'Linear' representations of the polycyclic monoid

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The polycyclic monoids

Definition

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} | a_i^{-1} a_i = 1, a_i^{-1} a_j = 0, i \neq j \rangle$$

Required properties

They are congruence-free. (If $n > 1$.)

Representations of inverse semigroups

Definition

A representation of an inverse monoid M is a homomorphism $M \to \mathcal{I}(X)$ for some set X.

Or...

Definition2

Just vertices and arrows labelled by the generating set.

Representations of the polycyclic monoids

Observation

Since the polycyclics are congruence-free, if $\varphi\colon P_n \to \mathcal{I}(X)$ is a homomorphism, then $x_1\varphi, \ldots, x_n\varphi$ are total maps with disjoint images.

The representation is *strong* if X is the union of these images.

A transitive representation is *primitive* if every morphism from X to another transitive space is an equivalence.

Classification of representations of the polycyclic monoids

The list of transitive representations of the polycyclic monoids

- 1-infinite trees: all of these are equivalent. These are equivalent to the action of P_n on A_n^*. These actions are neither strong nor primitive.
- 2-infinite trees: these are characterized by an infinite string. These actions are strong. They are primitive if and only if the infinite string is aperiodic.
- 'Cycled' trees: these are described by the cycle's word - conjugate words correspond to equivalent representations. The representation is primitive if and only if the word is primitive. These representations are always strong.

Multiplicity-freeness

A representation is *multiplicity-free* if all of its orbits are of different type.

Definition

Let N be a non-singular $\nu \times \nu$ matrix over $\mathbb{Z}$, and let $|\det(N)| = n$. Then there exist $\mathbf{d}_1, \ldots, \mathbf{d}_n \in \mathbb{Z}^\nu$ such that

$$\mathbb{Z}^\nu = \bigcup_{i=1}^{n} N\mathbb{Z}^\nu + \mathbf{d}_i,$$

where the union is disjoint.

The representation

Given such a system, let

$$f_i \colon \mathbb{Z}^\nu \to \mathbb{Z}^\nu,\ x \mapsto Nx + \mathbf{d}_i.$$

Then $x_i \mapsto f_i$ extends to a representation of P_n.

One-dimensional case

In this case we have a natural number N and a complete residue system $d_1 < \ldots < d_N$ modulo N.

Some results

- The orbits of the resulting representations are all of the 3rd type, primitive and multiplicity-free, so they are characterized by a finite set of Lyndon words.
  We would like to calculate the number of these words (that is, the number of the orbits, or so-called cycles), and their total lengths (the number of atoms).
- The systems $(d_1, \ldots, d_n)$, $(d_1 - (n-1), \ldots, d_n - (n-1))$ and $(-d_1, \ldots, -d_n)$ give rise to equivalent representations.

One-dimensional case, $n = 2$

That is, we have only one parameter: an odd number $p > 0$.

x is an atom if and only if $-p \leq x \leq 0$. The length of the cycle of x is $\text{ord}_{\frac{p}{(x,p)}}(2)$. The cycle belonging to x is the periodic part of the fraction $\frac{x}{p}$.

If $p, q > 0$ are odd numbers and the representations corresponding to p and q are equivalent then $p = q$.

For any finite set of Lyndon words, there exists a parameter $p > 0$ such that the set of Lyndon words corresponding to the representation determined by p contains the given set.

'Smooth' one-dimensional case

The parameters are of the following form:
$k, k+M, \ldots, k+(N-1)M$ for some $0 \leq k < N-1$ and $M \in \mathbb{N}$.

x is an atom if and only if $-M \leq x \leq \frac{k}{N-1}$.

$$(N^l - 1)x = k\frac{N^l - 1}{N-1} + Mz$$

where $0 \leq z \leq N^l - 1$.

A conjecture

If the representations arising from the pairs (k, M) and (k', M') are equivalent then $(k, M) = (k', M')$.

One-dimensional case, $n \neq 2$

This case is completely different from the previous one!

If x is an atom then $-\frac{d_n}{N-1} \leq x \leq -\frac{d_1}{N-1}$.

A half lemma

If $d_1, \ldots, d_{N-1}$ are fixed and $d_n \to \infty$ then the number of atoms divided by d_N tends to 0.

Lemma

When $n = 3$, there exist different triples yielding equivalent representations.

More dimensions

So we have an $\nu \times \nu$ non-singular integer matrix N, and a set of vectors $d_1, \dots, d_n$. Furthermore, we suppose that the eigenvalues of N have absolute value greater than one.

In this case the resulting representation is multiplicity-free, and it has finitely many atoms.

Let

$$T = \left\{ \sum_{j=1}^{\infty} N^{-j} d_{i_j} | d_{i_j} \in \{d_1, \dots, d_n\} \right\}.$$

The set T

Properties of T

- It is the unique compact subset of $\mathbb{R}^\nu$ such that

$$N \cdot T = \bigcup_i (T + d_i).$$

- It has a non-empty interior and integer Lebesgue measure.
- If it has Lebesgue-measure 1 then it is a $\mathbb{Z}^\nu$-periodic tile.