Inverse semigroups acting on graphs and trees

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Outline

$G-$sets

Let X be a $G-$set.

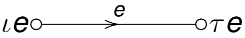
- The $G-$*stabilizer* of $x \in X$ is the set of elements of G that 'fix' x, i.e.
$$G_x = \{g \in G : gx = x\}$$
G_x is a subgroup of G
$g \in G, G_{gx} \simeq gG_xg^{-1}$
- G is said the act *freely* on X if $G_x = \{1\}$ for all $x \in X$
- The $G-$*orbit* of x is the set $Gx = \{gx : g \in G\}$ which is a $G-$subset of X
- The *quotient set* for the $G-$set X is the set of $G-$orbits, $G \backslash X = \{Gx : x \in X\}$ which clearly has a natural map $X \rightarrow G\backslash X, x \mapsto Gx$.
- A $G-$*transversal in* X is a subset Y of X which contains exactly one element of each $G-$orbit of X.
Hence the composite $Y \subseteq X \rightarrow G\backslash X$ is a bijection.

$G-$sets

A $G-$graph (X, V, E, ι, τ) is a non-empty $G-$set X with disjoint non-empty $G-$subsets V and E such that

$$X = V \,\dot{\cup}\, E$$

and two $G-$maps $\iota, \tau : E \to V$.
Pictorially we have,



The *quotient graph*, $G\backslash X$, is the graph

$$(G\backslash X, G\backslash V, G\backslash E, \bar{\iota}, \bar{\tau})$$

where

$$\bar{\iota}(Ge) = G\iota e, \bar{\tau}(Ge) = G\tau e$$

for all $Ge \in G\backslash E$.

$G-$sets

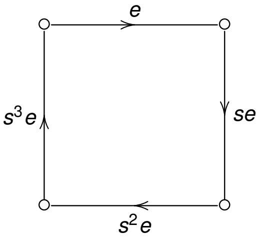
If $G \backslash X$ is connected then it can be shown that there exist subsets

$$Y_0 \subseteq Y \subseteq X$$

such that Y is a $G-$transversal in X, Y_0 is a subtree of X with $VY_0 = VY$ and for each $e \in EY, \iota(e) \in VY$.
In this case, Y is called a *fundamental transversal* in X.

Cayley Graphs

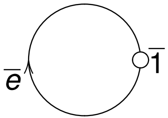
The *Cayley graph* of G with respect to a subset T of G is the $G-$graph, $X(G, T)$, with vertex set $V = G$, edge set $E = G \times T$ and incidence function $\iota(g, t) = g, \tau(g, t) = gt$ for all $(g, t) \in E$. For example, consider the cyclic group $C_4 = \langle s : s^4 \rangle$ and $T = \{s\}$ then the Cayley graph can be represented as



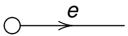
where $e = (1, s) \in G \times T$.

Cayley Graphs

The quotient graph is



and a corresponding fundamental $G-$transversal is



Graphs of Groups

A *graph of groups* $(G(-), Y)$, is a connected graph

$$(Y, V, E, \bar{\iota}, \bar{\tau})$$

together with a function $G(-)$ which assigns to each $v \in V$ a group $G(v)$ and to each edge $e \in E$ a subgroup

$$G(e) \subseteq G(\bar{\iota}e)$$

and a group monomorphism

$$t_e : G(e) \to G(\bar{\tau}e).$$

Graph of Groups - Standard example

- $G-$graph X such that $G\backslash X$ is connected
- fundamental transversal Y with subtree Y_0

For each edge e in EY, there are unique vertices $\bar{\iota}e \in G\iota e, \bar{\tau}e \in G\tau e$ in VY. In fact $\bar{\iota}e = \iota e$.
$\bar{\iota}, \bar{\tau} : EY \to VY$ make Y into a graph (isomorphic to $G\backslash X$)
For each e in EY, τe and $\bar{\tau}e$ belong to the same $G-$orbit and so there exists t_e in G such that

$$t_e\bar{\tau}e = \tau e$$

if $e \in Y_0$ then $\bar{\tau}e = \tau e$ and we take $t_e = 1$.

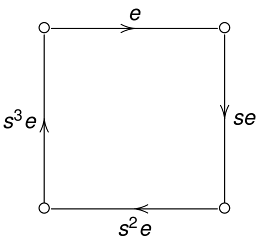
$$G_{\tau e} = t_e G_{\bar{\tau}e} t_e^{-1}$$

$G_e \subseteq G_{\iota e}, G_{\tau e}$ and so there is an embedding

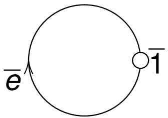
$$G_e \to G_{\bar{\tau}e}$$

given by $g \mapsto t_e^{-1} g t_e$.

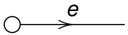
Graph of Groups - Standard example



The quotient graph is



with fundamental $G-$transversal



$$\iota(e) = 1, \tau(e) = s, \overline{\tau}(e) = 1, G_1 = \{1\}, G_s = \{1\}$$

The Fundamental Group

The *fundamental group* $\pi(G(-), Y, Y_0)$ is the group with generating set

$$\{t_e : e \in E\} \cup \bigcup_{v \in V} G(v)$$

and relations :
the relations for $G(v)$, for each $v \in VY$;

$$t_e^{-1} g t_e = t_e(g) \text{ for all } e \text{ in } EY \backslash EY_0;$$

$$t_e = 1 \text{ for all } e \in EY_0$$

The Fundamental Group

Given $G = \pi(G(-), Y, Y_0)$, we construct a *standard* $G-$graph as follows:
Let T be the $G-$set generated by Y and relations

$$gy = y, \text{for each } y \in Y, g \in G(y)$$

Then T has $G-$subsets $VT = GV$ and $ET = GE$.
Define $\iota, \tau : ET \to VT$ by

$$\iota(ge) = g\bar{\iota}e, \tau(ge) = gt_e\bar{\tau}e$$

Then T is a $G-$graph with fundamental transversal Y.

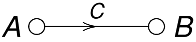
The Fundamental Group

The graph of groups associated to this $G-$graph is isomorphic to the original graph of groups.

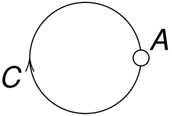
Conversely, given a group, G, acting on a *tree* we can form the graph of groups and the fundamental group, π, is then isomorphic to G and the standard graph is isomorphic to the original $G-$tree.

$G-$sets

The two classic examples of fundamental groups arise from the following two graphs of groups:



and



In the former case, the fundamental group is the amalgamated free product $A *_C B$ while in the later case it is the HNN-extension $A *_C t$.

Inverse semigroup actions

Throughout, S will denote an inverse semigroup. By a (left) $S-$act, X, we mean an (partial) action of S on the set X such that $(st)x$ exists if and only if $s(tx)$ exists and then

$$(st)x = s(tx).$$

In addition, we require that whenever $sx = sy$ then $x = y$. Right $S-$acts are defined dually and bi-acts can be defined in a fairly obvious way.

Example

Let S be an inverse semigroup and let $s \in S$. Define $s \cdot x = sx$ for $x \in \{s^{-1}sS\}$. This is the act induced by the Preston-Wagner representation of S.

Inverse semigroup actions

We denote by $D_s = \{x \in X : sx \in X\}$ the *domain* of the element s.

Lemma

If $s \in S, e \in E(S), x, y \in X$ then

1. *If $sx = y$ then $x = s^{-1}y$;*
2. *if $x \in D_s$ then $s^{-1}sx = x$;*
3. *if $x \in D_e$ then $ex = x$.*

Define $D^x = \{s \in S : x \in D_s\}$ the *domain* of the element x.
x is said to be *effective* if $D^x \neq \emptyset$.
X is *transitive* if for all $x, y \in X$, there exists $s \in S, y = sx$.

Stabilisers and $\omega-$cosets

For an inverse semigroup S define the *natural partial ordering* $\leq$ on S by

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S)$$

Let H be a subset of an inverse semigroup S. Denote by $H\omega$ the set

$$H\omega = \{s \in S : s \geq h, \text{for some } h \in H\}.$$

This is called the *closure* of H and we say that H is *closed* if $H\omega = H$.

If H is an inverse subsemigroup of S then the sets $(sH)\omega$, for $s \in D_H$, are called the *left* $\omega-$*cosets* of H in S. The set of all left $\omega-$cosets is denoted by S/H.

Stabilisers and $\omega-$cosets

As before define the *stabiliser* of $x \in X$ as

$$S_x = \{s \in S : sx = x\}.$$

Theorem

For all $x \in X$, S_x is either empty or a closed inverse subsemigroup of S.

If H is a closed inverse subsemigroup of an inverse semigroup S then S/H is a left $S-$act with action given by $s \cdot X = (sX)\omega$ whenever $X, sX \in S/H$. Moreover, it is easy to establish that $S_H = H$.

Stabilisers and $\omega-$cosets

If H and K are two closed inverse subsemigroups of S then we say that H and K are *conjugate* if $S/H \cong S/K$ (as $S-$acts).

Theorem

Let H and K be closed inverse subsemigroups of an inverse semigroup S. Then H and K are conjugate if and only if there is an element $s \in S$ such that

$$(s^{-1}Hs)\omega = K \text{ and } (sKs^{-1})\omega = H.$$

Stabilisers and $\omega-$cosets

If $ss^{-1} \in H$ then $s^{-1}Hs$ is an inverse subsemigroup of H.

Theorem

Let S be an inverse semigroup, $s \in S$ and suppose that H is a closed inverse subsemigroup with $ss^{-1} \in H$. Then there exists an embedding $\phi' : H \to s^{-1}Hs$ if and only if ss^{-1} is the identity of H.

Notice that ss^{-1} is the identity of H if and only if $H \subseteq sSs^{-1}$, if and only if $e = ss^{-1} \in H \subseteq eSe$.

Stabilisers and $\omega-$cosets

Theorem

Let S be an inverse semigroup and X a left $S-$act. Let $s \in S$ and $x \in D_s$. Then sS_xs^{-1} is an inverse subsemigroup of S.

Theorem

Let S be an inverse semigroup and X an $S-$act. Let $s \in S$ and $x \in D_s$. Then S_x and S_{sx} are conjugate.

graded actions

Let X be a left $S-$act. Say that X is *graded* if there exits a function $p : X \to E(S)$ such that

1. for all $e \in E(S), D_e = P^{-1}([e])$;
2. for $x \in X, t \in D^x$ if $t^{-1}t = p(x)$ then $tt^{-1} = p(tx)$.

Theorem

Let X be a left $S-$act. Then X is graded if and only if X is effective and for each $x \in X, S_x$ contains a minimum idempotent.

In fact it turns out that condition (2) is unnecessary.

$S-$graphs

If X is a $S-$biact, the (left) *Schützenberger* graph of X with respect to a subset T of S, is denoted $\Gamma = \Gamma(X, T)$, and is the (left) $S-$graph with vertex set $V = X$, edge set $E = \{(x, t) \in X \times T : xt \text{ exists and } xt \neq x\}$ and incidence functions $\iota(x, t) = x, \tau(x, t) = xt$ for all $(x, t) \in E$. The action is that induced by the left action of S on X.

$$x \xrightarrow{t} xt$$

In particular, we are interested in the case $X = {}_S S_S$.

Theorem

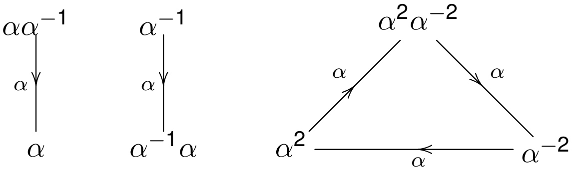
Let S be an inverse semigroup with generating set T. Then S is bisimple if and only if $S \setminus \Gamma = S \setminus \Gamma(S, T)$ is a connected graph.

$S-$graphs

For example, let S be the inverse subsemigroup of $\mathcal{I}_{\{1,2,3,4,5\}}$ generated by

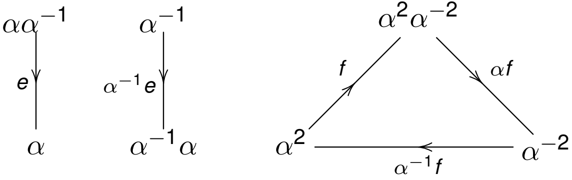
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \end{pmatrix}.$$

$S = \{\alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}$ and
$E(S) = \{\alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}$.
Let $T = \{\alpha\}$. Then the Schützenberger graph, $\Gamma = \Gamma(S, T)$, of S with respect to T is



$S-$graphs

The action of S (induced by the Preston-Wagner representation) on the graph is as follows



Also, we can calculate the orbits of Γ. The edge orbits are

$$Se = \{e, \alpha^{-1}e\}, Sf = \{f, \alpha f, \alpha^{-1}f\}$$

while the vertex orbits are

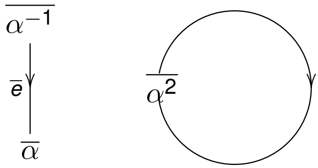
$$S \cdot \alpha = \{\alpha, \alpha^{-1}\alpha\},\ S \cdot \alpha^{-1} = \{\alpha^{-1}, \alpha\alpha^{-1}\}$$

and

$$S \cdot \alpha^2 = \{\alpha^2, \alpha^{-2}, \alpha^2\alpha^{-2}\}.$$

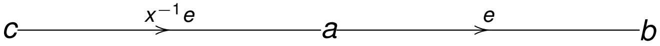
$S-$graphs

The quotient graph $S \setminus \Gamma$, then looks like

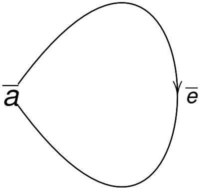


First the good news

Let S be the free inverse semigroup on one generators $\{x\}$. Let $V = \{a, b, c\}$ and define an action on V from the representation $S \to \mathcal{I}_V$ generated by $x \to \rho_x$ where $\rho_x = \begin{pmatrix} a & c \\ b & a \end{pmatrix}$. Define an $S-$graph G, as follows

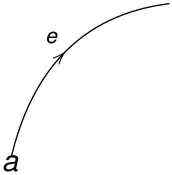


and note that the quotient graph, $S \setminus G$ is



First the good news

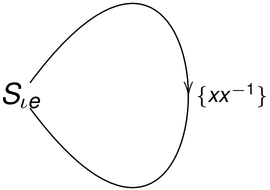
with a fundamental transversal Y



To construct the associated graph of inverse semigroups, notice that $\iota e = a, \tau e = b, \overline{\tau} e = a$,
$S_e = \{xx^{-1}\}, S_{\iota e} = \{x^{-1}x, xx^{-1}, xx^{-1}x^{-1}x\} = S_{\overline{\tau} e}$,
$S_{\tau e} = \{x^2x^{-2}, xx^{-1}\}$ and that xx^{-1} is the identity of $S_{\tau e}$.

First the good news

Hence the graph of inverse semigroups is given by



and there is an embedding $\{xx^{-1}\} \to S_{\iota e}$ given by $xx^{-1} \mapsto x^{-1}(xx^{-1})x = x^{-1}x$.

First the good news

Let $(S(-), Y)$ be a graph of inverse semigroups in which for each $e \in EY$, $S(e)$ is a monoid. Choose a spanning subtree Y_0 of Y. It follows that $VY_0 = VY$. The *'fundamental inverse semigroup'* $\pi(S(-), Y, Y_0)$ is the inverse semigroup defined by

1. The generating set is $\{t_e : e \in EY\} \cup \bigcup_{v \in VY} S(v)$.
2. The relations are
   - (a) the relations for $S(v)$, for each $v \in VY$;
   - (b) $t_e^{-1} s t_e = t_e(s)$ for all e in EY, $s \in S(e) \subseteq S(\bar{\iota} e)$;
   - (c) $t_e t_e^{-1}$ is the identity in $S(e)$ for all e in EY;
   - (d) $t_e = t_e^2$ for all e in EY_0.

First the good news

So in our example that equates to

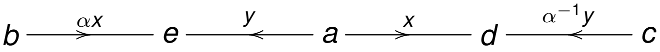
$$\begin{aligned}\pi &= Inv\langle x, y, z, t | x^2 = x, y^2 = y, z = yx, t^{-1}yt = x, tt^{-1} = y\rangle \\ &= Inv\langle t\rangle\end{aligned}$$

... now the bad news

Consider the previous example where S be the inverse subsemigroup of $\mathcal{I}_{\{1,2,3,4,5\}}$ generated by

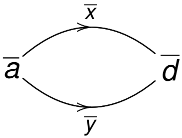
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \end{pmatrix}.$$

Then $S = \langle \alpha | \alpha\alpha^{-2} = \alpha^2 \rangle$ and that if we put $V = \{a, b, c, d, e\}$ then the representation $S \to \mathcal{I}_V$ given by
$\alpha \mapsto \rho_\alpha = \begin{pmatrix} a & b & c & d \\ b & c & a & e \end{pmatrix}$ generates a $S-$action on V.
Consider the $S-$tree $T = (S, V, E)$ given by

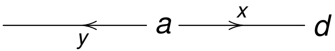


... now the bad news

The quotient graph is



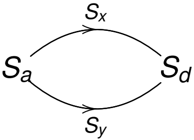
with an $S-$transversal Y



The stabilisers are given by $S_a = \{\alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}, S_d = \{\alpha^{-1}\alpha\} = S_x, S_e = \{\alpha\alpha^{-1}\} = S_y$.

... now the bad news

The graph of inverse semigroups is



with connecting monomorphism $S_y \to S_d$ given by $\alpha\alpha^{-1} \mapsto \alpha^{-1}\alpha\alpha^{-1}\alpha = \alpha^{-1}\alpha$. The 'fundamental inverse semigroup' is then given by the presentation

$$\pi = \langle \beta, \gamma, \delta, t | \beta^2 = \beta, \gamma^2 = \gamma, \delta^2 = \delta, \beta = \delta, t^{-1}\gamma t = \delta, tt^{-1} = \gamma \rangle$$

which reduces to $\langle t | \rangle$ with $\gamma = tt^{-1}, \beta = t^{-1}t$.

... now the bad news

The standard graph on π is then

$$ta \xrightarrow{tx} td \xleftarrow{y} a \xrightarrow{x} d \xleftarrow{t^{-1}y} t^{-1}a$$

Notice that we cannot recover the original action from this as t^2 does not act on a. However, the stabilisers are $S_a = \{tt^{-1}, t^{-1}t, tt^{-1}t^{-1}t\}, S_d = \{t^{-1}t\}, S_x = \{t^{-1}t\}, S_y = \{tt^{-1}\}$ and the graph of inverse semigroups is isomorphic to the previous one.

Ordered Graphs

A Yamamura (2004)

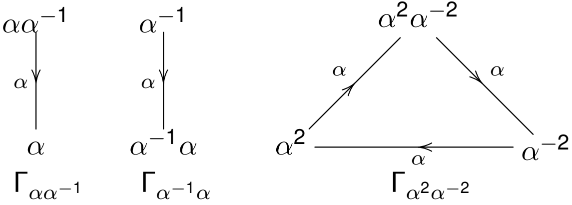
- X a graph, E a semilattice
- For each $e \in E$ there is a unique connected component X_e of X and $X = \dot{\bigcup} X_e$
- for each $f \leq e$, graph morphism $\rho_f^e : X_e \to X_f$ satisfying
    1. $\rho_e^e = 1_{X_e}$
    2. $\rho_f^d = \rho_f^e \circ \rho_e^d$, $f \leq e \leq d$.

$v_1, v_2 \in VX$ define $v_1 \leq v_2$ if there exists $f \leq e$ with $v_1 \in V(X_f), v_2 \in V(X_e)$ and $v_1 = \rho_f^e(v_2)$ - similarly for edges. This defines an ordering on the graph.

Schützenberger graphs are ordered ....

Ordered Graphs

$S = \{\alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}$ and
$E(S) = \{\alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}$.
Let $T = \{\alpha\}$. Then the Schützenberger graph, $\Gamma = \Gamma(S, T)$, of S with respect to T is



if $f \leq e$ then $\rho^e_f : \Gamma_e \to \Gamma_f, \quad x \mapsto fx$ In this case
$\alpha^2\alpha^{-2} \leq \alpha\alpha^{-1}, \alpha^{-1}\alpha$.

Actions on Ordered Graphs

- S and inverse monoid and $E = E(S)$, X a graph ordered by E
- $[X_e]$ = order ideal generated by X_e
  $[X_e] = \{x \in X : x \leq y, y \in X_e\}$
- $T_{e,f}$ = set of all graph isomorphisms $[X_e] \to [X_f]$
- $T_X = \bigcup T_{e,f} \subseteq \mathcal{I}_X$ and an action of S on X is given by a homomorphism $\theta : S \to T_X$
- $\theta_s : [X_{s^{-1}s}] \to [X_{ss^{-1}}]$, $sX_e = X_{ses^{-1}}$ if $e \leq s^{-1}s$ plus a few other axioms

Under *certain conditions* Yamamura has shown that similar results from the Bass-Serre theory of groups carry over to inverse monoids acting on ordered forests.