Tropical Representations of Plactic Monoids

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(mostly) joint with Marianne Johnson

York, 3 July 2019

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In fact $x \oplus y$ *is either* x *or* y*.*

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Tropical geometry is (roughly!) algebraic geometry where the base field is replaced by the tropical semiring.

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- (Mostly Enumerative) Algebraic Geometry
- **Semigroup Theory** (carrier for representations)

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Philosophy

The algebra of $M_n(\mathbb{T})$ mirrors the geometry of **tropical convex sets**.

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- ... $AB = A$ if and only if it is a left-zero semigroup.

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Theorem (Izhakian & Merlet 2018, building on ideas of Shitov)

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See `arXiv:1904.06094` for more details.

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(Entries in each column strictly decreasing, entries in each row weakly increasing, row lengths weakly increasing.)

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Schensted's algorithm (1961) constructs tableaux from words.

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- Corresponding answer is "yes" for right patience sorting (= Bell) monoids and "no" for left patience sorting monoids (Cain, Malheiro & F. M. Silva 2018)
- Corresponding answer is "yes" for hypoplactic, sylvester, Baxter, stalactic and taiga monoids (Cain & Malheiro 2018)
- Again conjectured "yes" for all finite n (Cain & Malheiro 2018)
- Recent preprint of Okniński on $n \geq 4$ withdrawn.

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Both the above representations generalise naturally to higher rank but do **not** remain faithful. e.g. in $\mathbb{P}_4$ they do not separate:

4	4		
2	3	4	
1	2	3	3

and

4			
2	3	4	4
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Remark

"d" from the previous slide is the longest chain length in this partial order.

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The Thing You Expect Me To Say

The map $\rho_n : \mathbb{P}_n \to UT_{2^n}(\mathbb{T})$ is a faithful representation of $\mathbb{P}_n$.

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Theorem (Daviaud, Johnson & K. 2018)

Any chain-structured tropical matrix semigroup of chain length d satisfies the same identities as $UT_d(\mathbb{T})$.

Further details

- M. Johnson & M. Kambites, *Tropical representations of plactic monoids*, `arXiv:1906.03991`

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- M. Kambites, *Free objects in triangular matrix varieties and quiver algebras over semirings*, `arXiv:1904.06094`
- L. Daviaud, M. Johnson & M. Kambites, *Identities in upper triangular tropical matrix semigroups and the bicyclic monoid*, J. Algebra Vol.501 pp.503–525 (2018).