PROPER WEAKLY LEFT AMPLE SEMIGROUPS

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ABSTRACT. Much of the structure theory of inverse semigroups is based on constructing arbitrary inverse semigroups from groups and semilattices. E-unitary (or proper) inverse semigroups are known to be P-semigroups (McAlister), or inverse subsemigroups of semidirect products of a semilattice by a group (O'Carroll) or C_u-semigroups built over an inverse category acted upon by a group (Margolis and Pin). On the other hand, every inverse semigroup is known to have an E-unitary inverse cover (McAlister).

The aim of this paper is to develop a similar theory for proper weakly left ample semigroups, a class with properties echoing those of inverse semigroups. We show how the structure of semigroups in this class is based on constructing semigroups from unipotent monoids and semilattices. The results corresponding to those of McAlister, O'Carroll and Margolis and Pin are obtained.

INTRODUCTION

The relation $\widetilde{\mathcal{R}}$ is defined on a semigroup S by the rule that $a \,\widetilde{\mathcal{R}}\, b$ if and only if a and b have the same set of idempotent left identities, that is, for all $e \in E(S)$, $ea = a$ if and only if $eb = b$. Green's relation $\mathcal{R}$ is contained in $\widetilde{\mathcal{R}}$, indeed $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$ where $a \,\mathcal{R}^*\, b$ if and only if $a \,\mathcal{R}\, b$ in some oversemigroup of S. When restricted to the regular elements of S all three relations $\widetilde{\mathcal{R}}$, $\mathcal{R}^*$ and $\mathcal{R}$ coincide.

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A semigroup S is said to be *left [semi]abundant* if each [$\widetilde{\mathcal{R}}$-class] $\mathcal{R}^*$-class contains an idempotent. A left [semi]abundant semigroup S such that $E(S)$ is a semilattice is said to be *left [semi]adequate*. It is easy to show that if S is left [semi]adequate then the idempotent in the $\mathcal{R}^*$-class [$\widetilde{\mathcal{R}}$-class] of $a \in S$ is unique: we denote this idempotent by a^+. Notice that in a left adequate semigroup $\mathcal{R}^* = \widetilde{\mathcal{R}}$ and so there is no ambiguity in this notation. Notice also that a^+ is the least element in the set of idempotents that are left identities of a.

In a semigroup S both equivalence relations $\mathcal{R}^*$ and $\mathcal{R}$ are left compatible but that may not be the case for $\widetilde{\mathcal{R}}$ (see [8]). A semigroup S is said to satisfy condition (CL) if $\widetilde{\mathcal{R}}$ is a left congruence on S.

A left semiadequate semigroup S that satisfies condition (CL) and in which, for all $a \in S$ and $e \in E(S)$,

$$ae = (ae)^+ a \qquad \text{(AL)}$$

is said to be *weakly left ample*.

In papers [9] and [10] we give a number of examples of left semiabundant and weakly left ample semigroups. We content ourselves here by remarking that any inverse semigroup is weakly left ample, but the class of weakly left ample semigroups is much wider, containing, for example, all unipotent monoids, and all Bruck-Reilly extensions or Brandt extensions of such.

When S is left adequate condition (CL) holds and S is said to be *left ample* (formerly left type-A, see for example [6]) when it satisfies condition (AL). Notice that the class of all inverse semigroups is properly contained in the class of all left ample semigroups. On a semigroup S with semilattice of idempotents $E(S)$ the relation σ defined as follows, for all $a,\ b \in S$,

$$a \,\sigma\, b \text{ if and only if } ea = eb, \text{ for some } e \in E(S),$$

is a right compatible equivalence. It is well known that when S is inverse σ is the least group congruence on S [14] and that when S is left ample σ is the least right cancellative monoid congruence on S [3].

In a sister paper [10], we show that when S is a weakly left ample *monoid* then σ is the least unipotent congruence on S. This proof can easily be modified to prove that if S is a weakly left ample *semigroup*, then σ is the least unipotent *monoid* congruence on S.

Inspired by the fact that an inverse semigroup S is E-unitary (formerly, proper) if and only if $\mathcal{R} \cap \sigma = \iota$, the identity relation on S [12, 13, 15], Fountain [3] introduced the class of proper left ample semigroups. A [weakly] left ample semigroup is said to be *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \iota$. Such semigroups are necessarily E-unitary but the converse, as proved in [3], is not true. Proper left ample semigroups have been widely studied, see for example [2, 3, 6, 16].

In the present paper our objective is to extend to proper weakly left ample semigroups the techniques of Margolis and Pin, introduced in [11]. These are used in [11] to describe E-unitary E-dense semigroups in terms of groups acting on categories. They are further developed by Fountain and Gomes [5, 6, 7] to study proper left ample monoids in terms of right cancellative monoids acting on categories. We aim to characterise proper weakly left ample semigroups by means of unipotent monoids acting on semigroupoids. By a semigroupoid we mean a category possibly without local identities [17] (or a "quiver" in the sense of [1]). Subsequently, we describe proper weakly left ample semigroups as certain subsemigroups of the semidirect product of a semilattice by a unipotent monoid. Clearly, by analogy with the inverse and the left ample cases we aim to finish by proving the existence of proper covers for arbitrary weakly left ample semigroups, an aim we achieve in Theorem 5.2.

Section 1 contains the main definitions and technical results used in the paper. Section 2 deals with the representation a proper weakly left ample semigroup S as a semigroup C_1 built from a proper 1-weakly left ample semigroupoid $\mathcal{C}$ acted upon 1-transitively and 1-injectively by a unipotent monoid M. We prove that, σ being the least unipotent monoid congruence on S, the semigroup S is isomorphic to the semigroup C_1 built from the derived semigroupoid of the canonical epimorphism $\theta : S \twoheadrightarrow S/\sigma,\ s \mapsto [s]$, acted upon by the unipotent monoid S/σ.

Section 3 answers the question of whether or not every proper weakly left ample semigroup can be embedded in the semidirect product of a semilattice by a unipotent monoid. Given a semigroupoid $\mathcal{C}$ and a unipotent monoid M under the above conditions, we prove that C_1 is embeddable into the wreath product $\mathcal{Y} \circledast M$, where $\mathcal{Y}$ is the semilattice of all ideals of $\mathrm{Mor}\,(1,1)$.

In Section 4 proper weakly left ample semigroups are described as strong $\mathcal{M}$-semigroups $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ is a semilattice, $\mathcal{Y}$ is a subsemilattice of $\mathcal{X}$ and M is a unipotent monoid acting on $\mathcal{X}$.

The last section deals with the construction of a proper weakly left ample cover of an arbitrary weakly left ample semigroup, following the techniques of [4] for E-dense monoids.

1. Preliminaries

Throughout this paper we regard weakly left ample semigroups [monoids] as algebras of type (2,1) [(2,1,0)], respectively. Here the unary operation is the map $a \mapsto a^+$, where a^+ is the unique idempotent in the $\widetilde{\mathcal{R}}$-class of a. With this signature, weakly left ample semigroups (monoids) are a quasivariety, axiomatised by Σ, $(\Sigma \cup \{x1 = 1x = x\})$, where Σ is the set

$$\begin{aligned}\{&(xy)z = x(yz); (x^2 = x \wedge y^2 = y) \Rightarrow xy = yx;\\ &(x^+)^2 = x^+;\ x^+x = x;\ (x^2 = x \wedge xy = y) \Rightarrow xy^+ = y^+;\\ &x^+ = y^+ \Rightarrow (zx)^+ = (zy)^+;\ x^2 = x \Rightarrow yx = (yx)^+y\}\end{aligned}$$

of quasi-identities. One of the uses of this approach is that, consequently, a morphism between weakly left ample semigroups must preserve $\widetilde{\mathcal{R}}$.

In what follows $\mathcal{C}$ is always a small *semigroupoid* in the sense of [17], that is, a category possibly without local identities, with set of *objects* $\mathrm{Obj}\,\mathcal{C}$ and set of *morphisms* $\mathrm{Mor}\,\mathcal{C}$. For all $v \in \mathrm{Obj}\,\mathcal{C}$, the set of all morphisms with *domain* [*codomain*] v is denoted by $\mathrm{Mor}\,(v, -)$ $[\mathrm{Mor}\,(-, v)]$. As in [11], we adopt additive notation for the composition of morphisms, though the operation is generally not commutative.

A morphism p is said to be *idempotent* whenever $p + p$ is defined and $p = p + p$. Clearly, if p is idempotent, then $p \in \mathrm{Mor}\,(u, u)$ for some object u. We denote by $E(\mathrm{Mor}\,(u, u))$ the set of all idempotents of the semigroup $\mathrm{Mor}\,(u, u)$ and by $E(\mathrm{Mor}\,\mathcal{C})$ the set of all idempotents of $\mathrm{Mor}\,\mathcal{C}$. Notice that as $\mathcal{C}$ is simply a semigroupoid $\mathrm{Mor}\,(u, u)$ may not be a monoid.

In a natural way we extend the definitions of left semiadequate and weakly left ample given for a semigroup. A semigroupoid $\mathcal{C}$ is said to be *left semiadequate* if and only if $E(\mathrm{Mor}\,(u, u))$ is a semilattice for any object u, and for every pair of objects u and v, and every $p \in \mathrm{Mor}\,(u, v)$, there exists an idempotent p^+ in $\mathrm{Mor}\,(u, u)$ such that

$$\begin{cases} p = p^+ + p, \quad \text{and} \\ p = r + p \quad \text{implies} \quad p^+ = r + p^+, \text{ for all } \ r \in E(\mathrm{Mor}\,(u, u))\,. \end{cases}$$

Since $E(\mathrm{Mor}\,(u, u))$ is a semilattice for each object u, it is easy to show that such an idempotent is necessarily unique. Notice also that $(p^+)^+ = p^+$ for any $p \in \mathrm{Mor}\,\mathcal{C}$, and that $p^+ = q^+$ implies $p, q \in \mathrm{Mor}\,(u, -)$ for some $u \in \mathrm{Obj}\,\mathcal{C}$.

A left semiadequate semigroupoid $\mathcal{C}$ is said to be *weakly left ample* if it satisfies the following conditions:

(CL) $p^+ = q^+$ implies $(r+p)^+ = (r+q)^+$,
for all $u \in \mathrm{Obj}\,\mathcal{C},\ p,q \in \mathrm{Mor}\,(u,-)$ and $r \in \mathrm{Mor}\,(-,u)$;

(AL) $p + s = (p+s)^+ + p$,
for all $u,v \in \mathrm{Obj}\,\mathcal{C},\ p \in \mathrm{Mor}\,(u,v)$ and $s \in E(\mathrm{Mor}\,(v,v))$.

The following lemma is easy to check and will be used frequently.

Lemma 1.1. *Let $\mathcal{C}$ be a weakly left ample semigroupoid and $p,q \in \mathrm{Mor}\,\mathcal{C}$ be such that $p+q$ is defined. Then*
(a) $(p+q)^+ = (p+q^+)^+$;
(b) $(p+q)^+ + p = p + q^+$.

A category is a semigroupoid with local identities and we may therefore consider weakly left ample categories. On the other hand, a monoid can be regarded as a category with a single object and in this case the definitions of weakly left ample monoid and weakly left ample category coincide. Also, a semigroup without identity can be looked upon as a semigroupoid with a single object and without a local identity. As in the monoid case, the definitions of weakly left ample for semigroupoids and for semigroups coincide. Clearly, in both cases, we may define the relation $\widetilde{\mathcal{R}}$ on $\mathrm{Mor}\,\mathcal{C}$ as follows: for all $p,q \in \mathrm{Mor}\,\mathcal{C}$, $p\ \widetilde{\mathcal{R}}\ q$ if and only if $p^+ = q^+$.

Notice that every left type-A category in the sense of [4], or *left ample* in the new terminology, is weakly left ample as for all $p,q \in \mathrm{Mor}\,\mathcal{C},\ p\ \mathcal{R}^*\ q$ if and only if $p^+ = q^+$.

By a 1-*weakly left ample semigroupoid* $\mathcal{C}$ we mean a weakly left ample semigroupoid with a distinguished object 1, such that $\mathrm{Mor}\,(1,1)$ is a semilattice. Such a semigroupoid is said to be *proper* if, for all $u \in \mathrm{Obj}\,\mathcal{C},\ p,q \in \mathrm{Mor}\,(1,u)$,

$$p^+ = q^+ \quad \text{implies} \quad p = q.$$

Lemma 1.2. *Let $\mathcal{C}$ be a proper 1-weakly left ample semigroupoid. Let $p,q \in \mathrm{Mor}\,(1,v)$, for some $v \in \mathrm{Obj}\,\mathcal{C}$. Then $p^+ + q = q^+ + p$.*

Proof. As $\mathcal{C}$ is a 1-weakly left ample semigroupoid, we have

$$(p^+ + q)^+ = p^+ + q^+ = q^+ + p^+ = (q^+ + p)^+,$$

with $p^+ + q,\ q^+ + p \in \mathrm{Mor}\,(1, v)$. Hence $p^+ + q = q^+ + p$, since $\mathcal{C}$ is proper.

A monoid M *acts* (on the left) on a semigroupoid $\mathcal{C}$ if there are maps $(t, u) \mapsto tu$ from $M \times \mathrm{Obj}\,\mathcal{C}$ into $\mathrm{Obj}\,\mathcal{C}$ and $(t, p) \mapsto tp$ from $M \times \mathrm{Mor}\,(u, v)$ into $\mathrm{Mor}\,(tu, tv)$, for all objects u, v of $\mathcal{C}$, such that the following conditions are satisfied, where $u, v, w \in \mathrm{Obj}\,\mathcal{C}$, $p \in \mathrm{Mor}\,(u, v)$, $q \in \mathrm{Mor}\,(v, w)$ and $t, h \in M$:

$$\begin{aligned} t(hu) &= (th)u, \\ 1u &= u, \\ t(p+q) &= tp + tq, \\ (th)p &= t(hp), \\ 1p &= p, \end{aligned}$$

and if $\mathcal{C}$ is a category $t0_u = 0_{tu}$.

When $\mathcal{C}$ is left semiadequate it is also required that

$$(tp)^+ = tp^+,$$

for all $t \in M$ and $p \in \mathrm{Mor}\,\mathcal{C}$.

When $\mathcal{C}$ has distinguished object 1, the action is said to be 1-*transitive* if for all $u \in \mathrm{Obj}\,\mathcal{C}$ there exists $t \in M$ such that $u = t1$, so that $\mathrm{Obj}\,\mathcal{C} = M1$; and it is said to be 1-*injective* if, for all $t, h \in M$,

$$t1 = h1 \quad \text{implies} \quad t = h\,.$$

Clearly, in the case when the action is both 1-transitive and 1-injective the correspondence $t \mapsto t1$ gives a bijection from M into $\mathrm{Obj}\,\mathcal{C}$ and so we may identify $\mathrm{Obj}\,\mathcal{C}$ with M.

2. E-UNITARY AND PROPER WEAKLY LEFT AMPLE SEMIGROUPS

In this section, we present a structure theorem for E-unitary and for proper weakly left ample semigroups in terms of semigroupoids acted upon by unipotent monoids. We recall that by an *E-unitary* semigroup S we mean a semigroup such that for all $a \in S$ and $e \in E(S)$, $ae \in E(S)$ or $ea \in E(S)$ implies $a \in E(S)$.

Let M be a monoid acting on a 1-weakly left ample semigroupoid $\mathcal{C}$. As in [11], the set

$$C_1 = \{(p,t) : t \in M,\ p \in \mathrm{Mor}\,(1,t)\}$$

is a semigroup under the operation defined by, for all $(p,t),(q,h) \in C_1$,

$$(p,t)(q,h) = (p + tq, th).$$

Lemma 2.1. *Let $\mathcal{C}$ be a 1-weakly left ample semigroupoid [category] and M a unipotent monoid acting on $\mathcal{C}$. Then C_1 is an E-unitary weakly left ample semigroup [monoid], with $E(C_1) \simeq \mathrm{Mor}\,(1,1)$.*

Proof. Once proved that

$$E(C_1) = \{(p,1) : p \in \mathrm{Mor}\,(1,1)\} \simeq \mathrm{Mor}\,(1,1)$$

and that, for any $(p,t) \in C_1$, we have $(p,t)^+ = (p^+,1)$, the rest of the proof is a routine matter.

Let S be a semigroup and M a monoid. Let $\phi : S \twoheadrightarrow M$ be a morphism from S onto M. As in [11] we define the (left) *derived semigroupoid* $\mathcal{D}$ of ϕ as follows:

$$\begin{aligned}&\mathrm{Obj}\,\mathcal{D} = M \text{ and, for } u,v \in \mathrm{Obj}\,\mathcal{D},\\ &\mathrm{Mor}\,(u,v) = \{(u,m,v) : m \in S, u(m\phi) = v\};\end{aligned}$$

composition is given by

$$(u,m,v) + (v,n,w) = (u,mn,w).$$

It is easy to prove that $\mathcal{D}$ is a semigroupoid and that, in particular, when S is a monoid $\mathcal{D}$ is a category.

We define an action (on the left) of M on $\mathcal{D}$ as follows: M acts on $\mathrm{Obj}\,\mathcal{D}$ by multiplication and for $(u,m,v) \in \mathrm{Obj}\,(u,v)$ and $t \in M$

$$t(u,m,v) = (tu,m,tv).$$

Let S be a weakly left ample semigroup. Recall from the Introduction that the relation σ defined on S by the rule that $a\,\sigma\,b$ if and only if $ea = eb$ for some $e \in E(S)$, is a right compatible equivalence relation. In [10] we prove that if S is a *monoid*, then σ is the least unipotent congruence on S. Minor observations yield the following result for *semigroups*.

Lemma 2.2. *Let S be a weakly left ample semigroup. Then σ is the least unipotent monoid congruence on S.*

Proof. As in [10], use of (AL) gives that σ is a congruence and S/σ is unipotent. For any $a \in S$, $a^+a = a$ and

$$aa^+ = (aa^+)^+a \; \sigma \; a^+a = a$$

so that the idempotent of S/σ is the identity. It is then clear that σ is the least unipotent *monoid* congruence on S.

As in [3] for right ample semigroups (the dual case), we say that a weakly left ample semigroup S is *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \iota$, the identity relation on S. In [3] it is proved that proper implies E-unitary, but the converse is not true.

The same arguments as in [6] for the left ample case can be used to prove the following lemma.

Lemma 2.3. *Let S be a weakly left ample semigroup. The following conditions are equivalent:*

- (a) *S is E-unitary,*
- (b) *for all $a \in S$, $e \in E(S)$, $ae \in E(S)$ $[ea \in E(S)]$ implies $a \in E(S)$,*
- (c) *$E(S)$ is a σ-class, the unique idempotent in S/σ.*

Next, we characterize the derived semigroupoid of the canonical epimorphism associated with σ.

Lemma 2.4. *Let S be an E-unitary $[$proper$]$ weakly left ample semigroup. Then the derived semigroupoid $\mathcal{D}$ of the canonical epimorphism associated with σ, is a $[$proper$]$ 1-weakly left ample semigroupoid. Moreover, the action of S/σ on $\mathcal{D}$ is both 1-transitive and 1-injective.*

Proof. First notice that, for any $u \in S/\sigma$

$$E(\mathrm{Mor}\,(u,u)) = \{(u,e,u) : e \in E(S)\} \simeq E(S).$$

In particular, when S is E-unitary by Lemma 2.3 we have

$$\mathrm{Mor}\,(1,1) = \{(1,e,1) : e \in E(S)\}.$$

Next, show that $(u,m,v)^+ = (u,m^+,u)$, for any $(u,m,v) \in \text{Mor}\,(u,v)$. It is then clear that the action of S/σ on $\mathcal{D}$ respects the unary operation $^+$. The rest of the proof is easy to check.

Theorem 2.5. *A semigroup S is an E-unitary [proper] weakly left ample semigroup if and only if S is isomorphic to a semigroup C_1, where $\mathcal{C}$ is a [proper] 1-weakly left ample semigroupoid acted upon 1-transitively and 1-injectively by a unipotent monoid M.*

Proof. Suppose that S is an E-unitary [proper] weakly left ample semigroup. In view of Lemma 2.4, it suffices to show that $\psi : S \to C_1$ defined by

$$m\psi = ((1,m,m\sigma),m\sigma)$$

is an isomorphism. This follows by the usual argument as in [5, 6].

Conversely, by Lemma 2.1 it remains to show that C_1 is proper whenever $\mathcal{C}$ is proper. Suppose that $\mathcal{C}$ is proper. Let $(p,t),(q,h) \in C_1$ be such that $(p,t)\ \widetilde{\mathcal{R}} \cap \sigma\ (q,h)$. Then $(p,t)^+ = (q,h)^+$ and $(r,1)(p,t) = (r,1)(q,h)$ for some idempotent $(r,1)$. Hence, $(p^+,1) = (q^+,1)$ and $t = h$. Thus, $p,q \in \text{Mor}\,(1,t)$ and $p^+ = q^+$, whence $p = q$ since $\mathcal{C}$ is proper.

In the particular case of S being a left ample monoid, our Theorem 2.5 gives a new version of Theorem 2.2 of [6], where the action is on the left instead of being on the right. There are very significant differences when we change the side of the action. Assuming the action on the right, in [6] it is proved that any proper left ample monoid is embeddable into a special submonoid of a semidirect product $T * Y$ where T is a right cancellative monoid acting on the right on a semilattice Y. On the other hand, in [2] every proper left ample semigroup is proved to be embeddable into a wreath product $Y \circledast T$, where T acts on the left on Y, on the lines of Theorem 4.10 of [5], which concerns the E-dense case. Notice however that, in both approaches, when dealing with the monoid case the embedding obtained is only a (2,1)-morphism. In the next section, having fixed the action on the left, we show how to extend Billhardt's result [2] for proper weakly left ample semigroups.

Another possible way of looking at E-unitary [proper] weakly left ample semigroups is as extensions of unipotent monoids by semilattices [injective on $\widetilde{\mathcal{R}}$-classes].

A semigroup S is an *extension of a unipotent monoid* M *by a semilattice* Y if and only if there exists a morphism $\phi : S \twoheadrightarrow M$ from S onto M such that $1\phi^{-1} \simeq Y$. In the next theorem we prove that every E-unitary weakly left ample semigroup S can be regarded as an extension of a unipotent monoid ($\simeq S/\sigma$) by a semilattice Y ($\simeq E(S)$).

Theorem 2.6. *Let S be a weakly left ample semigroup. The following conditions are equivalent:*

(a) *S is E-unitary [proper];*

(b) *there exists a morphism $\phi : S \twoheadrightarrow M$ from S onto a unipotent monoid M, such that $1\phi^{-1} = E(S)$ [and ϕ is injective on $\widetilde{\mathcal{R}}$-classes].*

Proof. Suppose that (a) holds. Consider the unipotent monoid $M = S/\sigma$. As S is E-unitary, by Lemma 2.3, $E(S)$ is the identity of M and the canonical epimorphism $\phi : S \twoheadrightarrow M,\ a \mapsto [a]$, is such that $1\phi^{-1} = E(S)$. Let S be proper and $a, b \in S$ be such that $a\phi = b\phi$ and $a \mathrel{\widetilde{\mathcal{R}}} b$. Then $a = b$, since $\widetilde{\mathcal{R}} \cap \sigma = \iota$.

Conversely, suppose that (b) holds. Let $e \in E(S)$ and $a \in S$ be such that $ea \in E(S)$. Then $(ea)\phi = 1 = a\phi$ since ϕ is a morphism and M is unipotent. Thus $a \in 1\phi^{-1} = E(S)$ and so S is E-unitary. Assume also that ϕ is injective on $\widetilde{\mathcal{R}}$-classes. First notice that $\sigma \subseteq \operatorname{Ker}\phi$ since $\operatorname{Ker}\phi$ is a unipotent monoid congruence on S and σ is the least such. Hence if $a, b \in S$ are such that $a \mathrel{\widetilde{\mathcal{R}} \cap \sigma} b$ then $a \mathrel{\widetilde{\mathcal{R}}} b$ and $a\phi = b\phi$. Thus $a = b$ and S is proper.

3. Semidirect products of unipotent monoids by semilattices

In this section we prove that any proper weakly left ample semigroup can be embedded in a semidirect product $Y * M$ of a unipotent monoid M by a semilattice Y, where M acts on the left on Y.

To obtain our result we use wreath products as in [2], however our proofs are based on graphical/categorical methods as in [6, 11].

Let Y be a semilattice and M a monoid. As usual, we say that M acts (on the left) on Y if we have a map $M \times Y \to Y$, $(t, a) \mapsto ta$, such that for all $t, h \in M$ and $a, b \in Y$,

$$\begin{aligned} t(a \wedge b) &= ta \wedge tb, \\ t(ha) &= (th)a, \\ 1a &= a. \end{aligned}$$

If M acts on Y then we can form the semidirect product $Y * M$ of Y by M, by defining an operation on $Y \times M$ as follows: for all $(a, t), (b, h) \in Y \times M$,

$$(a, t)(b, h) = (a \wedge tb, th).$$

It is easy to check that the following proposition holds.

Proposition 3.1. *Let $S = Y * M$ be the semidirect product of a semilattice Y by a unipotent monoid M, then*

a) $E(S) = \{(a, 1) : a \in Y\} \simeq Y$;

b) *for all* $(a, t), (b, h) \in S$, $(a, t) \widetilde{\mathcal{R}} (b, h)$ *if and only if* $a = b$;

c) *for all* $(a, t) \in S$, $(a, t)^+ = (a, 1)$;

d) *for all* $(a, t), (b, h) \in S$, $(a, t)\ \sigma\ (b, h)$ *if and only if* $t = h$;

e) $S/\sigma \simeq M$;

f) *S is a proper weakly left ample semigroup;*

g) *if Y has a greatest element w such that $tw = w$ for all $t \in M$, then S is a monoid.*

Corollary 3.2. *Let Y be a semilattice [with greatest element w] and M be a right cancellative monoid [such that $tw = w$ for all $t \in M$]. Then $S = Y * M$ is a proper left ample semigroup [monoid].*

Proof. In this case, we have $(a, t)\ \mathcal{R}^*\ (a, 1)$, for all $(a, t) \in S$, and it follows that $\widetilde{\mathcal{R}} = \mathcal{R}^*$. Thus the result is a consequence of the previous proposition.

Clearly, the next natural step is to show that every proper weakly left ample semigroup [monoid] is embeddable in the semidirect product of a semilattice [with greatest element] by a unipotent monoid. The usual technique of using a set of ideals as the semilattice [6, 11] does not seem to work here, essentially because the maximum unipotent monoid image of a proper weakly left ample semigroup need be neither left nor right cancellative. However we may reformulate Billhardt's wreath product [2] in terms of categories/semigroupoids and introduce the necessary readjustments to obtain the desired result.

First recall the definition of the wreath product $Y \circledast M$ of a semilattice Y by a monoid M.

Let $F = Y^M$ be the set of all maps from M into Y. On F we define an operation $\wedge$ as follows: for all $x \in M$ and $f, g \in F$,

$$x(f \wedge g) = xf \wedge xg.$$

Clearly $(F, \wedge)$ is a semilattice. Next, we can define an action (on the left) of M on F as follows: for all $t,\ x \in M$ and $f \in F$

$$x(tf) = (xt)f.$$

The *wreath product* $Y \circledast M$ is defined to be the semidirect product $F * M$ of F by M.

Let $\mathcal{C}$ be a proper 1-weakly left ample semigroupoid and M a unipotent monoid acting (on the left) 1-transitively and 1-injectively on $\mathcal{C}$. Let $\mathcal{Y}$ denote the set of all ideals of the semilattice $\mathrm{Mor}\,(1,1)$. Notice that as $\mathrm{Mor}\,(1,1)$ is a semilattice the operations of sum and intersection coincide on $\mathcal{Y}$ and under this operation $\mathcal{Y}$ is a semilattice. Further, if $p \in \mathrm{Mor}\,(1,1)$, the principal ideal $I(p)$ generated by p is $\mathrm{Mor}\,(1,1) + p$ and has unique generator p. In the following, for a subset P of $\mathrm{Mor}\,\mathcal{C}$, P^+ denotes the set $\{p^+ : p \in P\}$.

Theorem 3.5. *Let $\mathcal{C}$ be a proper 1-weakly left ample semigroupoid acted upon 1-transitively and 1-injectively by a unipotent monoid M. Then the proper weakly left ample semigroup C_1 is embeddable into the wreath product $\mathcal{Y} \circledast M$ of the semilattice $\mathcal{Y}$ of all ideals of* $\mathrm{Mor}\,(1,1)$ *by the monoid M.*

Proof. For each $(p,t) \in C_1$, we define a map

$$\begin{aligned} f_{(p,t)} : M &\longrightarrow \mathcal{Y} \\ h &\mapsto (\mathrm{Mor}\,(1,h) + hp)^+. \end{aligned}$$

In fact $(\mathrm{Mor}\,(1,h)+hp)^+ \subseteq \mathrm{Mor}\,(1,1)$ and

$$\mathrm{Mor}\,(1,1)+(\mathrm{Mor}\,(1,h)+hp)^+ \subseteq (\mathrm{Mor}\,(1,h)+hp)^+$$

since for all $r\in \mathrm{Mor}\,(1,1)$ and $q\in \mathrm{Mor}\,(1,h)$,

$$\begin{aligned} r+(q+hp)^+ &= (r+(q+hp)^+)^+ \\ &= (r+q+hp)^+. \end{aligned}$$

Thus $f_{(p,t)}$ is well defined and so we may consider the map

$$\begin{aligned} \psi:\quad C_1 &\longrightarrow \mathcal{Y}\circledast M \\ (p,t) &\mapsto (f_{(p,t)},t). \end{aligned}$$

To prove that ψ is injective let (p,t), $(q,h)\in C_1$ be such that $(p,t)\psi = (q,h)\psi$. Then $t=h$ and $f_{(p,t)}=f_{(q,t)}$. Thus, in particular, $1f_{(p,t)}=1f_{(q,t)}$, that is,

$$(\mathrm{Mor}\,(1,1)+p)^+ = (\mathrm{Mor}\,(1,1)+q)^+.$$

From Lemma 1.1, $I(p^+)=I(q^+)$ so that $p^+=q^+$. As $\mathcal{C}$ is proper, it follows that $p=q$. Therefore ψ is injective.

Next, we show that ψ is a morphism. Let (p,t), $(q,h)\in C_1$. Then

$$((p,t)(q,h))\psi = (p+tq,th)\psi = (f_{(p+tq,th)},th)$$

and

$$(p,t)\psi(q,h)\psi = (f_{(p,t)},t)(f_{(q,h)},h) = (f_{(p,t)}\wedge tf_{(q,h)},th).$$

To prove that $f_{(p+tq,th)} = f_{(p,t)}\wedge tf_{(q,h)}$, according to the definition of the wreath product, we need to show that, for all $m\in M$,

$$(\mathrm{Mor}\,(1,m)+m(p+tq))^+ = (\mathrm{Mor}\,(1,m)+mp)^+ + (\mathrm{Mor}\,(1,mt)+mtq)^+.$$

Now, since $\mathrm{Mor}\,(1,m)+mp\subseteq \mathrm{Mor}\,(1,mt)$ we have, by Lemmas 1.1 and 1.2,

$$\begin{aligned} &(\mathrm{Mor}\,(1,m)+mp)^+ + (\mathrm{Mor}\,(1,mt)+mtq)^+ \\ &= \big((\mathrm{Mor}\,(1,m)+mp)^+ + \mathrm{Mor}\,(1,mt)+mtq\big)^+ \\ &= ((\mathrm{Mor}\,(1,mt))^+ + \mathrm{Mor}\,(1,m)+mp+mtq)^+ \\ &\subseteq (\mathrm{Mor}\,(1,m)+m(p+tq))^+\,. \end{aligned}$$

Conversely, to prove the other inclusion, let $r \in \mathrm{Mor}\,(1, m)$. Then, using condition (AL),

$$\begin{aligned}(r + m(p + tq))^+ &= (r + mp + mtq)^+ = (r + mp + mtq^+)^+ \\ &= \big((r + mp + mtq^+)^+ + r + mp\big)^+ \\ &= (r + mp + mtq)^+ + (r + mp)^+ \\ &= (r + mp)^+ + (r + mp + mtq)^+ \\ &\in (\mathrm{Mor}\,(1, m) + mp)^+ + (\mathrm{Mor}\,(1, mt) + mtq)^+\end{aligned}$$

Thus the required equality. Hence ψ preserves multiplication.

It remains to prove that ψ respects the unary operation $+$. Let $(p, t) \in C_1$. Then

$$((p, t)\psi)^+ = (f_{(p,t)}, t)^+ = (f_{(p,t)}, 1)$$

and

$$(p, t)^+\psi = (p^+, 1)\psi = (f_{(p^+,1)}, 1).$$

For any $h \in M$,

$$hf_{(p,t)} = (\mathrm{Mor}\,(1, h) + hp)^+ = (\mathrm{Mor}\,(1, h) + hp^+)^+ = hf_{(p^+,1)}.$$

Thus $((p, t)\psi)^+ = (p, t)^+\psi$ and ψ is an embedding.

Finally, we can state our main result, that follows from Theorems 2.5 and 3.5.

Theorem 3.6. *Every proper weakly left ample semigroup is embeddable into a semidirect product of a semilattice by a unipotent monoid.*

Corollary 3.7. *Every proper left ample semigroup is embeddable into a semidirect product of a semilattice by a right cancellative monoid.*

The semigroups that arise in Theorem 3.6 are rather large as compared with the original semigroup. If S is a proper weakly left ample semigroup, then we have shown that S is embeddable in $\mathcal{Y}^M * M$, where $\mathcal{Y}$ is (isomorphic to) the semilattice of ideals of $E(S)$ and $M = S/\sigma$. The cardinality of $\mathcal{Y}^M$ is potentially as large as $2^{|S|}$. However one can always find a *subsemilattice*

$\mathcal{Z}$ of $\mathcal{Y}^M$ such that $M\mathcal{Z} \subseteq \mathcal{Z}$, if S is infinite $|Z| \leq |S|$, and S is embedded in $\mathcal{Z} * M$. For if $\theta : S \to \mathcal{Y}^M * M$ is an embedding, let

$$\mathcal{X} = \{\alpha \in \mathcal{Y}^M : (\alpha, u) \in \text{ im } \theta \text{ for some } u \in M\}.$$

Then $|M\mathcal{X}| \leq |S|$; denoting now by $\mathcal{Z}$ the subsemilattice of $\mathcal{Y}^M$ generated by $M\mathcal{X}$, we have that $M\mathcal{Z} \subseteq \mathcal{Z}$ and $|\mathcal{Z}| \leq |S|$. Clearly im $\theta \subseteq \mathcal{Z} * M$.

Alternatively we can construct a semilattice $\mathcal{U}$ *directly* from S, without the use of wreath product, such that S is embedded in a semidirect product $\mathcal{U}*M$ and this embedding is 'universal' in the sense that if S is embedded in a semidirect product $\mathcal{V}*N$, where $\mathcal{V}$ is a semilattice acted upon by a unipotent monoid N, then there is a morphism from $\mathcal{U} * M$ to $\mathcal{V} * N$. Further, if S is infinite, then $|\mathcal{U}| \leq |S|$. The construction of $\mathcal{U}$ has the advantages of directness and universality but looses the connection with ideals. This latter connection clarified the proof of Theorem 3.6; indeed we use Theorem 3.6 to show that S is *embedded* in $\mathcal{U} * M$.

Let S and M be as above; put $E = E(S)$. Certainly M acts on the direct product $E \times M$ by $m(e, n) = (e, mn)$, for all $m, n \in M$ and $e \in E$. Let $\mathcal{F}$ be the free semilattice on $E \times M$, so that $\mathcal{F}$ consists of finite subsets of $E \times M$ under union. Clearly the action of M on $E \times M$ extends to an action on $\mathcal{F}$. Let ρ be the congruence on $\mathcal{F}$ generated by X, where

$$X = \{(\{(p^+, t), (q^+, t[p])\}, \{((pq)^+, t)\}) : p, q \in S, t \in M\}.$$

Notice that if $(A, B) \in X$, then $(mA, mB) \in X$ for all $m \in M$. It follows that M acts on the quotient semilattice $\mathcal{U} = \mathcal{F}/\rho$, where $m[A] = [mA]$.

Define $\nu : S \to \mathcal{U} * M$ by

$$s\nu = ([\{(s^+, 1)\}], [s]).$$

Lemma 3.8 *The function ν is a morphism.*

Proof. Let $p, q \in S$. Then

$$\begin{aligned} p\nu q\nu &= ([\{(p^+, 1)\}], [p])([\{(q^+, 1)\}], [q]) \\ &= ([\{(p^+, 1)\}] \wedge [p][\{(q^+, 1)\}], [p][q]) \\ &= ([\{(p^+, 1), (q^+, [p])\}], [pq]) \\ &= ([\{((pq)^+, 1)\}], [pq]) \\ &= (pq)\nu. \end{aligned}$$

Further, using Proposition 3.1,

$$(p\nu)^+ = ([\{(p^+,1)\}],[p])^+ = ([\{(p^+,1)\}],1)$$
$$= ([\{((p^+)^+,1)\}],[p^+]) = p^+\nu$$

so that ν is a morphism.

If S is infinite, the cardinality of $\mathcal{U}$ above is no greater than that of $\mathcal{F}$, which is bounded above by $|S|$.

Proposition 3.9 *Let S be a proper weakly left ample semigroup with $M = S/\sigma$, and let $\nu : S \to \mathcal{U} * M$ be defined as above.*

Let $\mathcal{V}$ be a semilattice acted upon by a unipotent monoid N and suppose that $\phi : S \to \mathcal{V} * N$ is a morphism. Then there exists a morphism $\overline{\phi} : \mathcal{U} * M \to \mathcal{V} * N$ such that $\nu\overline{\phi} = \phi$.

Proof. Suppose that $\mathcal{V}, N$ and ϕ exist as given. Let $\psi : S \to N$ be the composition of ϕ with the projection onto the second coordinate. Then ψ is a morphism, so that as N is unipotent $\sigma \subseteq \ker \psi$ and there is a morphism $\tau : M \to N$ such that $\sigma^\natural\tau = \psi$.

For any $s \in S$ we write $s\phi = (A_s, s\psi)$. Notice that as ϕ preserves $+$,

$$(A_{s^+}, 1) = s^+\phi = (s\phi)^+ = (A_s, s\psi)^+ = (A_s, 1),$$

using Proposition 3.1. Thus $A_s = A_{s^+}$. An easy computation using the fact that ϕ is a semigroup morphism yields

$$A_{(st)^+} = A_{s^+} \wedge (s\psi)A_{t^+} \qquad (*),$$

for any $s, t \in S$.

Recalling that $\mathcal{F}$ is the *free* semilattice on $E \times M$, we may define $\alpha : \mathcal{F} \to \mathcal{V}$ by

$$\{(e,m)\}\alpha = (m\tau)A_e.$$

Then for any $m \in M$ and $f \in \mathcal{F}$ we have that

$$(mf)\alpha = (m\tau)(f\alpha) \qquad (**).$$

Using $(*)$ we see that $X \subseteq \ker \alpha$, so that $\rho \subseteq \ker \alpha$ and there is a (semigroup) morphism $\beta : \mathcal{U} = \mathcal{F}/\rho \to \mathcal{V}$ such that $\rho^\natural\beta = \alpha$.

We now define $\overline{\phi} : \mathcal{U} * M \to \mathcal{V} * N$ by

$$([f], m)\overline{\phi} = ([f]\beta, m\tau).$$

Let $([f], m), ([g], n) \in \mathcal{U} * M$. Then

$$(([f], m)([g], n))\overline{\phi} = ([f] \wedge m[g], mn)\overline{\phi}$$

$$= (([f] \wedge m[g])\beta, (mn)\tau) = (([f] \wedge [mg])\beta, (mn)\tau) = ([f]\beta \wedge [mg]\beta, (mn)\tau).$$

But, using $(**)$,

$$[mg]\beta = (mg)\rho^\natural\beta = (mg)\alpha = (m\tau)(g\alpha) = (m\tau)[g]\beta.$$

Hence

$$(([f], m)([g], n))\overline{\phi} = ([f]\beta \wedge (m\tau)[g]\beta, m\tau n\tau)$$
$$= ([f]\beta, m\tau)([g]\beta, n\tau) = ([f], m)\overline{\phi}\,([g], n)\overline{\phi}.$$

It is clear from Proposition 3.1 that $\overline{\phi}$ preserves $+$. Thus $\overline{\phi}$ is a morphism.

To see that $\nu\overline{\phi} = \phi$, let $s \in S$. Then

$$s\nu\overline{\phi} = ([\{(s^+, 1)\}], [s])\overline{\phi} = ([\{(s^+, 1)\}]\beta, [s]\tau)$$

$$= (\{(s^+, 1)\}\alpha, s\psi) = (A_{s^+}, s\psi) = s\phi$$

as required.

Corollary 3.10 *Let S be a proper weakly left ample semigroup. Put $M = S/\sigma$ and let $\mathcal{U}, \nu$ be defined as above. Then $\nu : S \to \mathcal{U} * M$ is an embedding.*

Proof. It remains to show that ν is one-one.

By Theorem 3.6, there is an embedding, say ϕ, of S into a semidirect product of a semilattice by a unipotent monoid. By Proposition 3.9, $\nu\overline{\phi} = \phi$, so that ν is one-one as required.

4. $\mathcal{M}$-SEMIGROUPS

In this section we present a different characterisation of a proper weakly left ample semigroup [monoid] S as a strong $\mathcal{M}$-semigroup [$\mathcal{M}$-monoid] $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ is a semilattice, $\mathcal{Y}$ is a certain subsemilattice of $\mathcal{X}$

with an upper bound $\varepsilon \in \mathcal{X}$ and M is a unipotent monoid acting (on the left) on $\mathcal{X}$.

Let $\mathcal{X}$ be a partially ordered set and $\mathcal{Y}$ a subset of $\mathcal{X}$ with an upper bound $\varepsilon \in \mathcal{X}$. Let M be a unipotent monoid acting (on the left) on $\mathcal{X}$, in such a way that

(1) $\forall a \in \mathcal{X}, \quad 1a = a,$
(2) $\forall a, b \in \mathcal{X},\ \forall t \in M, \quad a \leq b \Rightarrow ta \leq tb,$
(3) $\forall a \in \mathcal{X},\ \forall h, t \in M, \quad (ht)a = h(ta)$

and

(A) $\forall t \in M,\ \exists a \in \mathcal{Y}, \quad a \leq t\varepsilon,$
(B) $\forall a, b \in \mathcal{Y},\ \forall t \in M, \quad a \leq t\varepsilon \Rightarrow a \wedge tb$ exists and is in $\mathcal{Y}$,
(C) $\forall a, b, c \in \mathcal{Y},\ \forall h, t \in M,$

$$a \leq t\varepsilon,\ b \leq h\varepsilon \Rightarrow (a \wedge tb) \wedge thc = a \wedge t(b \wedge hc).$$

A triple $(M, \mathcal{X}, \mathcal{Y})$ satisfying the above conditions is said to be an $\mathcal{M}$-*triple*. Given an $\mathcal{M}$-triple $(M, \mathcal{X}, \mathcal{Y})$, define

$$\mathcal{M}(M, \mathcal{X}, \mathcal{Y}) = \{(a, t) \in \mathcal{Y} \times M : a \leq t\varepsilon\}$$

with multiplication given by

$$(a, t)(b, h) = (a \wedge tb, th).$$

Notice that, for all $a, b \in \mathcal{Y}$, we have $a \wedge b \in \mathcal{Y}$ in view of condition (B) since $a \leq \varepsilon$. Thus $\mathcal{Y}$ is a subsemilattice of $\mathcal{X}$.

On the other hand, if $\varepsilon \in \mathcal{Y}$ then ε is the maximum element of $\mathcal{Y}$.

It is a routine matter to prove the following result.

Lemma 4.1 *Let $(M, \mathcal{X}, \mathcal{Y})$ be a $\mathcal{M}$-triple. Then $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ is a proper weakly left ample semigroup such that*

$$E(\mathcal{M}(M, \mathcal{X}, \mathcal{Y})) \simeq \mathcal{Y} \text{ and } \mathcal{M}(M, \mathcal{X}, \mathcal{Y})/\sigma \simeq M.$$

Moreover, if $\varepsilon \in \mathcal{Y}$ then $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ is a monoid with identity $(\varepsilon, 1)$.

Next, consider the particular case of $\mathcal{X}$ a semilattice rather than simply a partially ordered set. We suppose that the action of $\mathcal{M}$ over $\mathcal{X}$ satisfies the stronger condition

(2′) $\forall a, b \in \mathcal{X},\ \forall t \in M, \quad t(a \wedge b) = ta \wedge tb.$

Notice that in this case, condition (C) always holds.

Let $\mathcal{X}$ be a semilattice, $\mathcal{Y}$ a subsemilattice of $\mathcal{X}$ with an upperbound $\varepsilon \in \mathcal{X}$ and M a unipotent monoid acting on $\mathcal{X}$ in such a way that conditions (1), (2′), (3), (A) and (B) hold. Then the triple $(M, \mathcal{X}, \mathcal{Y})$ is said to be a *strong $\mathcal{M}$-triple* and $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ is a *strong $\mathcal{M}$-monoid.*

Observe that given a strong $\mathcal{M}$-triple $(M, \mathcal{X}, \mathcal{Y})$ the semigroup $\mathcal{M}(M, \mathcal{X}, \mathcal{Y})$ is a subsemigroup of the semidirect product $\mathcal{X} * M$.

In the next theorem we show that any proper weakly left ample semigroup [monoid] is isomorphic to a strong $\mathcal{M}$-semigroup [$\mathcal{M}$-monoid], obtaining the analogue of [3, Theorem 4.4] for proper left ample monoids.

Theorem 4.2. *A semigroup [monoid] is proper weakly left ample if and only if it is isomorphic to a strong $\mathcal{M}$-semigroup [monoid].*

Proof. Let S be a proper weakly left ample semigroup. Then, by Theorem 2.5, we have $S \simeq C_1$ where $\mathcal{C}$ is a proper 1-weakly left ample semigroupoid acted upon 1-transitively and 1-injectively by the unipotent monoid $M = S/\sigma$. At this point, notice that we may assume $\mathcal{C}$ to be such that, for all $t \in M$, $\mathrm{Mor}\,(1, t) \neq \emptyset$ since this is clearly true in the derived semigroupoid.

By Theorem 3.5, the map $\psi : C_1 \to \mathcal{Y}^M * M,\ (p, t) \mapsto (f_{(p,t)}, t)$, is an embedding.

We prove that the image $C_1\psi$ of the semigroup C_1 under ψ is isomorphic to the strong $\mathcal{M}$-semigroup $\mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$, where

$$\overline{\mathcal{Y}} = \{f_{(p,1)} : p \in \mathrm{Mor}\,(1, 1)\}$$

and $\varepsilon \in \mathcal{Y}^M$ is defined by $h\varepsilon = (\mathrm{Mor}\,(1, h))^+$, for all $h \in M$. Certainly $\varepsilon \in \mathcal{Y}^M$, since $(\mathrm{Mor}\,(1, h))^+ \in \mathcal{Y}$, for any $h \in M$. To see this, let $p \in \mathrm{Mor}\,(1, h)$ and let $q \in \mathrm{Mor}\,(1, 1)$. Then

$$p^+ + q = q + p^+ = (q + p^+)^+ = (q + p)^+ \in (\mathrm{Mor}\,(1, h))^+.$$

Recall that, by Theorem 3.5, for all $(p, t) \in C_1$ we have that $f_{(p,t)} = f_{(p^+,1)}$ with $p^+ \in \mathrm{Mor}\,(1, 1)$. Thus $C_1\psi \subseteq \overline{\mathcal{Y}} \times M$ and $\overline{\mathcal{Y}}$ is a subsemilattice of $\mathcal{Y}^M$ since $\overline{\mathcal{Y}} \simeq \overline{\mathcal{Y}} \times \{1\} = (E(C_1))\psi$.

We are assuming that M acts on $\mathcal{Y}^M$ as in the definition of the wreath product $\mathcal{Y} \circledast M$. Hence to prove that $(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$ is a strong $\mathcal{M}$-triple it remains to show that ε is an upper bound of $\mathcal{Y}$ and that conditions (A) and (B) hold.

First, let $p \in \text{Mor}\,(1,1)$ and $h \in M$. Then

$$hf_{(p,1)} = (\text{Mor}\,(1,h) + hp)^+ \subseteq (\text{Mor}\,(1,h))^+ = h\varepsilon$$

hence $f_{(p,1)} \leq \varepsilon$ and so ε is an upper bound of $\overline{\mathcal{Y}}$.

To prove that condition (A) holds, let $t \in M$. As $\text{Mor}\,(1,t) \neq \emptyset$ there exists $q \in \text{Mor}\,(1,t)$. Thus $q^+ \in \text{Mor}\,(1,1)$ and so $f_{(q^+,1)} \in \overline{\mathcal{Y}}$. Let $h \in M$. Then

$$hf_{(q^+,1)} = \big(\text{Mor}\,(1,h) + hq^+\big)^+ = (\text{Mor}\,(1,h) + hq)^+ \subseteq (\text{Mor}\,(1,ht))^+ = h(t\varepsilon).$$

Therefore $f_{(q^+,1)} \leq t\varepsilon$.

To prove that condition (B) holds, suppose that $t \in M$ and $p \in \text{Mor}\,(1,1)$ is such that $f_{(p,1)} \leq t\varepsilon$. Then, for all $h \in M$, we have

$$hf_{(p,1)} = (\text{Mor}\,(1,h) + hp)^+ \subseteq (\text{Mor}\,(1,ht))^+ = h(t\varepsilon).$$

In particular, putting $h = 1$, we obtain

$$\text{Mor}\,(1,1) + p = (\text{Mor}\,(1,1) + p)^+ \subseteq (\text{Mor}\,(1,t))^+.$$

Thus $p \in (\text{Mor}\,(1,t))^+$. Let $r \in \text{Mor}\,(1,t)$ be such that $p = r^+$. Then $f_{(p,1)} = f_{(r,t)}$. Now, let $q \in \text{Mor}\,(1,1)$. Then (r,t), $(q,1) \in C_1$ and, as ψ is a morphism,

$$\begin{aligned}(f_{(r+tq,t)}, t) &= (r+tq,t)\psi = ((r,t)(q,1))\,\psi \\ &= (r,t)\psi(q,1)\psi \\ &= (f_{(r,t)}, t)(f_{(q,1)}, 1) \\ &= (f_{(r,t)} \wedge tf_{(q,1)}, t).\end{aligned}$$

Hence

$$f_{(p,1)} \wedge tf_{(q,1)} = f_{(r,t)} \wedge tf_{(q,1)} = f_{(r+tq,t)} \in \overline{\mathcal{Y}},$$

and so condition (B) holds. Therefore $(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$ is a strong $\mathcal{M}$-triple and

$$\mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}}) = \{(f_{(p,1)}, t) \in \overline{\mathcal{Y}} \times M : f_{(p,1)} \leq t\varepsilon\}.$$

Next, we prove that $C_1\psi = \mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$. As in the verification of condition (A), $C_1\psi \subseteq \mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$. On the other hand, if $(f_{(p,1)}, t) \in$

$\mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$ then $f_{(p,1)} \leq t\varepsilon$ and, as proved before, there exists $r \in \mathrm{Mor}\,(1,t)$ such that $f_{(p,1)} = f_{(r,t)}$. Hence $(r,t) \in C_1$ and $(f_{(p,1)}, t) = (f_{(r,t)}, t) = (r,t)\psi$. Therefore C_1 is isomorphic to $\mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$.

If S is a monoid then C_1 is a monoid with identity $(0_1, 1)$. For all $h \in M$,

$$hf_{(0_1,1)} = (\mathrm{Mor}\,(1,h) + h0_1)^+ = (\mathrm{Mor}\,(1,h) + 0_h)^+ = (\mathrm{Mor}\,(1,h))^+ = h\varepsilon.$$

Hence $\varepsilon \in \overline{\mathcal{Y}}$. By Lemma 4.1, $(\varepsilon, 1)$ is the identity of $\mathcal{M}(M, \mathcal{Y}^M, \overline{\mathcal{Y}})$; of course, ψ is a $(2,1,0)$-morphism.

5. COVERS

In this section we prove that every weakly left ample semigroup has a proper weakly left ample cover. Our proof follows the technique presented in [4] by Fountain for E-dense monoids and used in [7] for left ample semigroups.

We leave to the reader the checking of specific details for the weakly left ample case.

Definition. Let S and T be weakly left ample semigroups [monoids]. We say that T is a *cover* of S if there exists an idempotent separating $(2,1)$-morphism [$(2,1,0)$- morphism] from T onto S.

We start by showing that every weakly left ample monoid has a monoid cover.

Let S be a weakly left ample monoid with identity 1_S and set of idempotents $E = E(S)$. Put $X = S \setminus \{1_S\}$.

Let X^* be the free monoid on X with identity 1. We write the non-identity elements as sequences $(x_1, \ldots, x_n)$, where $n \geq 1$ and $x_i \in X \quad (i = 1, \ldots, n)$. To each word $w \in X^*$ we associate a subset S_w of S in the following way:

$$S_w = \begin{cases} E & \text{if } w = 1 \\ Ex_1Ex_2E\ldots Ex_nE & \text{if } w = (x_1, \ldots, x_n). \end{cases}$$

Clearly, for all $v, w \in X^*$, we have

$$S_{vw} = S_vS_w.$$

Now, we define a category $\mathcal{C}$ as follows:

$$\mathrm{Obj}\,\mathcal{C} = X^*$$

and, for all $v, w \in X^*$,

$$\mathrm{Mor}\,(v, w) = \begin{cases} \{(v, s, w) : s \in S_{w_1}\}, & \text{if } w = vw_1, \text{ for some } w_1 \in X^*. \\ \emptyset, & \text{otherwise.} \end{cases}$$

The composition law is given by

$$(v, s, w) + (w, t, u) = (v, st, u).$$

Clearly, the composition is well defined and associative. Also, for any object u we have

$$\mathrm{Mor}\,(u, u) = \{(u, e, u) : e \in E\}.$$

Next, we consider a (left) action of the unipotent monoid X^* on the category $\mathcal{C}$ defined as follows, the action of X^* on Obj $\mathcal{C}$ is given by the multiplication on X^* and, for all $u \in X^*$ and $(v, s, w) \in \mathrm{Mor}\,\mathcal{C}$,

$$u(v, s, w) = (uv, s, uw).$$

Choose the empty word 1 as the distinguished object of $\mathcal{C}$.

As in [7] it is a routine matter to prove that $\mathcal{C}$ is a proper 1-weakly left ample category and that X^* acts 1-transitively and 1-injectively on $\mathcal{C}$. To prove this result we need as in the left ample case a technical lemma that follows from the fact that condition (CL) holds in a weakly left ample monoid.

Lemma 5.1. *Let S be a weakly left ample monoid and $s \in S$. If $s = e_0x_1e_1 \ldots e_{n-1}x_ne_n$, for some $n \in \mathbb{N}, x_i \in S \quad (i = 1, \ldots, n)$ and $e_j \in E(S)$ $(j = 0, \ldots, n)$, then*

$$s = s^+(x_1 \cdots x_n).$$

Our main result of this section is the following.

Theorem 5.2. *Every weakly left ample monoid has a proper weakly left ample cover.*

Proof. Let S be a weakly left ample monoid and consider the proper 1-weakly left ample category $\mathcal{C}$ defined above. By Theorem 2.5, C_1 is a proper weakly left ample monoid, where

$$C_1 = \{((1,s,u),u) : u \in X^*, s \in S_u\}$$

with multiplication given by

$$((1,s,u),u)\,((1,t,v),v) = ((1,st,uv),uv)\,.$$

Then the map θ defined as follows:

$$\begin{array}{rcl} \theta : \quad C_1 & \longrightarrow & S \\ ((1,s,u),u)) & \mapsto & s \end{array}$$

is an idempotent separating $(2,1,0)$-morphism from C_1 onto S. Hence C_1 is a proper weakly left ample cover of S.

Corollary 5.3. *Every weakly left ample semigroup has a proper weakly left ample cover.*

Proof. Let S be a weakly left ample semigroup without identity. Consider the monoid S^1. Clearly, S^1 is a weakly left ample monoid and we may consider the monoid cover C_1 together with the surjective idempotent separating $(2,1,0)$-morphism $\theta : C_1 \to S^1$, $((1,s,u),u)\theta = s$, defined above. As θ is, in particular, a $(2,1)$-morphism the inverse image $S\theta^{-1}$ is a $(2,1)$-subalgebra of C_1. Now, C_1 is certainly in the class of all proper weakly left ample semigroups. This class forms a quasivariety [10], so that $S\theta^{-1}$ is a proper weakly left ample semigroup. Then $\theta\big|_{S\theta^{-1}} : S\theta^{-1} \to S$ is the required surjective idempotent separating $(2,1)$-morphism.

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