The diameter of endomorphism monoids

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Right congruences

Definition

A **right congruence** on a semigroup S is an equivalence relation ρ such that for every $a, b, c \in S$,

$$a \,\rho\, b \Rightarrow ac \,\rho\, bc.$$

- If $U \subseteq S \times S$, then the **right congruence generated by** U, denoted $\langle U \rangle$, is the smallest right congruence containing U.

Definition

A semigroup S is **right Noetherian** if every right congruence is finitely generated (f.g).

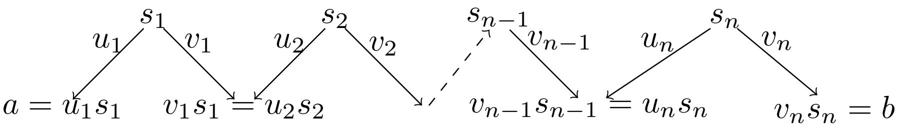
Generating right congruences

Lemma (Kilp, Knauer, Mikhalev, 2000)

Let $U \subseteq S \times S$. Then $a \langle U \rangle b$ if and only if either $a = b$ or there exists a U-path from a to b, that is,

$$a = u_1 s_1, v_1 s_1 = u_2 s_2, \ldots, v_n s_n = b$$

where $(u_i, v_i) \in U \cup U^{-1}$ and $s_i \in S^1$.



Pseudo-finite semigroups

- Pseudo-finite: the semigroup finiteness condition of the universal right congruence ω_r^S being finitely generated and there being a bound on the length of sequences required to relate any two elements.
- First studied by Dales and White in 2017 with regards to Banach algebras.
- Boring property for groups: pseudo-finite groups are finite.
- Kobyashi (2007): ω_r^M is f.g. if and only if M is of type right-FP1.
- I first joined the project for "Semigroups with finitely generated universal left congruence" (2019, Dandan, G, Q-G, Zenab).
- Clear picture for key classes including inverse semigroups, completely regular, and Rees matrix.
- Far more complex then first thought: there exists pseudo-finite regular semigroups without a completely simple minimal ideal "On minimal ideals in pseudo-finite semigroups" (2022, G,M, Q-G, R).
- Pseudo-finite transformation semigroups studied in "On the diameter of semigroups of transformations and partitions" (2023, E,G,Q-G,R).

Diameter

Lemma

Let S be a semigroup. Then ω_r^S is f.g. if and only if there exists a subset U of $S \times S$ such that for any $a, b \in S$, we have $a = b$ or there exists a U-path from a to b, that is,

$$a = u_1 s_1, v_1 s_1 = u_2 s_2, \ldots, v_n s_n = b$$

where $(u_i, v_i) \in U$ and $s_i \in S^1$.

Definition

Let S be a semigroup in which ω_r^S is f.g.

- If $\omega_r^S = \langle U \rangle$, then define $D_r(S; U) =$sup{length of the smallest U-path from a to b : $a, b \in S$}.
- The **right diameter of** S is then $D_r(S) =$min$\{D_r(S; U) : \omega_r^S = \langle U \rangle, |U| < \infty\}$.
- If $D_r(S)$ is finite, then S is called **right pseudo-finite**.

Diameter 1

- Having right diameter 1 is equivalent to the well-studied notion of the diagonal right act being f.g.
- For a semigroup S, the diagonal right S-act is the set $S \times S$ under the right action given by $(a, b)c = (ac, bc)$.
- First studied implicitly by Bulman-Fleming and McDowell (1990), and formalized by Robertson et al (2001).
- Gallagher and N (2005) studied this property for many natural semigroups, including subsemigroups of $\mathcal{T}_X$, endomorphisms of chains, and endomorphisms of independence algebras.
- Also considered the stronger property of the diagonal right act being **monogenic**, i.e. there exists $a, b \in S$ such that $S \times S = (a, b)S$.

Hierarchy of conditions

ω_r^S is finitely generated.

$$\Uparrow$$

S is pseudo-finite.

$$\Uparrow$$

$$D_r(S) = 1$$

$$\Updownarrow$$

The diagonal right S-act is finitely generated.

$$\Uparrow$$

The diagonal right S-act is monogenic.

The Meta Problem

Problem (Meta)

Which "characteristics" of a semigroup determines its left/right diameter.

- A "characteristic" of a semigroup could mean the existence of special elements, its properties (such as algebraic identities), or properties inherited from some other structure.
- Which characteristic is best suited for building a global theory of pseudo-finite semigroups?

Previous methods

Previous work can be broadly broken down into two methods depending on if we view a semigroup as a transformation semigroup or abstractly (e.g. variety of semigroups).

Transformation semigroups: Discussed in my talk last summer.

- Pro: Able to manipulate concrete elements to give bounds on the diameter. Great success when a degree of transitivity is added.
- Con: Global structure often mysterious or unhelpful.

Abstractly: Which properties hold for all semigroups of a particular diameter? Most widely used method.

- Pro: Global structure obtained by restricting to semigroups satisfying certain conditions. E.g. no infinite diameter 1 semigroup can be commutative (Gallagher).
- Con: Elements are not concrete, and so can be harder to manipulate.

Having our Baumkuchen and eating it

- Restrict to transformation monoids which have an inherited global structure to keep the benefits of both methods: Endomorphism monoids!

Definition

A **(first order) structure** $\mathbb{A} = (A; \mathfrak{K})$ is a set A together with a collection $\mathfrak{K}$ of textbfbasic relations and functions defined on A.

- A semigroup is considered as a set together with a binary (associative) operation.
- Both partially ordered sets (posets) and graphs can be considered as sets together with a single binary relation.
- A semilattice can also be considered as the structure $(Y; \wedge, \leq)$ where $a \leq b$ if and only if $a \wedge b = a$.

Endomorphisms

Definition

Let $\mathbb{A} = (A; \mathfrak{K})$ be a structure. Then a map $\theta\colon A \to A$ is an **endomorphism** of $\mathbb{A}$ if it preserves each function and relation from $\mathfrak{K}$, that is, for each function $f \in \mathfrak{K}$, relation $R \in \mathfrak{K}$, and $a_1, \ldots, a_n \in A$,

$$((a_1, \ldots, a_n)f)\theta = (a_1\theta, \ldots, a_n\theta)f,$$
$$(a_1, \ldots, a_n) \in R \Rightarrow (a_1\theta, \ldots, a_n\theta) \in R.$$

The set of all endomorphisms of $\mathbb{A}$ is denoted $\mathrm{End}(\mathbb{A})$, and forms a submonoid of $\mathcal{T}_A$.

E.g. If $\mathcal{Y} = (Y; \wedge, \leq)$ is a semilattice then $\theta \in \mathrm{End}(\mathcal{Y})$ if

$$(x \wedge y)\theta = x\theta \wedge y\theta \text{ and } x \leq y \Rightarrow x\theta \leq y\theta.$$

Endomorphisms

Warning: How we consider our structure (its signature) can change its endomorphism monoid. E.g. If Y is a semilattice then:

$$\mathrm{End}(Y;\wedge) = \mathrm{End}(Y;\wedge,\leq) \subseteq \mathrm{End}(Y;\leq).$$

Endomorphisms

The philosophy behind this method is that the properties of $\mathrm{End}(\mathbb{A})$ often depend solely on those of the underlying structure $\mathbb{A}$, which is easier to work with.

- Pro: Global structure inherited from $\mathbb{A}$?
- Pro: Local structure (concrete maps) inherited from $\mathbb{A}$ and the *closure* property.
- Con: Not all transformation monoids are the endomorphism monoid of some structure.

Theorem

Given a monoid monoid $M \leq \mathcal{T}_X$, t.f.a.e.:

(1) *M is the endomorphism monoid of some (first order) structure;*

(2) *M is the endomorphism monoid of some relational structure;*

(3) *M is closed in the topology of pointwise convergence. That is, whenever $\alpha \in \mathcal{T}_X$ is such that for each finite $A \subseteq X$ there exists $\gamma \in M$ with $\alpha|_A = \gamma|_A$ then $\alpha \in M$.*

Consequence: Suffices to consider relational structures!

Original motivation

- Tackling the problem via endomorphism monoids was briefly examined by Gallagher and R. in the diameter 1 case.
- (Mostly) classified those independence algebras with endomorphism monoids being of diameter 1.
- No infinite chain (totally ordered set) can have endomorphism monoid of left or right diameter 1.

Problem (Motivation)

Determine why chains cannot have endomorphisms of diameter 1 (or stronger). What determines their left/right diameter?

Chains Part 1:

Explaining the lower bound of 2.

Left and right units

- The diagonal right act of $\mathcal{T}_X$ is monogenic and generated by any injective maps α, β with disjoint images (Gallagher, R).
- The injective maps of $\mathcal{T}_X$ correspond its *right units* i.e. elements $a \in S$ such that there exists $b \in S$ with $ab = 1$.
- The submonoid of right units of a monoid is the $\mathcal{R}$-class R_1 of the identity.
- The **group of units** is the $\mathcal{H}$-class H_1.

Proposition

$R_1 = H_1 \Leftrightarrow L_1 = H_1 \Leftrightarrow J_1 = H_1 \Leftrightarrow S$ *does not contain a copy of the bicyclic monoid* $B = \langle a, b \,|\, ab = 1 \rangle$.

A monoid satisfying one (and hence all) of these conditions is called **Dedekind-finite**. E.g. $\mathcal{T}_X$ is not Dedekind-finite.

Lemma (EGMQ-GR)

A Dedekind-finite monoid has right/left diameter 1 if and only if it is finite.

Left and right units

Problem

Does the right diameter of $\mathrm{End}(\mathbb{A})$ *depend only on its right units?*

- We restrict our attention to relational structures in which there is an easy way to pass from the endomorphisms to its elements.

Reflexive structures

- Given an n-ary relation R of a set A, we define an R-**loop** to be an element $x \in A$ with $(x, x, \ldots, x) \in R$.
- We call R **reflexive** if each $x \in A$ is an R-loop.
- A relational structure $\mathbb{A}$ is called **reflexive** if each of its basic relations are reflexive.

Lemma

If $\mathbb{A}$ is reflexive then the constant map $c_x \colon A \to A$ $(a \mapsto x)$ is an endomorphism of $\mathbb{A}$ for each $x \in A$. Moreover, $\mathcal{C}_A = \{c_x : x \in A\}$ is the minimum ideal of $\mathrm{End}(\mathbb{A})$.

Example

Posets $(P; \leq)$, chains, prosets, looped graphs, and bands(!) are all reflexive.

Posets

We restrict to posets - the results extend to any reflexive structure (but with added ugliness).

Proposition (EGMQ-GR)

Let $\mathcal{P}$ be a non-trivial poset. If $S = \mathrm{End}(\mathbb{P})$ has monogenic diagonal right act then there exists $\alpha, \beta \in R_1$ such that their images are unrelated under $\leq$, i.e. there exists no $x, y \in P$ with $x\alpha \geq y\beta$ or $x\alpha \leq y\beta$.

Proof.

- Suppose $x\alpha \leq y\beta$.
- Fix any $u, v \in P$.
- Then $(c_u, c_v) = (\alpha, \beta)\delta$ for some $\delta \in S$.
- Hence

$$u = xc_u = (x\alpha)\delta \leq (y\beta)\delta = yc_v = v.$$

- u and v chosen arbitrarily, so P is trivial, a contradiction.

□

Posets

Proposition

Let $\mathbb{P}$ *be a non-trivial poset. If* $S = \mathrm{End}(\mathbb{P})$ *has monogenic diagonal right act then there exists* $\alpha, \beta \in R_1$ *such that their images are unrelated under* $\leq$*,*

Corollary

If $\mathbb{P}$ *is a non-trivial chain then* $\mathrm{End}(\mathbb{P})$ *does not have monogenic diagonal right act.*

Conjecture

Let $\mathbb{P}$ *be a poset. If* $S = \mathrm{End}(\mathbb{P})$ *has right diameter 1, then there exists right units which are "finitely related".*

Chains Part 2:

Finding upper-bounds

Higher diameters: left

Lemma (EGMQ-GR)

Let $S = \mathrm{End}(\mathbb{A})$ *for some reflexive structure* $\mathbb{A}$*. Then* $D_\ell(S) \leq 2$*.*

Proof.

For any $x \in A$ we have $D_\ell(S; \{(1, c_x)\}) \leq 2$: If $\theta, \psi \in S$ then

$$\theta = \theta \circ 1, \theta \circ c_x = c_x = \psi \circ c_x, \psi \circ 1 = \psi.$$

Succinct:

$$(\theta, c_x) = \theta(1, c_x), \psi(c_x, 1) = (c_x, \psi).$$

□

Corollary

Let $\mathbb{P}$ *be an infinite chain. Then* $D_\ell(\mathrm{End}(\mathbb{P})) = 2$*.*

Higher diameters: right

- Let $S = \mathrm{End}(\mathbb{A})$ for some infinite reflexive structure $\mathbb{A}$.
- Recall $\mathcal{C} = \mathcal{C}_A$ is a (right zero) minimum ideal of S, and thus $\omega = \omega_r^S|_{\mathcal{C}\times\mathcal{C}}$ is a right congruence of S.
- Let $D_r^S(\mathcal{C})$ denote the right diameter of $\mathcal{C}$ corresponding to $\omega_r^S|_{\mathcal{C}\times\mathcal{C}}$.

Lemma (EGMQ-GR)

Let $S = \mathrm{End}(\mathbb{A})$ for some reflexive structure $\mathbb{A}$ and let $\mathcal{C} = \mathcal{C}_A$. Then

$$D_r^S(\mathcal{C}) \leq D_r(S) \leq D_r^S(\mathcal{C}) + 2.$$

Swapping to orbits

- Let $S = \mathrm{End}(\mathbb{A})$ for some reflexive $\mathbb{A}$ and let $\mathcal{C} = \mathcal{C}_A$.
- Paths in $\omega_r^S|_{\mathcal{C}\times\mathcal{C}} = \langle U \rangle$ are of the form $c_u\delta = c_v\delta'$ $(\delta, \delta' \in S)$.
- But $c_u\delta = c_v\delta'$ if and only if $u\delta = v\delta'$.
- Hence any U-path from c_x to c_y is equivalent to a $\bar{U} = \{(u, v) \in A^2 \colon (c_u, c_v) \in U\}$-path

$$x = u_1\delta_1, v_1\delta_1 = u_2\delta_2, \ldots, v_n\delta_n = y.$$

Lemma

$D_r^S(\mathcal{C}) = 1$ *if and only if there exists a finite collection* $(u_1, v_1), \ldots, (u_n, v_n) \in A \times A$ *such that for each* $x, y \in A$ *there exists* $1 \leq i \leq n$ *and* $\delta \in \mathrm{End}(\mathbb{A})$ *with* $(x, y) = (u_i, v_i)\delta$*.*

We arrive at a central concept to transformation monoids: (2-)oligomorphicity!

Oligomorphic transformation monoid

- Let $S \leq \mathcal{T}_X$ be a transformation monoid.
- S acts on the right of X^n by $(x_1, \ldots, x_n)\theta = (x_1\theta, \ldots, x_n\theta)$.
- We call $(x_1, \ldots, x_n)S$ an n-**orbit**.

Definition

S is called n-**oligomorphic** if it has only finitely many n-orbits. If S is n-oligomorphic for each n then S is **oligomorphic**.

- Studied extensively by P. Cameron, M. Pech, J. Nešetřil, D. Mašulović etc.
- Oligomorphic groups are central in a number of model theoretic conceps e.g. ω-categoricity, quantifier elimination, and homogeneity.
- Oligomorphic transformation monoids give an endomorphism-dual to these concepts.

Corollary (EGMQ-GR)

Let $\mathbb{A}$ be a reflexive structure. Then $D_r^S(\mathcal{C}_A) = 1$ if and only if $\mathbb{A}$ is 2-oligomorphic. In which case $D_r(\mathrm{End}(\mathbb{A})) \leq 3$.

Oligomorphic transformation monoid

Theorem (Mašulović, 2007)

The endomorphism of a tree (and hence a chain) is oligomorphic. Not all posets have oligomorphic endomorphism monoids.

Lemma (EGMQ-GR)

The endomorphism monoid of a poset is 2-oligomorphic, with at most eight 2-orbits.

Corollary (EGMQ-GR)

If $\mathbb{P}$ is a poset then $D_r(\mathrm{End}(\mathbb{P})) \leq 3$. If, further, $\mathbb{P}$ is a chain we have $D_r(\mathrm{End}(\mathbb{P})) \in \{2, 3\}$.

Chains Part 3:

The dichotomy on the right

Classifying chains

- Let C be a chain with minimum z and let $S = \mathrm{End}(C)$.
- We call C **min-shiftable** if there exists a right unit $\alpha \in S$ with $z\alpha > z$.
- Then for each $\theta \in S$ we may construct $\delta_\theta \in S$ such that $\theta = \alpha\delta_\theta$ and $z\delta_\theta = z$.
- (α, c_z)-paths of length 2:

$$\theta = \alpha\delta_\theta, c_z\delta_\theta = c_z = c_z\delta_\psi, \alpha\delta_\psi = \psi.$$

Theorem (GEM+QG)

Let C be an infinite chain and $S = \mathrm{End}(C)$. Then $D_r(S) = 2$ if and only if C is either min-shiftable or max-shiftable. Otherwise, $D_r(S) = 3$.

- $D_r(\mathrm{End}(\mathbb{N})) = 2$.
- $D_r(\mathrm{End}(1 + \mathbb{Z})) = 3$.

The rough idea

Monogenic diagonal right act: $\exists\ \alpha, \beta \in R_1$ with unrelated images (chains discounted).

$$\Downarrow$$

$D_r(\mathrm{End}(\mathbb{P})) = 1$: $\exists \alpha, b \in R_1$ with only finitely entangled images(?) (chains still discounted).

$$\Downarrow$$

$D_r(\mathrm{End}(\mathbb{P})) = 2$: $\exists \alpha \in R_1$ of a special type (chains: move min or max).

$$\Downarrow$$

Otherwise: $D_r(\mathrm{End}(\mathbb{P})) = 3$.

Thank you!