Separability in Semigroups

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Joint work with Craig Miller, Martyn Quick and Nik Ruškuc

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  - an algebra $A \in \mathcal{C}$,
  - an element $x \in A$,
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- A point x can be separated from $S \in A \backslash \{x\} \iff$ if there exists a finite index congruence ρ on A such that $[x]_\rho \neq [s]_\rho$ for all $s \in S$.

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- These are all finiteness conditions.

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- **Complete Separability**: Semigroups with finitely divisible subsets

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- Subsemigroups inherit any of these properties.

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- Finitely presented and weakly subsemigroup separable $\implies$ generalised word problem solvable.

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- Note that $\mathbb{N} \times \mathbb{Z}$ is weakly subsemigroup separable but $\mathbb{Z}$ isn't.

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- S is not completely separable.

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- Can extend this argument to show all free semigroups and all free commutative semigroups are completely separable.

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- A semigroup is completely separable $\iff$ the profinite topology is discrete.

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- The action of $\Gamma(H)$ on H is regular, i.e. transitive and free.

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- Every Schützenberger group of a **commutative** weakly subsemigroup separable semigroup is itself weakly subsemigroup separable.

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- **Answer:** No!
- **Example:** $\mathbb{Z}$. We cannot separate -1 from $\mathbb{N}$ in a finite quotient.

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- **Question:** Is every f.g. commutative semigroup strongly subsemigroup separable?
- **Answer:** No!
- **Example:** $\mathbb{Z}$. We cannot separate -1 from $\mathbb{N}$ in a finite quotient.
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- **Example:** $\mathbb{Z}$. We cannot separate -1 from $\mathbb{N}$ in a finite quotient.
- $\mathbb{Z}$ isn't even weakly subsemigroup separable.
- **Question:** Can we say when a f.g. commutative semigroup has one of our separability properties?

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- If we have an infinite $\mathcal{H}$-class H, then $\Gamma(H)$ is an infinite finitely generated abelian group - it contains a copy of $\mathbb{Z}$!
- We can use the action of $\mathbb{Z}$ to show that S cannot be weakly subsemigroup separable.
- For a finitely generated commutative semigroup to have one of our separability properties it is necessary for all $\mathcal{H}$-classes to be finite.

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- A f.g. commutative semigroup is strongly subsemigroup separable $\iff$ it is completely separable.
- For a f.g. commutative semigroup S the following are equivalent:
  - S is strongly subsemigroup separable;
  - if $a, b \in S$ such that $a \in b^n S$ for all $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $a = b^m a$.

The Main Theorem

Theorem

Let S be a finitely generated commutative semigroup. Then the following are equivalent:

1. *S is completely separable;*
2. *S is strongly subsemigroup separable;*
3. *S is weakly subsemigroup separable;*
4. *every $\mathcal{H}$-class of S is finite.*

The End

Thanks for listening