MODEL COMPANIONS OF S-SYSTEMS

By VICTORIA GOULD

[Received 9 January 1986]

1. Introduction

THIS paper is concerned with the model theory of S-systems over a monoid S. An S-system is simply a set upon which the monoid S acts. It is the analogue in semigroup theory of an R-module in ring theory. The specific aspects of model theory which we consider are definability problems for classes of S-systems and the question of when a theory of S-systems has a model companion and how can the model companion be described when it does exist. The corresponding problems for modules have been considered by Eklof and Sabbagh [5] and Bouscaren [2], [3].

The relevant algebraic definitions are given in Section 2, but we assume a basic knowledge of semigroup theory and model theory including the construction of ultraproducts. As far as possible we follow the notation and terminology of [9] for semigroup theory and [4] for model theory. We adopt the convention that an ordinal is the set of smaller ordinals.

To a given class $\mathscr{C}$ of algebraic structures there corresponds at least one first order language L. One can then ask whether a property P, defined for members of $\mathscr{C}$, is expressible in the language L. In other words, is there a set of sentences Π such that a member $\mathscr{M}$ of $\mathscr{C}$ has property P if and only if all sentences in Π are true in $\mathscr{M}$. If the set Π exists we say that P is definable in L. Further, $\mathscr{D}$ is *axiomatisable* in L and Π *axiomatises* $\mathscr{D}$, where $\mathscr{D}$ is the subclass of $\mathscr{C}$ whose members have property P.

In Sections 3 and 4 we consider the question of the axiomatisability of classes of α-injective S-systems, for various cardinals α. The answers are given in terms of coherency properties of S. Theorem 3 is concerned with the case $1 < \alpha \leq \aleph_0$, when a direct proof involving ultraproducts suffices. If $\alpha = \aleph_0$ then the result says that the class of weakly f-injective S-systems is axiomatisable if and only if S is weakly coherent. This, together with some model theory, gives that the class of weakly injective S-systems ($\gamma(S)$-injective S-systems) is axiomatisable if and only if S is weakly coherent and satisfies the ascending chain condition on right ideals. Here $\gamma(S)$ is any cardinal such that every right ideal of S has a generating set of fewer than $\gamma(S)$ elements. The work of these sections is analogous to that of [5], which considers the corresponding case of modules over a ring. We note that for modules, the notions of injectivity and weak injectivity coincide, whereas this is not true of S-systems [1].

An S-system A is absolutely pure if every finite consistent system of equations with constants from A, has a solution in A. The proof of Theorem 3 relies on some results of [7] which connect the property of α-injectivity with the existence of solutions of certain restricted systems of equations. In Section 5 we use the techniques of Theorem 3 to characterise those monoids for which the class of absolutely pure S-systems is axiomatisable.

Important notions in model theory are those of the model completion and the model companion of a given first order theory. In our case, where the theory is the theory of S-systems Th(S), for some monoid S, these concepts coincide. Further, a general result tells us that the model completion of Th(S) exists if and only if the class $\mathscr{E}_S$ of existentially closed S-systems is axiomatisable.

If S is coherent we can axiomatise by a given set of sentences Π the class $\mathscr{E}(2)$ of S-systems that are 'one variable' existentially closed, that is, $A \in \mathscr{E}(2)$ if and only if every finite consistent system of equations and inequations in one variable, with constants from A, has a solution in A. In the last section we use some general model theoretic results to show that Π is in fact the model completion of Th(S). It follows from this that S satisfies a notion of coherency which seemed superficially to be stronger than the first.

I would like to thank Dr. J. B. Fountain for teaching me most of the model theory I know and for much helpful advice with regard to this paper. I am also grateful to Dr. M. Prest for explaining some points that are folklore among model theorists, but which are very difficult to track down in the literature.

2. Definitions and preliminary results

Throughout this paper S will denote a given monoid, that is, a semigroup with an identity. A set A is a *right S-system* if there is a map $\phi: A \times S \rightarrow A$ satisfying

$$\phi(a, 1) = a$$

and

$$\phi(a, st) = \phi(\phi(a, s), t)$$

for any element a of A and any elements s, t of S. For $\phi(a, s)$ we write as and we refer to right S-systems simply as *S-systems*. One has the obvious definitions of S-subsystem and S-homomorphism.

For a monoid S we denote by L_S the first order language with equality, which has no constant or relation symbols and which has a unary function symbol ρ_s for each element s of S. We write xs for $\rho_s(x)$ and we regard S-systems as L_S-structures in the obvious way.

For any elements s, t of S, we denote by $\psi_{s,t}$ the sentence

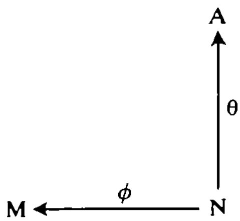
$$(\forall x)(x(st) = (xs)t)$$

of L_S. Put

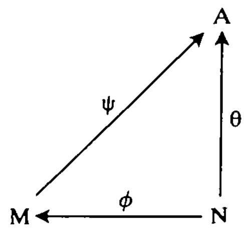
$$\Sigma_S = \{(\forall x)(x1 = x)\} \cup \{\psi_{s,t}\colon\ s, t \in S\}.$$

Clearly an L_S-structure $\mathcal{M}$ is an S-system if and only if $\mathcal{M}$ is a model of Σ_S. Thus Σ_S axiomatises the class of S-systems in the language L_S. It follows from the Completeness Theorem that if Th(S) is the theory of S-systems, that is, the set of sentences of L_S true in all S-systems, then Th(S) is the deductive closure of Σ_S.

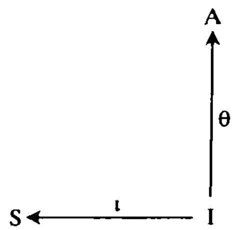
An S-system A is *injective* if given any diagram of S-systems and S-homomorphisms



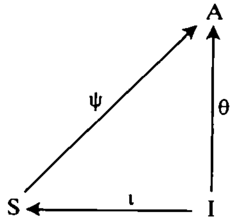
where $\phi\colon N \to M$ is an injection, there exists an S-homomorphism $\psi\colon M \to A$ such that



is commutative. By imposing conditions on M and N we weaken this definition to obtain the concept of α-injectivity, as follows. Let α be any cardinal strictly greater than 1. Then as S-system A is *α-injective* if given any diagram of the form



where I is a right ideal of S with a generating set of fewer than α elements, ι: $I \to S$ is the inclusion mapping and θ: $I \to A$ is an S-homomorphism, then there exists an S-homomorphism ψ: $S \to A$ such that



is commutative.

It is clear that an injective S-system is α-injective for any α and an α-injective S-system is β-injective for any cardinal β such that $1 \leq \beta \leq \alpha$. Let $\gamma = \gamma(S)$ be a cardinal such that every right ideal of S has a generating set of fewer than γ elements. The usual terminology for γ-injective is *weakly injective*. Further, we write *weakly f-injective* for $\aleph_0$-injective and *weakly p-injective* for 2-injective.

Given a system of equations and inequations Σ with constants from an S-system A, then any equation in Σ has one of the following three forms

$$\text{(I) } xs = xt \qquad \text{(II) } xs = yt \qquad \text{(III) } xs = a$$

where $s, t \in S$ and $a \in A$. Similarly, any inequation in Σ has one of the forms

$$\text{(I)}' \ xs \neq xt \qquad \text{(II)}' \ xs \neq yt \qquad \text{(III)}' \ xs \neq a.$$

If Σ has a solution in some S-system containing A, then Σ is *consistent*. If A is an S-subsystem of B, then A is *pure* (*existentially closed*) in B if every finite system of equations (and inequations) with constants from A, which is soluble in B, has a solution in A. An *absolutely pure* (*existentially closed*) S-system is one which is pure (existentially closed) in every S-system containing it. Equivalently, an S-system A is absolutely pure (existentially closed) if every finite consistent system of equations (and inequations), with constants from A, has a solution in A.

Let Σ be a system of equations with constants from an S-system A. If all the equations in Σ are of the form (III) and the same variable appears in each, then Σ is said to be an *α-system over A*, where α is any cardinal larger than that of Σ. Thus Σ is an α-system over A if and only if Σ has the form

$$\Sigma = \{xs_j = a_j \colon \ j \in J, \ |J| < \alpha, \ s_j \in S, \ a_j \in A\}.$$

Our interest in α-systems stems from the following proposition.

PROPOSITION 2.1. [7]. *Let $\alpha > 1$ be a cardinal and let A be an S-system. Then every consistent α-system over A has a solution in A if and only if A is α-injective.*

Essential to the proofs of this paper are criteria for consistency of various systems of equations over an S-system. The most general result of this nature which we shall need is provided in Lemma 2.3 below. This can be specialised as necessary, to apply to more restricted systems of equations.

At this point we establish some notation.

For an S-system A and a subset H of $A \times A$, then by $\rho(H)$ we denote the congruence generated by H, that is, the smallest congruence relation ν over A such that $H \subseteq \nu$.

LEMMA 2.2 [13]. *The ordered pair (a, b) is in $\rho(H)$ if and only if $a = b$ or there exists a natural number n and a sequence*

$$a = c_1t_1, \qquad d_1t_1 = c_2t_2, \ldots, d_{n-1}t_{n-1} = c_nt_n, \qquad d_nt_n = b,$$

where $t_1, \ldots, t_n$ are elements of S and for each $i \in \{1, \ldots, n\}$ either (c_i, d_i) or (d_i, c_i) is in H.

A sequence as in Lemma 2.2 will be referred to as a *$\rho(H)$-sequence of length n*. For any congruence ρ on A the set of congruence classes of ρ can be made into an S-system, with the obvious action of S. This S-system is written as A/ρ. The ρ-class of an element a of A is denoted by $a\rho$ or $[a]_\rho$ or simply $[a]$ where there is no ambiguity.

Let $\{x_i: i \in \mathbb{N}\}$ and $\{x_i^j: i, j \in \mathbb{N}\}$ be sets of variables. Let r, m be natural numbers and let

$$f: \{1, \ldots, r\} \to \mathbb{N}$$
$$g, h: \{1, \ldots, m\} \to \mathbb{N}$$

be functions such that there exists a natural number n with

$$\{1, \ldots, n\} = \{1, \ldots, r\} \cup g(\{1, \ldots, m\}) \cup h(\{1, \ldots, m\}).$$

We define a *standard form* to be a set of equations

$$X = \{x_ip_i^j = x_i^j: 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant f(i)\}$$
$$\cup \{x_{g(k)}s_k = x_{h(k)}t_k: 1 \leqslant k \leqslant m\}$$

where p_i^j, s_k, t_k are elements of S. If A is an S-system, then a system Σ of equations over A which results from substituting elements a_i^j of A for the variables x_i^j in X is said to be *in standard form*. We introduce the concept of standard form to provide a notation for labelling a system of equations. To avoid excessively complicated notation, we shall, however, sometimes be rather casual about a system of equations being in standard form. For example, if Σ is an α-system over an S-system A, then writing

$$\Sigma = \{x_1 p_1^j = a_1^j\colon\ 1 \leq j \leq f(1)\}$$

we still consider Σ to be in standard form. In this case the set H defined in Lemma 2.3 below is the empty set and so the corresponding congruence $\rho(H)$ is the identity congruence.

LEMMA 2.3. *Let A be an S-system and let Σ be a system of equations over A in standard form. Let $F = z_1S \cup \cdots \cup z_nS$ be the free S-system on $z_1, \ldots, z_n$ and define the subset H of $F \times F$ by*

$$H = \{(z_{g(k)}s_k,\ z_{h(k)}t_k)\colon\ 1 \leq k \leq m\}.$$

Then Σ is consistent if and only if for all $s, s' \in S$, $i, i' \in \{1, \ldots, r\}$, $j \in \{1, \ldots, f(i)\}$ and $j' \in \{1, \ldots, f(i')\}$

$$z_i p_i^j s\, \rho(H)\, z_{i'} p_{i'}^{j'} s' \Rightarrow a_i^j s = a_{i'}^{j'} s'.$$

Proof. Suppose first that Σ is consistent. If $s, s' \in S$ and $z_i p_i^j s\, \rho(H)\, z_{i'} p_{i'}^{j'} s'$, then a straightforward argument using Lemma 2.2 and the fact that Σ has a solution in an S-system containing A, gives that $a_i^j s = a_{i'}^{j'} s'$.

Conversely, suppose that for all elements s, s' of S

$$z_i p_i^j s\, \rho(H)\, z_{i'} p_{i'}^{j'} s' \Rightarrow a_i^j s = a_{i'}^{j'} s'.$$

Put $B = A \cup F$. Then B is an S-system, with the obvious action. Define a subset M of $B \times B$ by

$$M = \{(z_i p_i^j, a_i^j)\colon\ 1 \leq i \leq r, 1 \leq j \leq f(i)\} \cup H.$$

If a, b are many elements of A such that $a\rho(M)b$, then $a = b$ or a, b are connected via a $\rho(M)$-sequence. In the latter case a straightforward argument using induction on the length of $\rho(M)$-sequences again gives that $a = b$. Hence we may identify any $a \in A$ with $[a] \in B/\rho(M)$ and consider A as an S-subsystem of $B/\rho(M)$. Clearly $([z_1], \ldots, [z_n])$ is a solution of Σ in $B/\rho(M)$ and so Σ is consistent.

COROLLARY 2.4. *Let A be an S-system and let*

$$\Sigma = \{xs_j = a_j\colon\ 1 \leq j \leq n,\ s_j \in S,\ a_j \in A\}$$

be an $(n+1)$-system over A. Then Σ is consistent if and only if for all elements h, h' of S and for all $i, j \in \{1, \ldots, n\}$,

$$s_i h = s_j h' \Rightarrow a_i h = a_j h'.$$

To prove our results we rely heavily on the use of ultraproducts, and in particular Łoś's theorem.

THEOREM 2.5. (Loś, [4]). *Let L be a first order language and let $\mathscr{C}$ be a class of L-structures. If $\mathscr{C}$ is axiomatisable, then $\mathscr{C}$ is closed under ultraproducts.*

3. Axiomatising α-injective S-systems

For a cardinal α we denote by $\mathscr{I}(\alpha)$ the class of α-injective S-systems.

THEOREM 3. *Let α be countable cardinal. Then the following conditions are equivalent for a monoid S:*

(i) *the class $\mathscr{I}(\alpha)$ is axiomatisable;*
(ii) *any ultraproduct of α-injective S-systems is α-injective;*
(iii) *any ultraproduct of $\gamma(S)$-injective S-systems is α-injective;*
(iv) *for every natural number n less than α, S satisfies the following property (C_n):*
(C_n). The kernel of every S-homomorphism from $F_n = z_1S \cup \cdots \cup z_nS$ to S is finitely generated, where F_n is the free S-system on $z_1, \ldots, z_n$.

Proof. (i) ⇒ (ii). This is a direct application of Theorem 2.5.

(ii) ⇒ (iii). This is true, since a $\gamma(S)$-injective S-system is α-injective for any cardinal α.

(iii) ⇒ (iv). Let ϕ: $F \to S$ be an S-homomorphism where

$$F = z_1S \cup \cdots \cup z_nS$$

is the free S-system with free generators $z_1, \ldots, z_n$ and $n < \alpha$. Suppose that $\operatorname{Ker}\phi$ is not finitely generated. Let β be the smallest cardinal such that $\operatorname{Ker}\phi$ can be generated by a subset of cardinality β. Let $T = \{(b_\tau, b'_\tau)\colon \tau < \beta\}$ be a set of generators for $\operatorname{Ker}\phi$ and for each $v \leq \beta$ define ρ_v to be the congruence on F generated by $\{(b_\tau, b'_\tau)\colon \tau < v\}$. The minimality of β gives that for each $v < \beta$, there is an ordinal $\mu < \beta$ such that $(b_\mu, b'_\mu) \notin \rho_v$. We shall show, however, that our assumptions lead to the existence of an ordinal λ such that $\lambda < \beta$ and $b_\tau \rho_\lambda b'_\tau$ for every τ with $\tau < \beta$.

For each $v < \beta$ we can embed the quotient S-system F/ρ_v in a weakly injective S-system G_v. For example, we may take G_v to be the injective hull of F/ρ_v[1]. Let D be a uniform ultrafilter on β, that is, D is an ultrafilter such that all sets in D have cardinality β. The existence of such a D is shown in [4]. Let $\mathscr{U}$ be the ultraproduct $\prod_{v<\beta} G_v/D$. By assumption, $\mathscr{U}$ is α-injective.

We define a system of equations, Σ, in the single variable x, as follows. For $i \in \{1, \ldots, n\}$, let $\phi(z_i) = s_i$. For each z in F we let $\bar{z}$ denote the equivalence class modulo D of the element of $\prod_{v<\beta} G_v$ whose vth coordinate is $z\rho_v$. Put

$$\Sigma = \{xs_i = \bar{z}_i\colon\ 1 \leq i \leq n\}.$$

To show that Σ is consistent we will make use of Corollary 2.4. Suppose that $s_i u = s_j v$ where $i, j \in \{1, \ldots, n\}$ and $u, v \in S$. Then $\phi(z_i u) = \phi(z_j v)$, that is, $(z_i u, z_j v) \in \operatorname{Ker} \phi$. Thus either $z_i u = z_j v$, in which case we clearly have $\bar{z}_i u = \bar{z}_j v$, or there is a $\rho(T)$-sequence

$$z_i u = c_1 t_1, \qquad d_1 t_1 = c_2 t_2, \ldots, d_k t_k = z_j v.$$

In this case, $\{c_1, d_1\} = \{b_\tau, b'_\tau\}$ for some $\tau < \beta$ and so $c_1 \rho_\sigma d_1$ for any σ with $\tau < \sigma < \beta$. Since ρ_σ is a congruence, we have $c_1 t_1 \rho_\sigma d_1 t_1$. Now D is uniform and it follows that $\{\sigma\colon \tau < \sigma < \beta\} \in D$ and consequently,

$$\bar{z}_i u = \overline{z_i u} = \overline{c_1 t_1} = \overline{d_1 t_1}.$$

If $k = 1$, this clearly gives $\bar{z}_i u = \bar{z}_j v$. If $k > 1$, then induction on the length of $\rho(T)$-sequence gives the same result. Thus by Corollary 2.4, Σ is consistent.

Now, by hypothesis, $\mathcal{U}$ is α-injective and since $n < \alpha$, Proposition 2.1 gives that Σ has a solution in $\mathcal{U}$. Let a_D be such a solution so that for $i \in \{1, \ldots, n\}$, $a_D s_i = \bar{z}_i$. For each i the set $\{v < \beta\colon a(v)s_i = z_i \rho_v\}$ is a member of D and hence so is the intersection J of these sets. Let λ be an element of J. Then $a(\lambda)s_i = z_i \rho_\lambda$ for all i.

We now show that $b_\tau \rho_\lambda b'_\tau$ for any $\tau < \beta$. For some $h, i \in \{1, \ldots, n\}$ and $w, w' \in S$ we have $b_\tau = z_h w$, $b'_\tau = z_i w'$. Thus

$$\begin{aligned} b_\tau \rho_\lambda &= (z_h w)\rho_\lambda = (z_h \rho_\lambda) w = a(\lambda) s_h w = a(\lambda)\phi(b_\tau) \\ &= a(\lambda)\phi(b'_\tau) = a(\lambda) s_i w' = (z_i \rho_\lambda) w' = (z_i w')\rho_\lambda \\ &= b'_\tau \rho_\lambda. \end{aligned}$$

This is our promised contradiction.

Thus $\operatorname{Ker} \phi$ is finitely generated and so (iv) holds.

(iv)$\Rightarrow$(i). Let S_α denote the set of finite non-empty sequences X of elements of S, having strictly less than α. We define for every element $X = (s_1, \ldots, s_n)$ of S_α a sentence ψ_X of L_S such that an S-system A is α-injective if and only if A is a model of $\{\psi_X\colon X \in S_\alpha\}$.

Let $X \in S_\alpha$ where $X = (s_1, \ldots, s_n)$. Let $z_1 S \cup \cdots \cup z_n S$ be the free S-system on $z_1, \ldots, z_n$ and define $\phi = \phi_X\colon z_1 S \cup \cdots \cup z_n S \to S$ by $\phi(z_i) = s_i$. Let $K = K_X$ denote the kernel of ϕ; by assumption, K is finitely generated and we let

$$G = G_X = \{(z_{j_1} t_{j_1}, z_{j_2} t_{j_2})\colon 1 \leq j \leq m\}$$

be a finite generating set for K.

We now define ψ_X to be the sentence

$$(\forall x_1 \cdots x_n)\left(\bigwedge_{j=1}^{m} (x_{j_1} t_{j_1} = x_{j_2} t_{j_2}) \to (\exists x)\left(\bigwedge_{i=1}^{n} (x s_i = x_i)\right)\right).$$

Let A be an S-system and let

$$\Sigma = \{xs_i = a_i\colon\ 1 \leqslant i \leqslant n\}$$

be an α-system over A. Putting $X = (s_1, \ldots, s_n)$, we may find K_X and G_X as above. It is not difficult to see that Σ is consistent if and only if

$$\bigwedge_{j=1}^{m} (a_{j_1}t_{j_1} = a_{j_2}t_{j_2})$$

holds. Since A is α-injective if and only if all consistent α-systems over A have a solution in A, it follows that A is α-injective if and only if A is a model of $\{\psi_X\colon\ X \in S_\alpha\}$. Thus $\mathscr{I}(\alpha)$ is an axiomatisable class.

4. Coherent monoids

We say that an S-system A is *finitely presented* if it is isomorphic to F/ρ where F is a finitely generated free S-system and ρ is a finitely generated congruence on F. A standard argument gives that A is finitely presented in the above sense if and only if it is finitely presented in the sense of Wheeler [**18**]. Moreover it follows from this, and the comment on p. 326 of [**18**], that if A is finitely presented and A has generators $a_1, \ldots, a_n$, then there is a finitely generated congruence ρ on the free S-system $F = z_1S \cup \cdots \cup z_nS$ such that $\phi\colon\ F/\rho \to A$ defined by $\phi([z_is]) = a_is$ is a well-defined S-isomorphism.

The notion of coherency for rings is firmly established. We recall that a ring R is (right) coherent if every finitely generated right ideal of R is finitely presented. It is well known that this is equivalent to the condition which states that every finitely generated (right) R-submodule of a finitely presented R-module is finitely presented.

For monoids there are three obvious candidates for the definition of coherence. We will say that a monoid S is

(a) *weakly coherent* if every finitely generated right ideal of S is finitely presented;

(b) *coherent* if every finitely generated S-subsystem of every finitely presented cyclic S-system is finitely presented;

(c) *strongly coherent* if every finitely generated S-subsystem of every finitely presented S-system is finitely presented.

We note that an S-system is finitely presented cyclic if and only if it is isomorphic to S/ρ for some finitely generated right congruence ρ on S.

The theory Th(S) is a universal Horn theory and it follows that Th(S) has finite presentations. Thus a monoid S is strongly coherent if and only if Th(S) is coherent in the sense of Wheeler [**18**]. Clearly every strongly coherent monoid is coherent and every coherent monoid is weakly coherent. We show in Theorem 6 that in fact every coherent monoid is strongly coherent.

LEMMA 4.1. *Let n be a natural number. The following conditions are equivalent for an S-system A:*

(i) *every S-subsystem of A with a generating set of n elements is finitely presented;*

(ii) *the kernel of every S-homomorphism ϕ: $F \to A$, where F is the finitely generated free S-system on n generators, is finitely generated.*

Proof. (i)$\Rightarrow$(ii). Let A be an S-system satisfying (i) and let ϕ: $z_1S \cup \cdots \cup z_nS \to A$ be an S-homomorphism. Put $\phi(z_i) = a_i$, $1 \leq i \leq n$. Then $\operatorname{Im} \phi = a_1S \cup \cdots \cup a_nS$ is an S-subsystem of A with a generating set of n elements and so by assumption, $\operatorname{Im} \phi$ is finitely presented. In view of the comments at the beginning of this section there is a free S-system $F = y_1S \cup \cdots \cup y_nS$ on n generators $y_1, \ldots, y_n$, a finitely generated congruence ρ on F and an isomorphism θ: $\operatorname{Im} \phi \to F/\rho$ given by $\theta(a_is) = [y_is]_\rho$. Now clearly, $(z_is, z_jt) \in \operatorname{Ker} \phi$ if and only if $y_is\rho y_jt$ and consequently, $\operatorname{Ker} \phi$ is finitely generated.

(ii)$\Rightarrow$(i). This is quite straightforward.

COROLLARY 4.2. *A monoid S is weakly coherent if and only if it satisfies condition (C_n) of Theorem 3 for every natural number n.*

The following proposition is an immediate consequence of Theorem 3 and Corollary 4.2.

PROPOSITION 4.3. *The following conditions are equivalent for a monoid S:*

(i) *the weakly f-injective S-systems are an axiomatisable class;*

(ii) *S is weakly coherent.*

Let L be a first order language and $\mathscr{A}$ and L-structure with universe A. For any subset X of A we may augment the language L by adding a new constant a for each $a \in X$. The resulting language is denoted $L(X)$. Further, $(\mathscr{A}, a)_{a \in X}$ is the $L(X)$-structure obtained from A by interpreting each new constant of $L(X)$ by the appropriate element of X. As usual we make no distinction between the new constants in the language $L(X)$ and the corresponding elements of X. Then $\operatorname{Th}((\mathscr{A}, a)_{a \in X})$ is the first order theory in $L(X)$ consisting of those sentences of $L(X)$ which are true in the model $(\mathscr{A}, a)_{a \in X}$.

If $\{\phi(x)\}$ is a set of formulae of any first order language L, in one free variable x, then $\{\phi(x)\}$ is *realised* if there is an L-structure $\mathscr{B}$ and an element b in the universe of $\mathscr{B}$ such that $\mathscr{B} \models \{\phi(b)\}$. Let κ be a cardinal. Then an L-structure $\mathscr{A}$ is *κ-saturated* if every set of formulae $\{\phi(x)\}$ of $L(X)$, in one free variable x, where $X \subseteq A$, the universe of $\mathscr{A}$, and

$|X| < \kappa$, that is realised in some model of $\mathrm{Th}((\mathscr{A}, a)_{a \in X})$, is realised in $(\mathscr{A}, a)_{a \in X}$. From Theorem 16.4 of [16], $\mathscr{A}$ is an elementary substructure of κ-saturated L-structures for arbitrarily large regular cardinals κ.

LEMMA 4.4. *Let S be a weakly coherent monoid. Then every weakly f-injective S-system is an elementary substructure of a weakly injective S-system.*

Proof. Let A be a weakly f-injective S-system and let κ be a regular cardinal larger than $\gamma(S)$. Then A is an elementary substructure of a κ-saturated S-system B. In particular, $A \equiv B$ and since $\mathscr{I}(\aleph_0)$ is axiomatisable, B is weakly f-injective.

Now let $\Sigma = \{xs_j = b_j\colon j \in J\}$ be a consistent $\gamma(S)$-system over B. Then every finite subset of Σ is realised in B and so every finite subset of $\Pi = \mathrm{Th}((B, b)_{b \in B}) \cup \Sigma$ has a model. By the Completeness Theorem, Π has a model and so Σ is realised in some elementary extension C of B. Certainly C is a model of $\mathrm{Th}((B, b_j)_{j \in J})$ and so Σ is realised in B. Thus B is weakly injective.

We say that a monoid S is *noetherian* if S satisfies the ascending chain condition on right ideals and *strongly noetherian* if every right congruence over S is finitely generated. It is not difficult to see that a strongly noetherian monoid is noetherian. Normak shows in [**14**] that every strongly noetherian monoid is strongly coherent. This corresponds to the fact that every noetherian ring is coherent. However it is not clear whether a noetherian monoid is coherent or even weakly coherent.

In view of the above comments, Theorem 4.5 is an analogue of Proposition 3.19 of [**5**].

THEOREM 4.5. *The following conditions are equivalent for a monoid S:*

(i) *the weakly injective S-systems are an axiomatisable class*;

(ii) *all weakly f-injective S-systems are weakly injective and S is weakly coherent*;

(iii) *S is noetherian and weakly coherent.*

Proof. (i)$\Rightarrow$(ii). Every ultraproduct of weakly injective S-systems is weakly injective and so is certainly weakly f-injective. By Theorem 3, the class $\mathscr{I}(\aleph_0)$ is axiomatisable and so by Proposition 4.3, S is weakly coherent.

Let A be a weakly f-injective S-system. By Lemma 4.4, A is elementarily equivalent to a weakly injective S-system. But then A itself is weakly injective.

(ii)$\Rightarrow$(iii). This is immediate from Corollary 3.7 of [**8**].

(iii)$\Rightarrow$(i). Again by Corollary 3.7 of [**8**], $\mathscr{I}(\aleph_0)$ coincides with $\mathscr{I}(\gamma(S))$. Since S is weakly coherent, $\mathscr{I}(\gamma(S))$ is axiomatisable.

It is reported in [**10**] that Kuzičeva has obtained in [**12**] a characterisation of those monoids for which $\mathcal{T}(\gamma(S))$ is axiomatisable. However, we have been unable to obtain a copy of this paper, or any further details of its contents.

5. Axiomatising algebraically closed S-systems

For the purposes of this section it is convenient to define the notion of an α-algebraically closed (α-existentially closed) S-system and an α-coherent monoid.

Let α be a cardinal with $1<\alpha\leqslant\aleph_0$. Then an S-system A is *α-algebraically closed* (*α-existentially closed*) if every finite consistent system of equations (and inequations) with constants from A, in less than α variables, has a solution in A. A monoid S is *α-coherent* if every finitely generated S-subsystem of any S-system A, where A is finitely presented and has a generating set of fewer than α elements, is finitely presented.

It is clear that an S-system A is $\aleph_0$-algebraically closed ($\aleph_0$-existentially closed) if and only if it is absolutely pure (existentially closed). Further, for a monoid S the notions of 2-coherency and coherency coincide, as do the notions of $\aleph_0$-coherency and strong coherency. Actually, for any cardinals α, β with $1<\alpha$, $\beta\leqslant\aleph_0$, it turns out that a monoid is α-coherent if and only if it is β-coherent: this follows from Theorem 6 and the obvious fact that a strongly coherent monoid is α-coherent for any α such that $1<\alpha\leqslant\aleph_0$.

We point out that in the next theorem, the cardinal α plays an essentially different role to the cardinal α appearing in the statement of Theorem 3.

THEOREM 5.1. *Let α be a cardinal with $1<\alpha\leqslant\aleph_0$. Then the following conditions are equivalent for a monoid S:*

(i) *the class of α-algebraically closed S-systems is axiomatisable;*
(ii) *every ultraproduct of α-algebraically closed S-systems is α-algebraically closed;*
(iii) *every ultraproduct of injective S-systems is α-algebraically closed;*
(iv) *every ultraproduct of α-existentially closed S-systems is α-algebraically closed;*
(v) *S is α-coherent.*

Proof. (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iv). These implications are trivial.

(ii)$\Rightarrow$(iii). This follows immediately from the fact that every injective S-system is absolutely pure [13].

(iii)$\Rightarrow$(v), (iv)$\Rightarrow$(v). We show with one proof that both these implications are true. The argument closely resembles the proof of (iii)$\Rightarrow$(iv) in Theorem 3 and so we do not include every detail.

Let A be a finitely presented S-system which can be generated by n

elements where $n < \alpha$. By Lemma 4.1, it suffices to show that the kernel of an S-homomorphism ϕ: $F \rightarrow A$ is finitely generated where F is free with free generators $z_1, \ldots, z_m$. We may suppose that $A = F'/\rho$ where F' is free with free generators $y_1, \ldots, y_n$ and ρ is a finitely generated congruence on F'.

Suppose that Ker ϕ is not finitely generated and let β be the smallest cardinal such that Ker ϕ can be generated by a subset of cardinality β. We now proceed as in the proof of (iii)$\Rightarrow$(iv) in Theorem 3 and aim for the same contradiction.

Rather than embed F/ρ_ν in a weakly injective S-system G_ν, we choose G_ν to be injective for the proof of (iii)$\Rightarrow$(v) and to be α-existentially closed for the proof of (iv)$\Rightarrow$(v). The latter choice is possible because Th(S) is inductive, that is, the union of every increasing chain of models of Th(S) is a model of Th(S). It is a well known consequence of this that every S-system can be embedded in an existentially closed S-system which is certainly α-existentially closed. Now the ultraproduct

$$\mathcal{U} = \prod_{\nu<\beta} G_\nu/D$$

where D is a uniform ultrafilter on β is by hypothesis, α-algebraically closed.

We define a system of equations, Σ, in variables $x_1, \ldots, x_n$ over $\mathcal{U}$ as follows. First, for $i \in \{1, \ldots, m\}$ we choose $j(i)$ in $\{1, \ldots, n\}$ and s_i in S such that $y_{j(i)}s_i$ is in the ρ-class of $\phi(z_i)$. As in Theorem 3, for each $z \in F$ we let $\bar{z}$ denote the equivalence class modulo D of the element of $\prod_{\nu<\beta} G_\nu$ whose νth coordinate is $z\rho_\nu$. Put

$$\Sigma_1 = \{x_{j(i)}s_i = \bar{z}_i\colon\ 1 \leqslant i \leqslant m\}.$$

Now ρ is finitely generated and so $\rho = \rho(H)$ for some finite subset H of $F' \times F'$. We define Σ_2 by

$$\Sigma_2 = \{x_iu_i = x_jv_j\colon\ (y_iu_i, y_jv_j) \in H\}$$

and put $\Sigma = \Sigma \cup \Sigma_2$.

The uniformity of D is used in the proof that Σ is consistent over $\mathcal{U}$. The argument is tedious but straightforward along similar lines to that in Theorem 3, relying on Lemma 2.3 rather than Corollary 2.4.

Since fewer than α-variables occur in Σ and $\mathcal{U}$ is α-algebraically closed, Σ has a solution in $\mathcal{U}$. If we let $(a_D^1, \ldots, a_D^n)$ be such a solution, then a slight extension of the argument in Theorem 3 gives the existence of an ordinal λ such that $a^i(\lambda)u_i = a^j(\lambda)v_j$ for every equation $x_iu_i = x_jv_j$ of Σ_2 and $a^{j(i)}(\lambda)s_i = z_i\rho_\lambda$ for each $i \in \{1, \ldots, m\}$.

We now show that $b_\tau\rho_\lambda b'_\tau$ for any $\tau < \beta$. For some h, $i \in \{1, \ldots, m\}$ and v, w in S we have $b_\tau = z_hv$, $b'_\tau = z_iw$. Now $\phi(b_\tau) = \phi(b'_\tau)$ so that

$y_{j(h)}s_h v \rho y_{j(l)} s_l w$. An argument using induction on the length of ρ-sequences gives that if $y_p s \rho y_q t$, then $a^p(\lambda)s = a^q(\lambda)t$. Thus

$$\begin{aligned} b_\tau \rho_\lambda = (z_h v)\rho_\lambda = (z_h \rho_\lambda)v = a^{j(h)}(\lambda)s_h v = a^{j(l)}(\lambda)s_l w \\ = (z_l \rho_\lambda)w = (z_l w)\rho_\lambda = b'_\tau \rho_\lambda. \end{aligned}$$

This is our promised contradiction. Thus Ker ϕ is finitely generated and S is α-coherent.

(v)$\Rightarrow$(i). Denote by $\mathscr{F}_\alpha$ the set of standard forms of finite systems of equations in fewer than α variables. We define for each $X \in \mathscr{F}_\alpha$ a sentence ψ_X of L_S such that an S-system A is α-algebraically closed if and only if A is a model of $\{\psi_X \colon X \in \mathscr{F}_\alpha\}$.

Let

$$\begin{aligned} X = \{x_i p_i^j = x_i^j \colon 1 \leq i \leq r, 1 \leq j \leq f(i)\} \\ \cup \{x_{g(k)} s_k = x_{h(k)} t_k \colon 1 \leq k \leq m\} \end{aligned}$$

be a standard form. We refer the reader to Section 2 for the notation. Note that we are assuming that n, the number of variables in Σ, is strictly less than α. Not also that $r \leq n$.

Let $F_X = y_1 S \cup \cdots \cup y_n S$ and let ρ_X be the congruence on F_X defined by $\rho_X = \rho(H_X)$ where

$$H_X = \{(y_{g(k)} s_k, y_{h(k)} t_k) \colon 1 \leq k \leq m\}.$$

Let G_X be the free S-system on the $f(1) + \cdots + f(r)$ symbols z_i^j $(1 \leq i \leq r, 1 \leq j \leq f(i))$ and define an S-homomorphism $\phi_X \colon G_X \to F_X/\rho_X$ by

$$\phi_X(z_i^j) = [y_i p_i^j].$$

By assumption, S is α-coherent and so ker ϕ_X is a finitely generated congruence on G_X. Thus ker $\phi_X = \rho(K_X)$ for some finite subset K_X of $G_X \times G_X$, say

$$K_X = \{(z_{v(j)}^{u(j)} w_j, z_{v'(j)}^{u'(j)} w'_j \colon 1 \leq j \leq s\}.$$

Now define the sentence ψ_X to be

$$\begin{aligned} (\forall x_1^1 \cdots x_1^{f(1)} x_2^1 \cdots x_r^{f(r)}) \Big(\Big(\bigwedge_{j=1}^{s} x_{v(j)}^{u(j)} w_j = x_{v'(j)}^{u'(j)} w'_j \Big) \\ \to (\exists x_1 \cdots x_n) \Big(\Big(\bigwedge_{i=1}^{r} \bigwedge_{j=1}^{f(i)} x_i p_i^j = x_i^j \Big) \wedge \Big(\bigwedge_{k=1}^{m} x_{g(k)} s_k = x_{h(k)} t_k \Big) \Big) \Big). \end{aligned}$$

A straightforward but tedious proof gives that A is α-algebraically closed if and only if A is a model of $\Pi_\alpha = \{\psi_X \colon X \in \mathscr{F}_\alpha\}$. Thus the α-algebraically closed S-systems form an axiomatisable class.

COROLLARY 5.2. *A monoid S is coherent if and only if the class of 2-algebraically closed S-systems is axiomatisable.*

For the remainder of this section we suppose that S is a coherent monoid. We show that $\mathcal{F}_2$ and the sentences ψ_X for $X \in \mathcal{F}_2$, simplify considerably. For $\mathcal{F}_2$ is the set of standard forms of systems of equations in one variable. If A is an S-system and Σ is a system of equations over A of form X, say, where $X \in \mathcal{F}_2$, then Σ must have the form

$$\Sigma = \{xp_j = a_j\colon\ 1 \leqslant j \leqslant r\} \cup \{xs_k = xt_k\colon\ 1 \leqslant k \leqslant m\}$$

where for the sake of simplicity, we have amended the notation slightly. Now $F_X = y_i S$, and $\rho_X = \rho(H_X)$ where

$$H_X = \{(y_1 s_k, y_1 t_k)\colon\ 1 \leqslant k \leqslant m\}.$$

We identify F_X with S and H_X with the corresponding congruence on S. Further, G_X is the free S-system on $z_1 S \cup \cdots \cup z_r S$ and the S-homomorphism $\phi_X\colon G_X \to S/\rho_X$ is defined by

$$\phi_X(z_r) = [p_r].$$

We may now write ker ϕ_X as $\rho(K_X)$ where K_X has the form

$$K_X = \{(z_{l(i)} w_i, z_{l'(i)} w_i')\colon\ 1 \leqslant i \leqslant s\}.$$

The sentence ψ_X of L_S has become

$$(\forall x_1 \cdots x_r)\left(\left(\bigwedge_{i=1}^{s} x_{l(i)} w_i = x_{l'(i)} w_i'\right)\right.$$
$$\left.\to (\exists x)\left(\left(\bigwedge_{j=1}^{r} xp_j = x_j\right) \wedge \left(\bigwedge_{k=1}^{m} xs_k = xt_k\right)\right)\right),$$

and from the proof of Theorem 5.1 we see that Σ is a consistent system of equations if and only if

$$\bigwedge_{i=1}^{s} (a_{l(i)} w_i = a_{l'(i)} w_i')$$

holds.

In fact we have developed enough machinery to write down a set of sentences of L_S axiomatising the 2-existentially closed S-systems.

For variables x, x_j, y_δ we shall be interested in *standard forms* consisting of finitely many equations and inequations,

$$Y = \{xp_j = x_j,\ xs_k = xt_k,\ xq_\delta \neq y_\delta,\ xu_\varepsilon \neq xv_\varepsilon\}$$

where the indices j, k, δ, ε are from finite sets. If A is an S-system, then by substituting elements a_j, b_δ for the variables x_j, y_δ respectively in Y, we obtain a finite system Π of equations and inequations in one variable x over A. Let Y^+, Π^+ be the subset of equations in Y, Π respectively. Putting $X = Y^+$ we may define H_X, ρ_X, G_X, ϕ_X and K_X as above. Further, from Lemma 2.3, Π^+ is a consistent system of equations if and only if for all $s, s' \in S$ and $j, j' \in \{1, \ldots, r\}$

$$p_j s \rho_X p_{j'} s' \Rightarrow a_j s = a_{j'} s'.$$

At this point we split the inequalities of Π into two sets. Let I_X be the S-subsystem of S/ρ_X generated by $[p_1], \ldots, [p_r]$. An inequality of the form $xq_\delta \neq b_\delta$ is said to be *simple* over Π^+ if $[q_\delta] \notin I_X$. An inequality of the form $xu_\varepsilon \neq xv_\varepsilon$ is *simple* over Π^+ if $[u_\varepsilon] \neq [v_\varepsilon]$ and not both $[u_\varepsilon]$, $[v_\varepsilon]$ are in I_X. Then Π' denotes the set of inequalities in Π which are simple over Π^+ and $\Pi'' = \Pi \backslash (\Pi^+ \cup \Pi')$.

LEMMA 5.3. *The system of equations and inequations* $\Pi^+ \cup \Pi'$ *is consistent if and only if* Π^+ *is a consistent system of equations.*

Proof. Clearly we need only show that if Π^+ is consistent, then $\Pi^+ \cup \Pi'$ is consistent.

If Π^+ is consistent, then the criterion of Lemma 2.3 is satisfied. Further, A is embedded under the canonical isomorphism in the S-system $B = (A \cup S)/\rho(M_X)$, where

$$M_X = \{(p_j, a_j)\} \cup H_X$$

and [1], the equivalence class of 1 with respect to $\rho(M_X)$, is a solution of Π^+ in B. But it is not too difficult to see that [1] is also a solution of $\Pi^+ \cup \Pi'$ in B.

Let Y be a standard form as defined above. We say that Y is *simple* if for any S-system A and any system Π of equations and inequations over A which results from Y, we have $\Pi' = \varnothing$. Let $\mathcal{H}$ denote the set of all simple standard forms.

For $Y \in \mathcal{H}$, define a sentence θ_Y of L_S by

$$(\forall x_1 \cdots x_r y_1 \cdots y_{r'})\Big(\Big(\bigwedge_{l=1}^{s} x_{l(i)} w_i = x_{l'(i)} w_i'\Big)$$

$$\rightarrow (\exists x)((\Lambda x p_j = x_j) \wedge (\Lambda x s_k = x t_k) \wedge (\Lambda x q_\delta \neq y_\delta) \wedge (\Lambda x u_\varepsilon \neq x v_\varepsilon))\Big).$$

PROPOSITION 5.4. *Let S be a coherent monoid. Then an S-system A is 2-existentially closed if and only if A is a model of* $\{\theta_Y\colon Y \in \mathcal{H}\}$.

Proof. Let A be a 2-existentially closed S-system and let $Y \in \mathcal{H}$. Let $a_1, \ldots, a_r, b_1, \ldots, b_{r'} \in A$ and let Π denote the system of equations and inequations corresponding to the form Y and elements $a_1, \ldots, a_r$,

$b_1, \ldots, b_{r'}$ of A. Let $\Sigma = \Pi^+$, so that Σ is the system of equations with form Y^+ and constants $a_1, \ldots, a_r$. Suppose that

$$\bigwedge_{i=1}^{s} a_{l(i)} w_i = a_{l'(i)} w'_i.$$

holds. Then Σ is a consistent system of equations over A and since all the inequalities of Π are simple over Σ, Lemma 5.3 gives that Π is a consistent system over A. But A is 2-existentially closed and so Π has a solution in A. It follows that $A \models \{\theta_Y\colon\ Y \in \mathscr{H}\}$.

Conversely, suppose that $A \models \{\theta_Y\colon\ Y \in \mathscr{H}\}$ and let Π be a consistent system of equations and inequations in one variable over A. Using the notation defined before Lemma 5.3, we have $\Pi = \Pi^+ \cup \Pi' \cup \Pi''$. Let $\Pi^+ \cup \Pi'$ have form $Y \in \mathscr{H}$ so that Π^+ has form $Y^+ (Y^+ \in \mathscr{F}_2)$. Since Π^+ is consistent,

$$\bigwedge_{i=1}^{s} a_{l(i)} w_i = a_{l'(i)} w'_i$$

holds in A and so as $A \models \theta_Y$, $\Pi^+ \cup \Pi'$ has a solution c, say, in A. But we are given that Π is a consistent system and so Π has a solution d in some S-system D containing A.

Let $xq_\delta \neq b_\delta \in \Pi''$. So $[q_\delta] \in [p_j]S$ for some j, that is, $q_\delta \rho_X p_j h$ for some j and some $h \in S$. If $q_\delta = p_j h$, then certainly $cq_\delta = cp_j h$ and $dq_\delta = dp_j h$. If there is a ρ_X-sequence

$$q_\delta = c_1 e_1, \qquad d_1 e_1 = c_2 e_2, \ldots, d_l e_l = p_j h,$$

then as $cc_i = cd_i$, $dc_i = dd_i$, $1 \leq i \leq l$, we again have that $cq_\delta = cp_j h$ and $dq_\delta = dp_j h$. Then

$$b_\delta \neq dq_\delta = dp_j h = a_j h = cp_j h = cq_\delta$$

and so c is a solution of $xq \neq b_\delta$. Similarly, if $xu_\varepsilon \neq xv_\varepsilon \in \Pi''$ then one can show that $cu_\varepsilon \neq cv_\varepsilon$. Thus c is a solution of Π in A, giving that A is 2-existentially closed.

6. Model completions of S-systems

We begin this section by giving some model theoretic definitions and results. Further details may be found in [4], [11] and [15].

Let L be a first order language and let T be a theory in L. Then T is *model complete* if for any models $\mathscr{A}, \mathscr{B}$ of T with $\mathscr{A} \subseteq \mathscr{B}$, we have $\mathscr{A} \preccurlyeq \mathscr{B}$, that is, $\mathscr{A}$ is an elementary substructure of $\mathscr{B}$. If T^* is another theory in L then T and T^* are *mutually model consistent* if every model of T is embeddable in a model of T^*, and vice versa.

The theory T^* is a *model companion* of T if T and T^* are mutually model consistent and T^* is model complete. In the case of an inductive theory having the amalgamation property, such as $\mathrm{Th}(S)$ for any monoid S, the notion of model companion coincides with that of *model completion,* which is in general a stronger concept.

For a theory T one defines in a natural way an existentially closed model of T, by analogy with an existentially closed field or an existentially closed S-system, for example. The class of existentially closed models of T is denoted by $\mathscr{E}(T)$. If T is an inductive theory, then T has a model companion if and only if $\mathscr{E}(T)$ is axiomatisable. If this is the case, then $\mathrm{Th}(\mathscr{E}(T))$ is a model companion of T.

Let S be a coherent monoid. We show by direct verification that $\Theta = \{\theta_{\bar{X}}: \bar{X} \in \mathscr{H}\} \cup \Sigma_S$, where Σ_S is the set of sentences of L_S axiomatising S-systems, is a model companion of $\mathrm{Th}(S)$.

Clearly every model of Θ is an S-system and since $\mathrm{Th}(S)$ is inductive, every S-system is embedded in an existentially closed S-system which is certainly a model of Θ. Thus $\mathrm{Th}(S)$ and Θ are mutually model consistent.

To show that Θ is model complete, we use some standard results (see for example **[11]**).

Let $\mathcal{M}$, $\mathcal{N}$ be models of a theory T in a first order language L. Suppose that $\emptyset \neq X \subseteq M$, where M is the universe of $\mathcal{M}$, X is finite and f is a function from X to N, the universe of $\mathcal{N}$. Then f is a *local isomorphism* if there exists an isomorphism from the substructure of $\mathcal{M}$ generated by X onto the substructure of $\mathcal{N}$ generated by $f(X)$ which extends f. Further, f is said to be *immediately extendible* if for each $a \in M$ there is an element b of N such that

$$\bar{f}: X \cup \{a\} \to f(X) \cup \{b\}$$

defined by

$$\bar{f} \mid X = f, \qquad \bar{f}(a) = b$$

is a local isomorphism and conversely, for each $b \in N$ there exists an element a of M such that

$$\bar{g}: f(X) \cup \{b\} \to X \cup \{a\}$$

defined by

$$\bar{g} \mid f(X) = f^{-1}, \qquad \bar{g}(b) = a$$

is a local isomorphism.

From Theorem 12 and Lemma 1 of **[11]**, to show that Θ is model complete it is enough to show that every local isomorphism between $\aleph_0$-saturated models of Θ is immediately extendible.

Let M, N be models of Θ, that is, M, N are 2-existentially closed

S-systems. Suppose in addition that M, N are $\aleph_0$-saturated. Let $U = u_1S \cup \cdots \cup u_nS$, $V = v_1S \cup \cdots \cup v_nS$ be locally isomorphic S-subsystems of M, N respectively, where $f(u_is) = v_is$ is an isomorphism.

Let $u_{n+1} \in M$ and define a set Λ of sentences of $L(\{u_1, \ldots, u_n\} \cup \{x\})$ by

$$\Lambda = \{\phi(x)\colon\ \phi(x) \text{ is atomic or negated atomic and } \phi(u_{n+1}) \text{ is true in } M\}.$$

Thus Λ is the system of all equations and inequations with constants from U, which are satisfied by u_{n+1}.

Let Π be a finite subset of Λ. Put $\Pi = \Pi^+ \cup \Pi' \cup \Pi''$ where Π^+, Π', Π'' are defined as in Section 5. We suppose that $\Pi^+ \cup \Pi'$ has standard form Y, Π^+ has standard form Y^+ and the constants in the equations of Π are $a_1, \ldots, a_r$. Let $\Lambda(V), \Pi(V), \ldots$ be the systems with constants from V, obtained from $\Lambda, \Pi, \ldots$ by replacing each u_i with its image v_i under f in V.

Since Π^+ is consistent,

$$\bigwedge_{i=1}^{s} a_{l(i)}w_i = a_{l'(i)}w_i'$$

is true in U (see Section 5 for the notation). But this gives that

$$\bigwedge_{i=1}^{s} f(a_{l(i)})w_i = f(a_{l'(i)})w_i'$$

holds in V. Thus $\Pi^+(V)$ is a consistent system of equations and so by Lemma 5.3, $\Pi^+(V) \cup \Pi'(V)$ is a consistent system over V. As N is 2-existentially closed, $\Pi^+(V) \cup \Pi'(V)$ has a solution $\bar{v}_{n+1}$ in N.

Suppose now that $xq_\delta \neq b_\delta \in \Pi''(V)$. Thus $q_\delta \rho_X p_i h$ for some $i \in \{1, \ldots, r\}$ and $h \in S$. It follows that $u_{n+1}q_\delta = u_{n+1}p_ih$ and $\bar{v}_{n+1}q_\delta = \bar{v}_{n+1}p_ih$. We have that $b_\delta = f(u_ks)$ for some $u_ks \in U$. Thus $u_ks \neq u_{n+1}q_\delta$. Also, $xp_i = a_i \in \Pi^+$ and $xp_i = f(a_i) \in \Pi^+(V)$ gives $u_{n+1}p_i = a_i$ and $\bar{v}_{n+1}p_i = f(a_i)$. Then

$$u_{n+1}q_\delta = u_{n+1}p_ih = a_ih \in U$$

giving

$$\begin{aligned} b_\delta &= f(u_ks) \neq f(u_{n+1}q_\delta) = f(a_ih) = f(a_i)h \\ &= \bar{v}_{n+1}p_ih = \bar{v}_{n+1}q_\delta. \end{aligned}$$

If $xu_\varepsilon \neq xv_\varepsilon \in \Pi''(V)$, then either $u_\varepsilon \rho_X v_\varepsilon$, or $u_\varepsilon \rho_X p_i h$ and $v_\varepsilon \rho_X p_j k$ where $i, j \in \{1, \ldots, r\}$ and $h, k \in S$. But if $u_\varepsilon \rho_X v_\varepsilon$, then $u_{n+1}u_\varepsilon = v_{n+1}v_\varepsilon$, a contradiction. So we must have $u_\varepsilon \rho_X p_i h$ and $v_\varepsilon \rho_X p_j k$, which gives that $u_{n+1}u_\varepsilon = u_{n+1}p_ih$, $u_{n+1}v_\varepsilon = u_{n+1}p_jk$, $\bar{v}_{n+1}u_\varepsilon = \bar{v}_{n+1}p_ih$, $\bar{v}_{n+1}v_\varepsilon = \bar{v}_{n+1}p_jk$.

In a manner similar to the above we obtain that $\bar{v}_{n+1}u_\varepsilon \neq \bar{v}_{n+1}v_\varepsilon$ and so $\bar{v}_{n+1}$ is a solution of $\Pi(V)$ in N.

Thus every finite subset of

$$\mathrm{Th}(N, \{v_1, \ldots, v_n\}) \cup \Lambda(V)$$

has a model and it follows from the Completeness Theorem that there is a model P of $\mathrm{Th}(N, \{v_1, \ldots, v_n\})$ and an element c of P such that $P \models \phi(c)$ for each $\phi(x) \in \Lambda$. Since N is $\aleph_0$-saturated, there is an element $v_{n+1} \in N$ such that $\phi(v_{n+1})$ is true for each $\phi(x) \in \Lambda$. It is then easy to see that

$$\bar{f}\colon \{u_1, \ldots, u_{n+1}\} \to \{v_1, \ldots, v_{n+1}\}$$

defined by $\bar{f}(u_i) = v_i$, $1 \leq i \leq n+1$, is a local isomorphism and it follows that f is immediately extendible. Hence Θ is model complete and so Θ is a model companion of $\mathrm{Th}(S)$.

THEOREM 6. *The following conditions are equivalent for a monoid S:*
(i) *S is coherent;*
(ii) $\mathrm{Th}(S)$ *has a model completion;*
(iii) *S is strongly coherent.*
If any of these conditions hold, then

$$\Theta = \{\theta_Y \colon Y \in \mathcal{H}\}$$

is defined and is a model completion of $\mathrm{Th}(S)$.

Proof. We need only show that (ii) implies (iii).

If $\mathrm{Th}(S)$ has a model completion, then this is a model companion and as $\mathrm{Th}(S)$ is inductive, the class of existentially closed S-systems is axiomatisable. Thus every ultraproduct of existentially closed S-systems is existentially closed and so certaintly is absolutely pure. From Theorem 5.1 we have that S is strongly coherent.

The equivalence of conditions (ii) and (iii) of the above theorem can be shown more directly by using some general results of Wheeler **[18]**. By Theorem 1 and Propositions 1 and 2 of **[18]** it is only necessary to show that $\mathrm{Th}(S)$ has the conservative congruence extension property for finite presentations. In fact this is not too difficult to do. However, this approach does not yield an explicit description of the model completion of $\mathrm{Th}(S)$, nor the equivalence of conditions (i) and (iii).

7. Axiomatising injective S-systems

In this final section we give a result for injective S-systems which corresponds to Theorem 4.5 for weakly injective S-systems.

For a monoid S it is shown in **[7]** that an S-system A is injective if and only if every consistent system of equations with constants from A, has a solution in A. Clearly every injective S-system is absolutely pure, but it is not known what conditions S must satisfy for all absolutely pure S-systems to be injective. If S is strongly noetherian, then the classes of absolutely pure and injective S-systems coincide [13], but this condition on S is too strong. For example, all S-systems are injective when S is a group with a 0 adjoined [6], but certainly S may have right congruences which are not finitely generated. On the other hand, the condition that S be noetherian is not strong enough: if all absolutely pure S-systems are injective, then S is noetherian, but the converse is not true [13].

In spite of this gap in our knowledge, we can still prove the following.

THEOREM 7.1. *The following conditions are equivalent for a monoid S:*
(i) *the injective S-systems form an axiomatisable class;*
(ii) *S is coherent and all absolutely pure S-systems are injective.*

Proof. (ii)$\Rightarrow$(i). This implication follows immediately from Theorems 5.1 and 6.

(i)$\Rightarrow$(ii). First we show that if A is a κ-saturated absolutely pure S-system, where κ is greater than the cardinality of S, then A is injective.

Let $\Sigma = \{xs = xt\colon\ s, t \in S\}$. Then every finite subset of Σ has a solution in A [7] and so $\Sigma \cup \mathrm{Th}(A)$ has a model. Since A is κ-saturated, Σ has a solution in A, that is, there is an element u of A such that $u = us$ for all $s \in S$.

Now left I be an ideal of S, ρ a right congruence on S and $I\rho$ the S-subsystem of S/ρ defined by $I\rho = \{[s]\colon\ s \in I\}$. Suppose that $\phi\colon\ I\rho \to A$ is an S-homomorphism. Define a system of equations with constants from A by

$$\Sigma = \{xu = xv\colon\ u\rho v\} \cup \{xs = a\colon\ s \in I,\ \phi[s] = a\}.$$

Let $s, t \in I$, $h, k \in S$ and suppose that $sh\rho tk$. Then $\phi[sh] = \phi[tk]$, giving $\phi[s]h = \phi[t]k$. As in the proof of Lemma 2.3, it follows that Σ is a consistent system of equations. Thus every finite subset of Σ has a solution in A and so it follows from the fact that A is κ-saturated that Σ has a solution in A.

Let a be the solution of Σ in A. Now define $\bar{\phi}\colon\ S/\rho \to A$ by $\bar{\phi}([r]) = ar$. It is easy to see that $\bar{\phi}$ is an S-homomorphism extending ϕ. We can now show that A is injective. Our proof is similar to that of (ii)$\Rightarrow$(i) of Proposition 2.1 in [6] and so we omit some details.

Let N be an S-subsystem of M and let $\theta\colon\ N \to A$ be an S-homomorphism. We wish to extend θ to an S-homomorphism $\bar{\theta}\colon\ M \to A$. Using a Zorn's Lemma argument we may suppose that θ can be extended to an S-homomorphism $\xi\colon\ P \to A$ where P is an S-subsystem of

M containing N and ξ cannot be extended to any S-subsystem of M properly containing P.

Assume that $P \neq M$: we aim for a contradiction. Choose $c \in M \backslash P$, put $I = \{s \in S\colon cs \in P\}$ and $\rho = \{(s, t) \in S \times S\colon cs = ct\}$. Note that ρ is a right congruence on S.

If $I = \varnothing$ then define $\xi'\colon P \cup cS \rightarrow A$ by $\xi'(p) = \xi(p)$ and $\xi'(cs) = u$ for all $p \in P$ and $s \in S$, where $u \in A$ and $u = us$ for all $s \in S$. Then ξ' is an S-homomorphism properly extending ξ, which is a contradiction. Hence $I \neq \varnothing$ and it follows that I is a right ideal of S.

Define $\phi\colon I\rho \rightarrow A$ by $\phi([s]) = \theta(cs)$. It is easy to see that ϕ is a well defined S-homomorphism. We showed above that ϕ can therefore be extended to an S-homomorphism $\bar{\phi}\colon S/\rho \rightarrow A$. Then we may properly extend θ to an S-homomorphism $\bar{\theta}\colon P \cup cS \rightarrow A$, where $\bar{\theta}$ is defined by $\bar{\theta}(p) = \theta(p)$ for all $p \in P$ and $\bar{\theta}(cs) = \bar{\phi}([s])$ for all $s \in S$. Again, this is a contradiction. Thus $P = M$ and it follows that A is injective.

Since every ultraproduct of injective S-systems is injective, Theorem 5.1 gives that S is coherent. Let B be an absolutely pure S-system. Then B is an elementary substructure of a κ-saturated S-system A, where κ is a cardinal larger than the cardinality of S. From Theorems 5.1 and 6, A is absolutely pure and so by the above argument, A is injective. But then B is also injective, for the injective S-systems are by assumption an axiomatisable class.

As an immediate consequence of the theorem we have the following corollary first observed by Skornjakov in **[17]**.

COROLLARY 7.2. *If S is a strongly Noetherian monoid, then the class of injective S-systems in axiomatisable.*

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Department of Mathematics
University of Manchester
Oxford Road
Manchester