ for some $e \in E_T$. It follows that $e \cdot \alpha\psi = (e \cdot \alpha)\psi = \big((\alpha\theta)^+ \cdot \alpha\big)\psi = \alpha\psi$ and similarly, $e \cdot \beta\psi = \beta\psi$, so that (U) holds.

With φ defined as above we have $\alpha\varphi = (\overline{\alpha}\psi, \alpha\theta)$. For any $\alpha \in \mathcal{Z}$ we have

$$(\alpha\theta)^+ \cdot \overline{\alpha}\psi = ((\alpha\theta)^+ \cdot \overline{\alpha})\psi = \overline{\alpha}\psi,$$

so that φ is well defined and is an injection into $S \rtimes^\lambda T$ under the given conditions. Since the action of T and θ both preserve $^+$ it is easy to see that $\overline{\alpha^+} = \overline{\alpha}^+$ and then since both ψ and θ preserve $^+$ we deduce that so does φ.

Condition (S3) immediately gives that φ also preserves multiplication and hence is an embedding. Finally, (S4) guarantees that φ is onto. □

We now consider the two sided version of Theorem 7.4.

Theorem 7.5 *Let $\mathcal{Z}$, S and T be restriction semigroups such that, regarded as left restriction semigroups, the conditions of Theorem 7.4 hold. Suppose in addition that T acts on the right of $\mathcal{Z}$ preserving * such that $\mathcal{Z} \circ T \subseteq K$, T acts doubly on K by restriction endomorphisms and (S5) and (S6) hold.*

Then T acts doubly on S by restriction endomorphisms, satisfying (U), such that $\mathcal{Z}$ is isomorphic as a restriction semigroup to $S \rtimes^\lambda T$ via $\alpha\varphi = (\overline{\alpha}\psi, \alpha\theta)$.

Conversely, if S and T are restriction semigroups such that T acts doubly on S by restriction endomorphisms satisfying (U), then with $\mathcal{Z} = S \rtimes^\lambda T$, we have that θ, ψ and the actions of T on $S \rtimes^\lambda T$ exist, satisfying the given conditions.

Proof We need only prove the direct part. We assume the given conditions hold for $\mathcal{Z}$, T and S and let the left action of T on S be as in Theorem 7.4. Define a map $S \times T \to S$ by

$$(\alpha\psi) \circ t = (\alpha \circ t)\psi.$$

Since ψ is a surjection we have that the map is everywhere defined and it is well defined by (S5); as T acts on $\mathcal{Z}$, it is easily seen to also be an action. That the right action preserves multiplication in S is similar to the proof for the left action in Theorem 7.4. It is then easy to show, as T acts doubly on K by restriction endomorphisms, that the same is true for the actions of T on S.

Finally we must show that φ preserves *. We use (S6) to show that

$$\overline{\alpha}^* \circ \alpha\theta = (\alpha\theta \cdot \alpha^*) \circ \alpha\theta = (\alpha\theta)^* \cdot \alpha^* = (\alpha^*\theta) \cdot \alpha^* = \alpha^* = \overline{\alpha^*}.$$

We then have

$$\begin{aligned}\alpha\varphi^* &= (\overline{\alpha}\psi, \alpha\theta)^* = (\overline{\alpha}\psi^* \circ \alpha\theta, (\alpha\theta)^*)\\ &= (\overline{\alpha}^*\psi \circ \alpha\theta, \alpha^*\theta) = ((\overline{\alpha}^* \circ \alpha\theta)\psi, \alpha^*\theta)(\overline{\alpha^*}\psi, \alpha^*\theta) = \alpha^*\varphi,\end{aligned}$$

as required. □

Theorems 7.4 and 7.5 are, as they stand, rather clumsy. The reader might hope that we would have a characterisation of $S \rtimes^\lambda T$ involving an internal notion that replaces the use of ψ. Certainly, for any $\alpha, \beta \in S \rtimes^\lambda T$ we have that $\alpha = \beta$ if and only if $\overline{\alpha} = \overline{\beta}$ and $\alpha\theta = \beta\theta$. We content ourselves here with the following observation in the special case that E_T is up-directed. In this case, for any $e \geqslant f$ we may define $\theta_{f,e} : K_f \to K_e$ by $(a, f)\theta_{f,e} = (a, e)$, noting this is a well-defined morphism, and the collection of the restriction monoids K_e, together with the morphisms $\theta_{f,e}, e \geqslant f$, form a directed system. The direct limit of this system $K/\sim$ is isomorphic to S in a natural way. Thus conditions on ψ could be replaced by conditions on the direct limit, in particular, $\overline{\alpha}\psi = \overline{\beta}\psi$ if and only if $\overline{\alpha} \sim \overline{\beta}$, so that $\alpha = \beta$ if and only if $\overline{\alpha} \sim \overline{\beta}$ and $\alpha\theta = \beta\theta$.

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