About the generalised star-height problem and profinite identities

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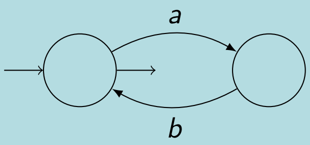
Rational languages

Automata



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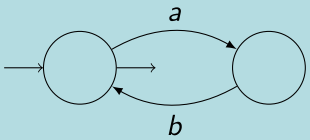


Rational languages

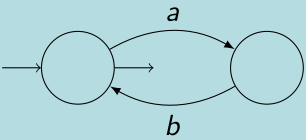
$$(ab)^*$$

Rational Expressions

Automata

Büchi

Logic



$MSO[<]$ on finite words

- starts with an a
- ends with a b
- the successor of an a is a b
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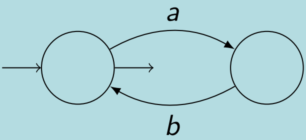

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- starts with an *a*
- ends with a *b*
- the successor of an *a* is a *b*
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Rational languages

$(ab)^*$

Finite monoids

$$\{a,b\}^* \xrightarrow{\varphi} M$$

- $M = \{1, a, b, ab, ba, 0\}$
- $a.b = ab, a.a = 0, \ldots$

$$\varphi^{-1}(ab) = (ab)^*$$

⟷ Kleene

Rational Expressions

Algebra

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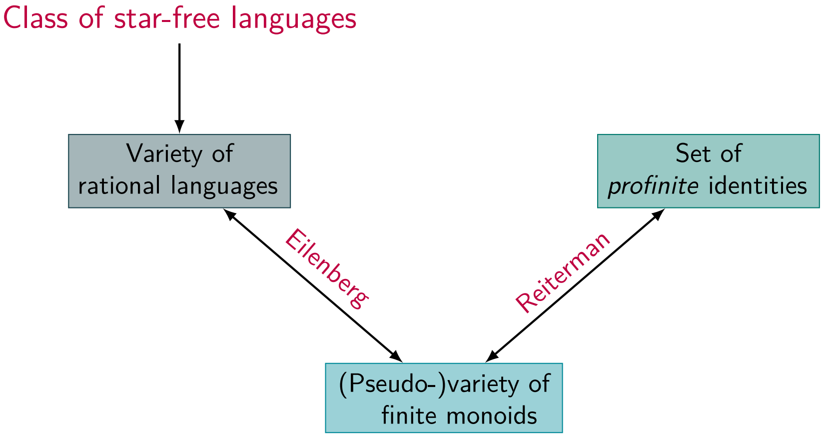
Varieties and identities



Varieties of languages

A variety of languages is a class of rational languages

$$\nu(A_1) \cup \nu(A_2) \cup \nu(A_3) \ldots$$

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- it is closed under inverse image: for each monoid morphism
  $\varphi : A_i^* \to A_j^*$, $L \in \nu(A_j)$ implies $\varphi^{-1}(L) \in \nu(A_i)$

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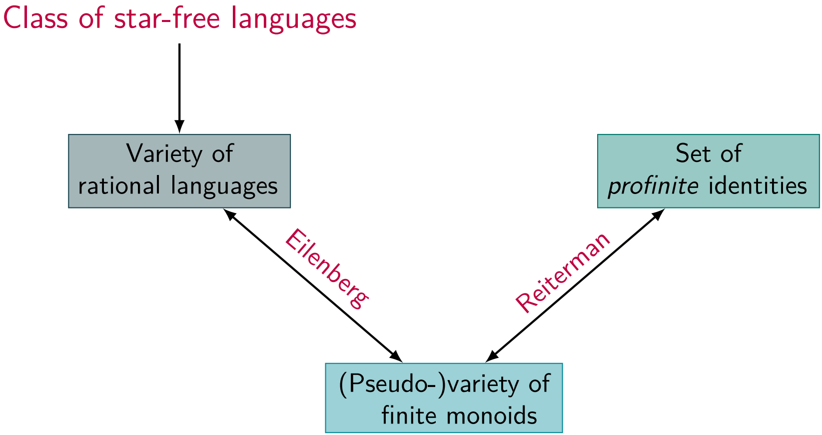

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d is an ultrametric distance:

- $d(u, v) = 0$ iff $u = v$
- $d(u, v) = d(v, u)$
- $d(u, v) \leqslant \max(d(u, w), d(w, v))$

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Example 3: $u \in A^*$, $n \in \mathbb{N}$ - $u^{n!}$ and $u^{(n+1)!}$?

The profinite world

Definition

Profinite monoid $\widehat{A^*}$:
completion of A^* with respect to the distance d.

- Monoid if u and v sequences of words, $(u.v)_n = u_n v_n$
- Metric space
- A^* dense subset
- Compact

VIP (very important profinite) words

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Idempotent power of $u \in A^*$

$$u^{\omega} = \lim_{n \to \infty} u^{n!}$$

Identities

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$ab = ba$

Languages with a zero

Zero (Reilly-Zhang 2000, Almeida-Volkov 2003)

$|A| \geqslant 2$

$u_0, u_1, \ldots$ an enumeration of the words of A^*

$v_0 = u_0, \quad v_{n+1} = (v_n u_{n+1} v_n)^{(n+1)!}$

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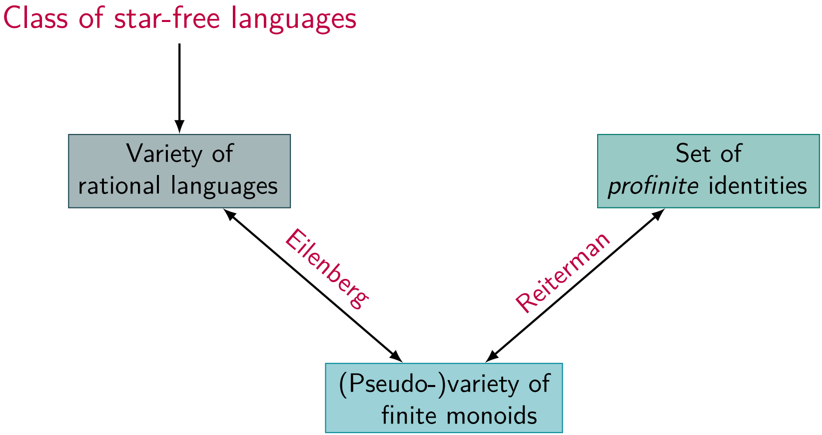
Languages with a sink state: $\rho_A u = u\rho_A = \rho_A$

Correspondance

Theorem

A class of languages is a variety if and only if it is defined by a set of profinite identities.

Varieties and identities



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$\rightarrow$ $(ab)^*$ is star-free.
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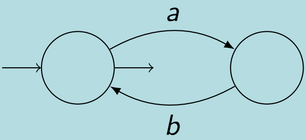

Counter-free automata

(McNaughton-Papert)

Logic

FO[<] on finite words

(McNaughton-Papert)

Star-free languages

(Schützenberger)

Smallest set:
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Rational Expressions

Aperiodic monoids

$$x^{\omega} = x^{\omega+1}$$

Algebra

The generalised star-height problem

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Examples of rational languages of a given generalised star-height?

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Examples of rational languages of a given generalised star-height?

$\longrightarrow$ OPEN : we do not even know if there exist a rational language with star-height at least 2.

Equations

Definition

Given two profinite words u, v, a rational language L satisfies

$$u \to v$$

if $u \in \overline{L}$ implies $v \in \overline{L}$

$a, b \in A$

Equation $ab \to aba$

$\{L \subseteq A^* \mid ab \notin L\} \cup \{L \subseteq A^* \mid ab, aba \in L\}$

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Equations

Theorem [Gehrke, Grigorieff, Pin 2008]

Classes of rational languages

- Lattice (union, intersection): $\rightarrow$
- Boolean algebra (lattice, complement): $\leftrightarrow$
- Lattice closed under quotient: $\leqslant$
- Boolean algebra closed under quotient: $=$

quotient : $u^{-1}Lv^{-1} = \{w \mid uwv \in L\}$

Equations for u^* [joint work with C.Paperman]

$$P_u = \bigcup_{p \text{ prefix of } u} u^* p \quad \text{and} \quad S_u = \bigcup_{s \text{ suffix of } u} s u^*$$

$$x^\omega y^\omega = 0 \text{ for } x, y \in A^* \text{ such that } xy \neq yx \qquad (E_1)$$

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DECIDABLE

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DECIDABLE Lattice closed under quotients

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DECIDABLE Boolean algebra closed under quotients

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DECIDABLE Boolean algebra

The Boolean algebra

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An example:

$$(a^2)^* - (a^6)^* = (a^6)^* a^2 \cup (a^6)^* a^4$$

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Equivalence relation over the integers

$r \equiv_m s$ if and only if $\gcd(r, m) = \gcd(s, m)$
$(u^m)^* u^r \subseteq L$ if and only if $(u^m)^* u^s \subseteq L$

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Equivalence relation over the integers

$r \equiv_m s$ if and only if $\gcd(r, m) = \gcd(s, m)$
$(u^m)^* u^r \subseteq L$ if and only if $(u^m)^* u^s \subseteq L$

$$2 \equiv_6 4 \text{ since } \gcd(2, 6) = 2 = \gcd(4, 6)$$
$$(u^6)^* u^2 \subseteq L \text{ if and only if } (u^6)^* u^4 \subseteq L$$

The Boolean algebra

$$x^\alpha \leftrightarrow x^\beta \text{ for all } (\alpha, \beta) \in \Gamma \qquad (E_7)$$

An example:

$$(a^2)^* - (a^6)^* = (a^6)^* a^2 \cup (a^6)^* a^4$$

$1 \quad a \quad a^2 \quad a^3 \quad a^4 \quad a^5 \quad a^6 \quad a^7 \quad a^8 \quad a^9 \quad a^{10} \quad a^{11} \quad a^{12} \quad a^{13} \quad a^{14} \quad \dots$

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$x^\alpha \leftrightarrow x^\beta$ for α and β representing sequences of integers $(km + r)_k$ and $(km + s)_k$ with $r \equiv_m s$...

The Boolean algebra

$$x^\alpha \leftrightarrow x^\beta \text{ for all } (\alpha, \beta) \in \Gamma \qquad (E_7)$$

An example:

$$(a^2)^* - (a^6)^* = (a^6)^* a^2 \cup (a^6)^* a^4$$

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Equivalence relation over the integers

$r \equiv_m s$ if and only if $\gcd(r, m) = \gcd(s, m)$
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$x^\alpha \leftrightarrow x^\beta$ for α and β profinite numbers in $\widehat{\mathbb{N}} = \widehat{\{a\}^*}$
satisfying some specific conditions...

The Boolean algebra

$$x^\alpha \leftrightarrow x^\beta \text{ for all } (\alpha,\beta)\in\Gamma \qquad (E_7)$$

An example:

$$(a^2)^* - (a^6)^* = (a^6)^* a^2 \cup (a^6)^* a^4$$

$1 \quad a \quad a^2 \quad a^3 \quad a^4 \quad a^5 \quad a^6 \quad a^7 \quad a^8 \quad a^9 \quad a^{10} \quad a^{11} \quad a^{12} \quad a^{13} \quad a^{14} \quad \ldots$

Γ is the set of all the pairs of profinite numbers $(dz^{\mathcal{P}}, dpz^{\mathcal{P}})$ s.t.:

- $\mathcal{P}$ is a cofinite sequence of prime numbers $\{p_1, p_2, \ldots\}$
- $z^{\mathcal{P}} = \lim_n (p_1 p_2 \ldots p_n)^{n!}$
- $p \in \mathcal{P}$
- if q divides d then $q \notin \mathcal{P}$

$$x^\alpha \leftrightarrow x^\beta \text{ for all } (\alpha,\beta)\in\Gamma \qquad (E_7)$$

The generalised star-height problem

- One can decide if a given rational language is star-free.
- $(aa)^*$ is not star-free.
- *Generalised star-height*: minimal number of nested stars in a generalised expression ($\cup$, $\cdot$, ^{c}, *) representing a rational language.

Examples of rational languages of a given generalised star-height?

$\longrightarrow$ OPEN : we do not even know if there exist a rational language with star-height at least 2.