Equations defining the polynomial closure of a lattice of languages

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Joint work with

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Univ. of York - December 6, 2010

Topics

- Regular languages

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- Semigroup equations

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- Varieties

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- Polynomial closure of a lattice of languages closed under quotients

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Star:

$$L \longmapsto L^* = \{u_1 \cdots u_n \mid u_1, \ldots, u_n \in L,\ n \in \mathbb{N}_0\}$$

the submonoid of A^* generated by L

Operations on Languages

Quotients ($a \in A$):

$$L \longmapsto a^{-1}L = \{u \mid au \in L\}$$
$$L \longmapsto La^{-1} = \{u \mid ua \in L\}$$

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$$(ab + ba)^* bbaabb(bba)^* + \left((aaa + bbb)^* + a^5\right)^* b$$

Finite automaton

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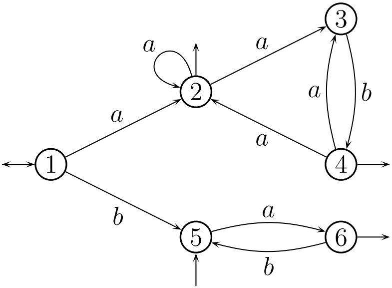
Automaton $\mathcal{A}$:



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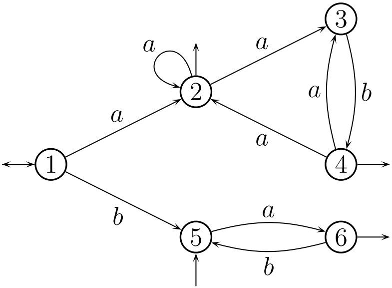

Words recognized by $\mathcal{A}$:

$$1,\ a,\ aa,\ a^3,\ a^4,\ a^2b,\ a^4baba^6b,\ ba,\ (ba)^2,\ aba,\ (ab)^2a,\ \ldots$$

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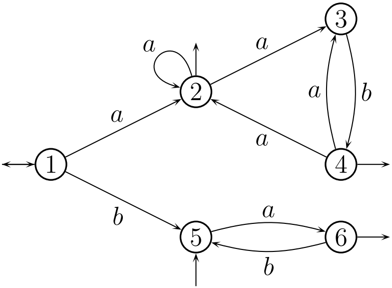

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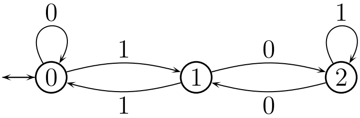

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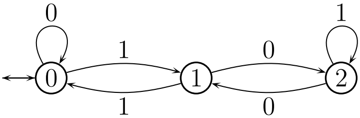

Words recognized by $\mathcal{A}$ are precisely the words that represent the multiples of 3 on base 2, for instance 0, 00, 11, 0011, 1001, 1000110100.

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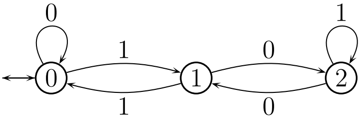

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Is there an algorithm to test whether a language belongs to $\mathrm{SF}(A^*)$?

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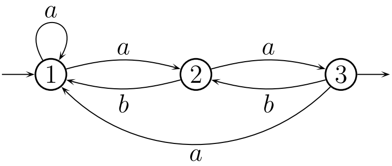
Transition monoid

Alphabet $A = \{a, b\}$

Automaton $\mathcal{A}$:



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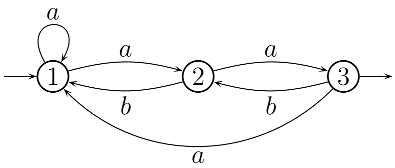

The transitions of $\mathcal{A}$ can be defined by the following two binary relations:

$$a \longmapsto \overline{a} = \{(1,1), (1,2), (2,3), (3,1)\}$$
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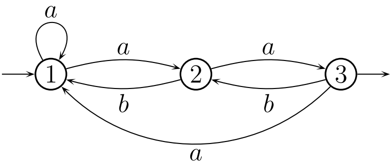

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For words, for instance:

$$babba \longmapsto \overline{babba} = \{(0,1),\ (1,0),\ (2,2)\}$$

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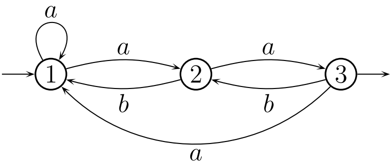
Alphabet $A = \{a, b\}$

Automaton $\mathcal{A}$:



The transitions of $\mathcal{A}$ can be defined by the following two binary relations:

$$a \longmapsto \bar{a} = \{(1,1), (1,2), (2,3), (3,1)\}$$
$$b \longmapsto \bar{b} = \{(2,1), (3,2)\}$$

For words, for instance:

$$babba \longmapsto \overline{babba} = \{(0,1), (1,0), (2,2)\}$$
$$= \bar{b} \circ \bar{a} \circ \bar{b} \circ \bar{b} \circ \bar{a}$$

Transition monoid

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Transition monoid of $\mathcal{A}$: $M(\mathcal{A}) = \{\overline{u} \mid u \in A^*\}$ with composition.
We have the morphism

$$\begin{array}{rccc} \varphi\colon & A^* & \longrightarrow & \big(M(\mathcal{A}), \circ\big) \\ & u & \longmapsto & \overline{u} \end{array}$$

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$$u \in L \iff (u)\varphi \in P$$

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For $L \subseteq A^*$, TFAE:

1. *L is recognized by a finite automaton, i.e. L is recognizable.*
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M recognizes $L \iff M(L)$ is homomorphic image of a submonoid of M.

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The answer is Yes.

Theorem (Schützenberger)

For $L \subseteq A^*$, TFAE:

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3. *$M(L)$ is finite and $\underbrace{\text{aperiodic}}_{\text{its subgroups are trivial}}$.*

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$[\Sigma]$ – class of all monoids that satisfy all identities of Σ.

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Then $\mathcal{V}$ is a variety of languages.

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How to caracterize the **M**-varieties by identities?

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with the conventions $\min \emptyset = -\infty$ and $2^{-\infty} = 0$.

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- $d(u,v) = 0$ if and only if $u = v$.
- $d(u,v) = d(v,u)$.
- $d(u,w) \leq \max\{d(u,v), d(v,w)\}$.
- $d(uu', vv') \leq \max\{d(u,v), d(u',v')\}$.

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The multiplication on A^* induces, in a natural way, an associative multiplication on $\widehat{A^*}$, which is continuous.

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Let $u \in A^*$. The sequence $(u^{n!})_n$ is a Cauchy sequence in A^*.

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Since M is finite, there exists k s.t. $(u\hat{\varphi})^k = e$, an idempotent.
It follows that if $n \geq k$, then $(u\hat{\varphi})^{n!} = e$, and so $\lim(u\hat{\varphi})^{n!} = e$, the idempotent power of $u\hat{\varphi}$.

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Examples:

M satisfies $xy = yx$ if and only if $\forall s, t \in M$, $st = ts$.

A finite *semigroup* S satisfies $x^\omega y x^\omega = x^\omega$ if and only if $\forall s \in S, e \in E(S)$, $ese = e$.

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$\mathbf{A} = [\![x^\omega = x^{\omega+1}]\!]$ – finite aperiodic monoids.

$\mathbf{LI} = [\![x^\omega y x^\omega = x^\omega]\!]$ – finite locally trivial *semigroups*.

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How to characterize these classes algebraically?

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Ordered monoid $(M, \leq)$: monoid M equipped with a partial order $\leq$ compatible with the product:

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How to characterize algebraically the classes $\mathcal{V}$ satisfying the following?

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Proposition (Gehrke, Grigorieff, Pin)

Let $L \subseteq A^*$ regular and $u \in \widehat{A^*}$. TFAE:

1. $u \in \overline{L}$.
2. $\hat{\varphi}(u) \in \varphi(L)$, *for every morphism* $\varphi\colon A^* \to M$, *where M is a finite monoid.*
3. $\hat{\eta}(u) \in \eta(L)$, *where* $\eta\colon A^* \to M(L)$ *is the syntactic morphism of L.*

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Notice that, by the previous proposition,

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\hat{\eta}(u) \leq \hat{\eta}(v) \quad &\iff \quad \forall s, t \in M(L) \left(s\hat{\eta}(v)t \in \eta(L) \Rightarrow s\hat{\eta}(u)t \in \eta(L) \right) \\
&\iff \quad \forall x, y \in A^* \left(\hat{\eta}(xvy) \in \eta(L) \Rightarrow \hat{\eta}(xuy) \in \eta(L) \right)
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How to characterize algebraically the classes $\mathcal{V}$ satisfying the following?

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Theorem (Gehrke, Grigorieff, Pin)

A set $\mathcal{L}$ of languages of A^* is a lattice of languages closed under quotients if and only if, for some set Σ of equations of the form $u \leq v$, with $u, v \in \widehat{A^*}$, $\mathcal{L}$ is the set of the languages of A^* that satisfy all equations of Σ.

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$\Sigma(\mathcal{L})$: the set of equations of the form $x^\omega y x^\omega \leq x^\omega$, where $x, y \in \widehat{A^*}$ are such that the equations $x = x^2$ and $y \leq x$ are satisfied by $\mathcal{L}$.

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