Strange semigroups and their presentations

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Free semigroups

Let $\mathscr{A}$ be a non-empty set called an *alphabet*, the elements of which are called *letters*.
A *non-empty word* w over $\mathscr{A}$ has the form $a_1 a_2 \ldots a_n$ where $a_i \in \mathscr{A}$ for all $i \in \underline{\mathbf{n}}$.
The set of all non-empty words over $\mathscr{A}$ is written as

$$\mathscr{A}^+ = \{a_1 a_2 \ldots a_n : a_i \in \mathscr{A}, n \in \mathbb{N}\}.$$

For any two words $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_m \in \mathscr{A}^+$, we define a binary operation $\cdot$ given by

$$(a_1 a_2 \ldots a_n) \cdot (b_1 b_2 \ldots b_m) = a_1 a_2 \ldots a_n b_1 b_2 \ldots b_m.$$

The pair $(\mathscr{A}^+, \cdot)$ forms a semigroup over the alphabet $\mathscr{A}$, which is in fact the *free semigroup* on $\mathscr{A}$.

Semigroup presentations

Let ρ be a relation on $\mathscr{A}^+$. For words $w, x \in \mathscr{A}^+$ we write $w\,\rho^{\#}\,x$ if and only if $w = x$ or there exists a finite sequence of the form

$$w = c_1 p_1 d_1 \quad c_1 q_1 d_1 = c_2 p_2 d_2 \quad \ldots \quad c_m q_m d_m = x,$$

such that $c_i, d_i \in \mathscr{A}^+$ and $(p_i, q_i) \in \rho \cup \rho^{-1}$ for all $i \in \underline{\mathbf{m}}$.

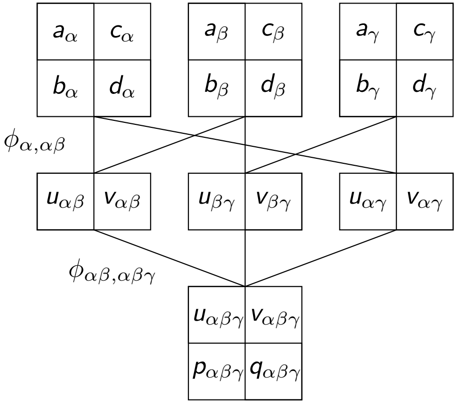
A *semigroup presentation* $\langle \mathscr{A} : \rho \rangle$ denotes the quotient

$$\mathscr{A}^+/\rho^{\#} = \{[w]_{\rho^{\#}} : w \in \mathscr{A}^+\}$$

where $[w]_{\rho^{\#}}$ (or $[w]$ where $\rho^{\#}$ is understood) is a typical element in the $\rho^{\#}$-class of $w \in \mathscr{A}^+$.

Strange semigroups

A semigroup S is *right strange* (*left strange*) if $aS \cap bS$ ($Sa \cap Sb$) is finitely generated for all $a, b \in S$, and there exists $a, b \in S$ such that this intersection is empty, principal and n-generated ($n \geq 2$).



A semigroup is *strange* if it is both left strange and right strange.

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Fix $n \in \mathbb{N}$ where $n \geq 2$. We define an alphabet $\mathscr{A}_n$ as

$$\mathscr{A}_n = \{a, b, u_i, v_i : i \in \underline{\mathbf{n}}\}$$

and define relations ρ_n and λ_n on $\mathscr{A}_n^+$ as

$$\rho_n = \{(au_i, bv_i) : i \in \underline{\mathbf{n}}\} \quad \lambda_n = \{(u_i a, v_i b) : i \in \underline{\mathbf{n}}\}.$$

(Whenever n is fixed, we will omit the subscript n from notation.)

The semigroup S_ρ (S_λ) with presentation $\langle \mathscr{A} : \rho \rangle$ ($\langle \mathscr{A} : \lambda \rangle$) is a cancellative right (left) strange semigroup.

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Fix $n \in \mathbb{N}$ where $n \geq 2$. Let γ be a relation on $\mathscr{A}^+$ where

$$\gamma = \rho \cup \{(xy, yx) : x, y \in \mathscr{A}\}.$$

The semigroup S_γ with presentation $\langle \mathscr{A} : \gamma \rangle$ is clearly commutative. It remains to show that it is strange!

The *rewriting system* on $\mathscr{A}^+$ with rewriting rules

$$bv_i \to au_i \quad av_iu_j \to au_iv_j$$

such that $i, j \in \underline{\mathbf{n}}$ and $i < j$ is noetherian and confluent.

Normal forms and balanced words

A word $w \in \mathscr{A}^+$ is in normal form if we cannot apply any rewriting rules...

$$a^{p_0} b^{q_0} u_1^{p_1} \dots u_n^{p_n}$$
$$a^{p_0} u_1^{p_1} \dots u_i^{p_i} v_i^{q_i} \dots v_n^{q_n}$$
$$u_1^{p_1} v_1^{q_1} \dots u_n^{p_n} v_n^{q_n}$$

Two words $w, x \in \mathscr{A}_n^+$ are *balanced* if and only if

$$w = a^{p_0} b^{q_0} u_1^{p_1} v_1^{q_1} \dots u_n^{p_n} v_n^{q_n} \quad x = a^{r_0} b^{s_0} u_1^{r_1} v_1^{s_1} \dots u_n^{r_n} v_n^{s_n}$$

such that $p_i + q_i = r_i + s_i$ for all $i \in \underline{\mathbf{n}}^\flat$.

To describe the intersection of principal ideals in S_γ, it is suffient to describe the intersection of balanced words of normal form.

Intersection of principal ideals

Let $\mathscr{T} = \{a, u_i : i \in \underline{\mathbf{n}}\}$. For any balanced words $w, x \in \mathscr{A}^+$ in normal form, we define

$$\delta(w, x) = \sum_{t \in \mathscr{T}} ||w|_t - |x|_t|.$$

For example if

$$w = a^2 b u_1 u_2^2 \quad x = a^3 u_1 u_2 v_2$$

we have that $\delta(w, x) = 2$.

Our goal: to show by induction on $\delta(w, x)$ that there are only finitely many possibilities for $h, k \in \mathscr{A}^+$ such that $wh \gamma^\# xk$!

Intersection of principal ideals

Suppose that $w, x \in \mathscr{A}^+$ are balanced words in normal form where

$$w = a^{p_0} b^{q_0} u_1^{p_1} \dots u_n^{p_n} \quad \text{and} \quad x = a^{r_0} b^{s_0} u_1^{r_1} \dots u_n^{r_n}$$
$$w = a^{p_0} b^{q_0} u_1^{p_1} \dots u_n^{p_n} \quad \text{and} \quad x = a^{r_0} u_1^{r_1} \dots u_j^{r_j} v_j^{s_j} \dots v_n^{s_n}$$

such that $p_0 < r_0$.

For w and x as above, have $wh \, \gamma^{\#} \, xk$ if

$$h = a^{\lambda_0} v_1^{\lambda_1} \dots v_n^{\lambda_n} \quad k = b^{\lambda_0} u_1^{\lambda_1} \dots u_n^{\lambda_n}$$

where $\lambda_0 + \ldots + \lambda_n = \delta(w, x)$ and each $\lambda_i \in \mathbb{N}^0$.

For example...

$$a^2 b u_1 u_2^2 \, (\mathbf{av_1}) \, \gamma^{\#} \, a^3 u_1 u_2 v_2 \, (\mathbf{bu_1}).$$

Intersection of principal ideals

Suppose that $w, x \in \mathscr{A}^+$ are balanced words in normal form where

$$w = a^{p_0} u_1^{p_1} \dots u_i^{p_i} v_i^{q_i} \dots v_n^{q_n} \quad \text{and} \quad x = a^{r_0} u_1^{r_1} \dots u_j^{r_j} v_j^{s_j} \dots v_n^{s_n}$$

such that $\sum_{i \in \underline{\mathbf{n}}} |w|_{u_i} \geq \sum_{i \in \underline{\mathbf{n}}} |x|_{u_i}$ and $p_0 \in \mathbb{N}$.

For w and x as above, have $wh \, \gamma^{\#} \, xk$ if

$$h = a^{\lambda_0} v_1^{\lambda_1} \dots v_n^{\lambda_n} \quad k = b^{\lambda_0} u_1^{\lambda_1} \dots u_n^{\lambda_n}$$

such that $\lambda_0 + \dots + \lambda_n = \delta(w, x)$ and each $\lambda_i \in \mathbb{N}^0$.

For example...

$$a^3 u_1 u_2^2 \, (\mathbf{v_1}) \, \gamma^{\#} \, a^3 u_1 u_2 v_2 \, (\mathbf{u_1}).$$

Intersection of principal ideals

Suppose that $w, x \in \mathscr{A}^+$ are balanced words in normal form where

$$w = u_1^{p_1} v_1^{q_1} \dots u_n^{p_n} v_n^{q_n} \quad \text{and} \quad x = u_1^{r_1} v_1^{s_1} \dots u_n^{r_n} v_n^{s_n}$$

such that $\sum_{i \in \underline{\mathbf{n}}} |w|_{u_i} \geq \sum_{i \in \underline{\mathbf{n}}} |x|_{u_i}$.

For w and x as above $wh \, \gamma^{\#} \, xk$ if $h = x$ and $k = w$ or

$$h = rv_1^{\lambda_1} \dots v_n^{\lambda_n} \quad k = ru_1^{\lambda_1} \dots u_n^{\lambda_n}$$

such that $r \in \{a, b\}$, $\lambda_1 + \ldots + \lambda_n = \sum_i \left(|w|_{u_i} - |x|_{u_i} \right)$ and each $\lambda_i \in \mathbb{N}^0$.

For example...

$$u_1 u_2^2 \, (\mathbf{av_1}) \, \gamma^{\#} \, u_1 u_2 v_2 \, (\mathbf{au_1}).$$

Commutative cancellative presentation

Fix some $n \in \mathbb{N}$ and let κ be a relation on $\mathscr{A}$ where

$$\kappa = \gamma \cup \{(u_i v_j, u_j v_i) : i, j \in \mathbf{n}\}.$$

The semigroup presentation $\langle \mathscr{A}_n : \kappa \rangle$ is clearly commutative. It remains to show that it is cancellative and strange!

The *rewriting system* on $\mathscr{A}^+$ with rewriting rules

$$bv_i \to au_i \quad v_i u_j \to u_i v_j$$

such that $i, j \in \underline{\mathbf{n}}$ and $i < j$ is noetherian and confluent.

Normal forms

This time, normal forms appear as either...

$$a^{p_0}\, b^{q_0}\, u_1^{p_1} \ldots u_n^{p_n}$$
$$a^{p_0}\, u_1^{p_1} \ldots u_i^{p_i}\, v_i^{q_i} \ldots u_n^{p_n}\, v_n^{q_n}$$

We have already looked at these cases!

Cancellativity was shown using induction!

Suppose w, x are in the same normal form, then $wa\,\kappa^{\#}\,xa$ implies $wa = xa$ in normal form...

What next?

Fix $n \in \mathbb{N}$ where $n \geq 2$. Let us define some relations

$$\sigma = \{(w, w^2) : w \in \mathscr{A}^+\}$$
$$\mu = \{([a]_{\sigma^\#}[u_i]_{\sigma^\#}, [b]_{\sigma^\#}[v_i]_{\sigma^\#}) : i \in \underline{\mathbf{n}}\}$$
$$\nu = \{([w]_{\rho^\#}, [w^2]_{\rho^\#}) : w \in S_\rho\}$$

Let $\mathscr{A}$ and S_ρ as before and $\mathscr{B}_{\mathscr{A}} = \mathscr{A}^+/\sigma^\#$. Is it true that

$$\mathscr{B}_{\mathscr{A}}/\mu^\# \cong S_\rho/\nu^\#?$$