Straight Left Orders

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Abstract

A subsemigroup S of Q is a *left order* in Q if every element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a^\sharp$ is the inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S lies in a subgroup of Q. In the case where one insists also that $a\mathcal{R}b$ in Q, S is said to be a *straight left order* in Q. We remark that if $\mathcal{H}$ is a congruence on Q, then any left order in Q is straight. The aim of this paper is to describe those semigroups that are straight left orders.
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1 Introduction

A subsemigroup S of a semigroup Q is a *left (right) order* in Q if every element of Q can be written as $a^\sharp b$ ($ba^\sharp$) where $a, b \in S$ and $a^\sharp$ is a group inverse of a and if, in addition, every square–cancellable element of S lies in a subgroup of Q. For the definitions of group inverse and square–cancellable element we refer the reader to Section 2, where further details of the terms used in this introduction may be found. If S is a left (right) order in Q then Q is a *semigroup of left (right) quotients* of S ; if S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S. These notions were introduced in [FP], although only in the context of orders in completely 0–simple semigroups.

Our approach to orders in semigroups is obviously inspired by the concept of a classical order in ring theory. Ore's theorem tells us that a ring R is a (classical) left order in some ring Q if and only if R contains a non zero divisor and satisfies the left Ore condition, that is, given any $a, b \in R$, where a is a non zero divisor, there exist elements $c, d \in R$ where c is a non zero divisor such that $cb = da$. Orders in various special classes of rings have also been described ; perhaps the best known example of a result of this sort is Goldie's celebrated theorem characterising left orders in simple artinian rings.

Fountain and Petrich give a description in [FP] of orders in completely 0–simple semigroups. A number of subsequent papers have characterised left orders in other well known classes of semigroups. But what of an analogue to Ore's theorem? What are the semigroups that can occur as left orders in any semigroup? This question is much more complex than the corresponding one for rings, essentially because we consider group inverses in *any* subgroup of a semigroup of left quotients, whereas for rings one concentrates on the group of units. A first step in the direction of an answer is made in [G4]. In that paper we characterise left orders in semigroups in the class of regular $\mathcal{H}$–semigroups, where a semigroup Q is an *$\mathcal{H}$–semigroup* if Green's relation $\mathcal{H}$ is a congruence on Q. A sequel, [G5], shows how this result yields as corollaries the previous characterisations of (left) orders in semigroups in particular classes, for all those classes considered had been classes of regular $\mathcal{H}$–semigroups.

If S is a left order in a regular $\mathcal{H}$–semigroup Q, then as shown in [G1], any element of Q may be written as $a^\sharp b$ where $a, b \in S$ and $a\mathcal{R}b$ in Q, a property which facilitates greatly consideration of products of elements in Q. In view of this we say that a left order S in a semigroup Q is *straight* if every element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a\mathcal{R}b$ in Q. *Straight right orders* and *straight orders* are defined in the obvious way. The aim of this paper is to characterise those semigroups that are straight left orders.

In Section 2 we give a number of definitions and preliminary results. Section 3 introduces and investigates the notion of a **–pair* $\mathcal{P} = (\leq_\ell, \leq_r)$ of preorders on a semigroup S. If S is a subsemigroup of a semigroup Q then defining $\leq_\ell$

$(\leq_r)$ on S by $a \leq_\ell b$ ($a \leq_r b$) if and only if $Q^1a \subseteq Q^1b$ ($aQ^1 \subseteq bQ^1$), then $\mathcal{P} = (\leq_\ell, \leq_r) = \mathcal{P}(Q)$ is always a *–pair, the **–pair induced by* Q. Additionally, if S is a straight left order in Q then $\mathcal{P}(Q)$ satisfies a number of properties : any *–pair with these properties we call an *embeddable *–pair*. Using the concept of an embeddable *–pair we describe in Theorem 4.1 of Section 4 straight left orders.

Given a straight left order S in Q, if $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}^{*S}$ and $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}^{*S}$ then S is *stratified* in Q ; stratified left orders have been investigated in a number of papers. If $\mathcal{P}(Q) = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$, then we shall say that S is *fully stratified* in Q. It is immediate from the definitions that a fully stratified left order is stratified, however, the converse is not always true. In Section 5 we apply Theorem 4.1 to characterise fully stratified left orders.

The final section shows how the description of left orders in regular $\mathcal{H}$–semigroups given in [G4] may also be deduced from Theorem 4.1.

2 Preliminaries

We assume a familiarity with the basic notions of semigroup theory, in particular, Green's relations. As far as possible we follow standard notation and terminology, as can be found in [H].

The relation $\leq_{\mathcal{L}}$ is defined on a semigroup S by the rule that for any elements a, b of S, $a \leq_{\mathcal{L}} b$ if and only if $S^1a \subseteq S^1b$. Clearly $\leq_{\mathcal{L}}$ is a preorder that is right compatible with multiplication and whose associated equivalence relation is $\mathcal{L}$. The dual relation to $\leq_{\mathcal{L}}$ is denoted by $\leq_{\mathcal{R}}$.

The relation $\leq_{\mathcal{L}^*}$ is defined on a semigroup S by the rule that for any elements a, b of S, $a \leq_{\mathcal{L}^*} b$ if and only if

$$bx = by \text{ implies that } ax = ay$$

for all $x, y \in S^1$. Then $\leq_{\mathcal{L}^*}$ is a preorder that is right compatible with multiplication, as is the associated equivalence relation $\mathcal{L}^*$. The preorder $\leq_{\mathcal{R}^*}$ and the equivalence relation $\mathcal{R}^*$ are defined dually. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ have the following alternative description.

Lemma 2.1 *[F] Let S be a semigroup and let $a, b \in S$. Then $a\mathcal{L}^*b (a\mathcal{R}^*b)$ if and only if $a\mathcal{L}b$ ($a\mathcal{R}b$) in some oversemigroup of S.*

The intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$ on any semigroup S is denoted by $\mathcal{H}^*$. It is easily seen that $\leq_{\mathcal{L}} \subseteq \leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}} \subseteq \leq_{\mathcal{R}^*}$, so that $\mathcal{L} \subseteq \mathcal{L}^*, \mathcal{R} \subseteq \mathcal{R}^*$ and $\mathcal{H} \subseteq \mathcal{H}^*$. If S is a regular semigroup then $\leq_{\mathcal{L}} = \leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}} = \leq_{\mathcal{R}^*}$ so that $\mathcal{L} = \mathcal{L}^*, \mathcal{R} = \mathcal{R}^*$ and $\mathcal{H} = \mathcal{H}^*$.

An element a of a semigroup S is *square–cancellable* if $a\mathcal{H}^*a^2$. Thus a is square–cancellable if and only if for all $x, y \in S^1$,

$$a^2x = a^2y \text{ implies that } ax = ay$$

and

$$xa^2 = ya^2 \text{ implies that } xa = ya.$$

Note that if S is a subsemigroup of Q and $a \in S$ lies in a subgroup of Q, then a is square–cancellable in S. By definition of (left, right) order, all such elements must lie in subgroups of any semigroup of (left,right) quotients. The set of square–cancellable elements of a semigroup S is denoted by $\mathcal{S}(S)$.

Let a be an element of a semigroup S. If a lies in a subgroup of S, then H_a is a subgroup, the maximum subgroup containing a. Thus if $a^\sharp$ is the inverse of a, in the sense of group theory, in a subgroup of S, then $a^\sharp$ is the inverse of a in H_a and so is well–defined. Given a semigroup S and $a \in S$, by writing $a^\sharp$ it will be implicit that a lies in a subgroup of S.

The notion of a left order in a group is much older than that of a left order in an arbitrary semigroup. We recall here that a semigroup is *right (left) reversible* if $Sa \cap Sb \neq \emptyset (aS \cap bS \neq \emptyset)$ for any $a, b \in S$. A theorem of Ore and Dubreil (Theorem 1.24 of [CP]) states that a semigroup S is a left order in a group G if and only if S is right reversible and cancellative.

This paper is concerned with a special sort of left orders : *straight* left orders. We summarize here some of their properties.

Proposition 2.2 *[G1] Let S be a straight left order in a semigroup Q.*

(i) If $a \in \mathcal{S}(S)$, then $H_a^Q \cap S$ is a left order in H_a^Q.

(ii) Every $\mathcal{H}$–class of Q contains an element of S ; if $q \in Q$ and $q = a^\sharp b$ where $a \in \mathcal{S}(S), b \in S$ and $a\mathcal{R}b$ in Q, then $q\mathcal{H}b$ in Q.

Considering only left orders that are straight imposes regularity on the semigroups of left quotients.

Lemma 2.3 *The following conditions are equivalent for a semigroup Q :*

(i)Q is regular;

(ii)Q is a straight left order in Q;

(iii)there is a straight left order in Q.

Proof That (i) implies (ii) is easy and that (ii) implies (iii) is immediate. Suppose that (iii) holds and S is a straight left order in Q . Then if $q \in Q, q = a^\sharp b$ for some $a \in \mathcal{S}(S), b \in S$ with $a\mathcal{R}b$ in Q. By Proposition 2.2, $q\mathcal{H}b$ in Q so that $q\mathcal{R}a\mathcal{H}aa^\sharp$ in Q. Thus every $\mathcal{R}$–class of Q contains an idempotent and so Q is regular.

Finally in this section we make a remark on notation. If (X)(ℓ) denotes a condition having an obvious left–right dual, then (X)(r) will denote the dual condition. In this case, '(X)' is shorthand for '(X)(ℓ) and (X)(r)'.

3 Embeddable *–pairs

An ordered pair $\mathcal{P} = (\leq_\ell (\mathcal{P}), \leq_r (\mathcal{P}))$ of preorders on a semigroup S is a *–*pair* if $\leq_\ell (\mathcal{P})$ is right compatible with multiplication, $\leq_r (\mathcal{P})$ is left compatible with multiplication , $\leq_\ell (\mathcal{P}) \subseteq \leq_{\mathcal{L}^*}$ and $\leq_r (\mathcal{P}) \subseteq \leq_{\mathcal{R}^*}$. Clearly $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ is a *–pair for any semigroup. Given a *–pair $\mathcal{P} = (\leq_\ell (\mathcal{P}), \leq_r (\mathcal{P}))$ for a semigroup S, then we denote by $\mathcal{L}'(\mathcal{P})$ and $\mathcal{R}'(\mathcal{P})$ the equivalence relations associated with $\leq_\ell (\mathcal{P})$ and $\leq_r (\mathcal{P})$ respectively. We remark that $\mathcal{L}'(\mathcal{P})$ is a right congruence and $\mathcal{R}'(\mathcal{P})$ is a left congruence. Where there is no danger of ambiguity $\leq_\ell (\mathcal{P}), \leq_r (\mathcal{P}), \mathcal{L}'(\mathcal{P})$ and $\mathcal{R}'(\mathcal{P})$ are written more simply as $\leq_\ell, \leq_r, \mathcal{L}'$ and $\mathcal{R}'$. The notation for the equivalence relation $\mathcal{L}'(\mathcal{P}) \cap \mathcal{R}'(\mathcal{P})$ is $\mathcal{H}'(\mathcal{P})$ or $\mathcal{H}'$ and the $\mathcal{H}'(\mathcal{P})$–class of an element a is $H'_a(\mathcal{P})$ or H'_a. If $a\mathcal{H}'(\mathcal{P})a^2$ then a is $\mathcal{P}$–*good* or simply *good*.

If S is a subsemigroup of a semigroup Q then $\mathcal{P}(Q) = (\leq_{\mathcal{L}^Q} \cap (S \times S), \leq_{\mathcal{R}^Q} \cap (S \times S))$ is clearly a *–pair for S : it is called the *–*pair for* S *induced by* Q.

We wish to consider the *–pair for a semigroup S induced by a semigroup Q in which S is a straight left order. To describe such *–pairs it will be convenient to use the following notions.

Let $\mathcal{P} = (\leq_\ell, \leq_r)$ be a *–pair for a semigroup S. For a good element a of S, put

$$L(a, \mathcal{P}) = L(a) = \{b \in S : b \leq_\ell a\}$$

and

$$\overline{L}(a, \mathcal{P}) = \overline{L}(a) = \{[b]_{\mathcal{L}'} : b \in L(a)\}$$

where $[b]_{\mathcal{L}'}$ is the $\mathcal{L}'$–equivalence class of b. Note that $L(a)$ is a union of $\mathcal{L}'$–classes and if $b \in L(a)$ then $ba \leq_\ell a^2 \mathcal{H}' a$ so that $ba \in L(a)$. Thus one may define a map $\rho_a(\mathcal{P}) = \rho_a : L(a) \to L(a)$ by $b\rho_a = ba$. Further, as $\mathcal{L}'$ is a right congruence, $\overline{\rho}_a(\mathcal{P}) = \overline{\rho}_a : \overline{L}(a) \to \overline{L}(a)$ given by $([b]_{\mathcal{L}'})\overline{\rho}_a = [ba]_{\mathcal{L}'}$ is a map. One defines in a dual manner $R(a, \mathcal{P}) = R(a), \overline{R}(a, \mathcal{P}) = \overline{R}(a)$ and maps $\lambda_a(\mathcal{P}) = \lambda_a : R(a) \to R(a), \overline{\lambda}_a(\mathcal{P}) = \overline{\lambda}_a : \overline{R}(a) \to \overline{R}(a)$.

The main theorem of this paper gives necessary and sufficient conditions on a *–pair $\mathcal{P}$ for a semigroup S such that S is a straight left order in a semigroup Q where $\mathcal{P} = \mathcal{P}(Q)$. Given such a *–pair $\mathcal{P}$, we know that Q is uniquely determined up to isomorphism.

Proposition 3.1 *Let S be a straight left order in semigroups Q and Q', where $\mathcal{P}(Q) = \mathcal{P}(Q')$. Then Q is isomorphic to Q' under an isomorphism which restricts to the identity map on S.*

Proof This is an immediate consequence of Theorem 3.1 of [G2].

A subsemigroup S of a semigroup Q is *very large in Q* if S has non–empty intersection with every $\mathcal{H}$–class of Q. From Proposition 2.2 we know that if S is a straight left order in Q then S must be very large in Q.

Proposition 3.2 *Let S be a very large subsemigroup of a regular semigroup Q and let $\mathcal{P}(Q) = \mathcal{P} = (\leq_\ell, \leq_r)$. Then S satisfies the following conditions with respect to $\mathcal{P}$.*
(Ei)$\mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}'$.
(Eii)(ℓ)For all $b, c \in S, b \leq_\ell c$ if and only if $b\mathcal{L}'dc$ for some $d \in S$;
(Eii)(r).
(Eiii)Every $\mathcal{L}'$–class and every $\mathcal{R}'$–class contains a good element.
(Eiv)For all good elements a, ρ_a is one–one and preserves $\mathcal{R}'$–classes, $\overline{\rho}_a$ is one–one, λ_a is one–one and preserves $\mathcal{L}'$–classes and $\overline{\lambda}_a$ in one–one.

Proof (Ei)(Eiii) These follow directly from the fact that S is very large in Q and $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ in Q.

(Eii)(ℓ) If $b\mathcal{L}'dc$ then $Qb = Qdc \subseteq Qc$ so that $b \leq_\ell c$. Conversely, if $b \leq_\ell c$ then $b = qc$ for some $q \in Q$. Now $q\mathcal{H}h$ for some $h \in S$ so that $b = qc\mathcal{L}hc$ in Q and $b\mathcal{L}'hc$ in S.

(Eiv) Given $b, c \in L(a)$, we know $Qb \subseteq Qa$ and $Qc \subseteq Qa$. Since a is good, $a^\sharp a = aa^\sharp \mathcal{H} a$ for some $a^\sharp \in Q$. Certainly $bQ = baa^\sharp Q \subseteq baQ \subseteq bQ$ so that $b\mathcal{R}'ba = b\rho_a$. If $b\rho_a = c\rho_a$ then $baa^\sharp = caa^\sharp$ from which we have that $b = c$. The proof the $\overline{\rho}_a$ is one–one is similar.

Dual arguments now complete the proof.

A *–pair $\mathcal{P}$ for a semigroup S is an *embeddable *–pair* if $\mathcal{P}$ satisfies conditions (Ei),(Eii),(Eiii) and (Eiv). We give below an equivalent formulation of these conditions and deduce a number of further properties held by embeddable *–pairs. It is convenient at this point to list the conditions under consideration.

Let $\mathcal{P}$ be a *–pair for a semigroup S.
(Ev)(ℓ)For all $a, b \in S$ where a is good, if $b \leq_\ell a$ then $ba\mathcal{R}'b$.
(Evi)(ℓ) For all $a, b, c \in S$ where a is good, if $b, c \leq_\ell a$ and $ba = ca$, then $b = c$.
(Evii)(ℓ) For all $a, b, c \in S$ where a is good, if $b, c \leq_\ell a$ and $ba\mathcal{L}'ca$, then $b\mathcal{L}'c$.
(Fi)(ℓ) For all $a, b, c \in S$ where a is good, if $b, c \leq_\ell a$ and $ba \leq_\ell ca$, then $b \leq_\ell c$.
(Fii)(ℓ)For all $a, b \in S$ where a is good, if $a\mathcal{L}'b$ then $ba\mathcal{H}'b$.
(Fiii) If $b \in S$ then there exists an $s \in S$ with

$$b\mathcal{R}'bs\mathcal{L}'s\mathcal{R}'sb\mathcal{L}'b;$$

further, given such an s, bs and sb are good, $bsb\mathcal{H}'b$ and $sbs\mathcal{H}'s$.
(Fiv) If a is good then H'_a is a subsemigroup all of whose elements are good.
(Fv)(ℓ) For all $a, b \in S$ where a is good, if $a \leq_\ell b$ then $ba\mathcal{L}'a$.

Lemma 3.3 *Let $\mathcal{P}$ be a *–pair for a semigroup S. Then $\mathcal{P}$ is an embeddable *–pair if and only if $\mathcal{P}$ satisfies conditions (Ei),(Eii),(Eiii),(Ev),(Evi) and (Evii).*

Proof This is just a matter of rewording.

Lemma 3.4 *Let $\mathcal{P} = (\leq_\ell, \leq_r)$ be an embeddable *–pair for a semigroup S. Then $\mathcal{P}$ satisfies conditions (Fi),...,(Fv).*

Proof (Fi)(ℓ) If $b, c \leq_\ell a$ where a is good and $ba \leq_\ell ca$, then we know from (Eii)(ℓ) that $ba\mathcal{L}'dca$ for some $d \in S$ and further, $dc \leq_\ell c \leq_\ell a$. Thus by (Evii)($\ell$), $b\mathcal{L}'dc$ and it follows that $b \leq_\ell c$.

(Fii)(ℓ) If $a\mathcal{L}'b$ where a is good then certainly by (Ev)(ℓ) we have $ba\mathcal{R}'b$. In addition, $ba\mathcal{L}'a^2\mathcal{H}'a\mathcal{L}'b$ so that $ba\mathcal{L}'b$.

(Fiii) If $b \in S$ then using condition (Eiii) there are good elements c, d in S with $c\mathcal{R}'b\mathcal{L}'d$. Then from (Ei) we may choose $s \in S$ with $c\mathcal{L}'s\mathcal{R}'d$. We claim that s is the element required. For using (Fii), $b\mathcal{H}'bd\mathcal{R}'bs$ and so by symmetry we obtain

$$b\mathcal{R}'bs\mathcal{L}'s\mathcal{R}'sb\mathcal{L}'b.$$

It is then easy to see that bs and sb are good. The last part of the condition is immediate from (Fii).

(Fiv) If a is good and $b, c \in H'_a$ then $bc\mathcal{R}'ba\mathcal{H}'b\mathcal{H}'a$, using (Fii). Dually, $bc\mathcal{L}'a$ so that $bc \in H'_a$ and H'_a is a subsemigroup, which clearly can consist only of good elements.

(Fv)(ℓ) If $a, b \in S$ where a is good and $a \leq_\ell b$ then $a\mathcal{H}'a^2 \leq_\ell ba \leq_\ell a$ so that $a\mathcal{L}'ba$.

4 Straight left orders

Given a straight left order S in Q, then by Propositin 2.2 and Lemma 2.3, Q is regular and S is very large in Q. Hence Proposition 3.2 gives that the *–pair $\mathcal{P}(Q)$ is an embeddable *–pair. In this section we give necessary and sufficient conditions on an embeddable *–pair $\mathcal{P}$ for a semigroup S such that S is a straight left order in a semigroup Q where $\mathcal{P} = \mathcal{P}(Q)$.

Theorem 4.1 *Let S be a semigroup having an embeddable *–pair $\mathcal{P} = (\leq_\ell, \leq_r)$. Then S is a straight left order in a semigroup Q such that $\mathcal{P} = \mathcal{P}(Q)$ if and only if S satisfies the following conditions with respect to $\mathcal{P}$.*
(Gi)$\mathcal{S}(S) = \{a \in S : a \text{ is good}\}$.
(Gii) If $a \in \mathcal{S}(S)$ then H'_a is right reversible.
(Giii) If $b, c \in S$ then $b \leq_\ell c$ if and only if there exist $h \in \mathcal{S}(S)$ and $k \in S$ with $h\mathcal{R}'k, b \leq_r h$ and $hb = kc$.
(Giv) If $a, c \in \mathcal{S}(S)$ and $b \in S$ with $a\mathcal{R}'b$, then there exist $u, h \in \mathcal{S}(S), v, k \in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, au\mathcal{R}'bc, v \leq_\ell c, u \leq_r a, k \leq_\ell a, hua = ka^2$ and $hvc^2 = kbc$.

Remark Given a *–pair $\mathcal{P} = (\leq_\ell, \leq_r)$ satisfying (Gi) and (Giii), condition (Evii)(ℓ) in the list of conditions $\mathcal{P}$ must satisfy to be an embeddable *–pair becomes superfluous. For in this case, if $b, c \in S, a \in \mathcal{S}(S), b, c \leq_\ell a$ and $ba\mathcal{L}'ca$, then using (Giii), $hba = kca$ for some $h \in \mathcal{S}(S)$ and $k \in S$ with $ba \leq_r h$ and $h\mathcal{R}'k$. By (Eii)(ℓ) and (Evi)(ℓ) we have $hb = kc$ and by (Ev)(ℓ) $b\mathcal{R}'ba \leq_r h$. Again by (Giii), $b \leq_\ell c$. Dually one obtains $c \leq_\ell b$ so that $b\mathcal{L}'c$.

Proof of Theorem 4.1

Let S be a straight left order in a semigroup Q where $\mathcal{P} = \mathcal{P}(Q)$. Then $a \in \mathcal{S}(S)$ if and only if $a\mathcal{H}a^2$ in Q. So $a \in \mathcal{S}(S)$ if and only if $a\mathcal{H}'a^2$ in S, that is, a is good. Given $a \in \mathcal{S}(S)$ then H_a^Q is a group and by Propositon 2.2, $H'_a = S \cap H_a^Q$ is a left order in H_a^Q. Theorem 1.24 of [CP] tells us that H'_a is right reversible.

To see that (Giii) holds, take $b, c \in S$ where $b \leq_\ell c$. Then $b = h^\sharp kc$ for some $h \in \mathcal{S}(S)$ and $k \in S$ where $h\mathcal{R}^Q k$. Then $bQ = h^\sharp kcQ \subseteq h^\sharp Q = hQ$ and $hb = hh^\sharp kc = kc$. Conversely, given elements $b, c, k \in S$ and $h \in \mathcal{S}(S)$ with $b \leq_r h$ and $hb = kc$, then $b = h^\sharp hb = h^\sharp kc$ so that $Qb \subseteq Qc$ and $b \leq_\ell c$ as required.

Finally we consider (Giv). Let $a, c \in \mathcal{S}(S)$ and $b \in S$ with $a\mathcal{R}'b$. Then $a^\sharp bc^\sharp = u^\sharp v$ for some $u \in \mathcal{S}(S)$ and $v \in S$ with $u\mathcal{R}'v$. This gives that

$$Qv = Qu^\sharp v = Qa^\sharp bc^\sharp \subseteq Qc^\sharp = Qc$$

and

$$uQ = u^\sharp vQ = a^\sharp bc^\sharp Q \subseteq aQ.$$

Now $bc^\sharp = au^\sharp v$ so that $au\mathcal{R}^Q au^\sharp v = bc^\sharp \mathcal{R}^Q bc$. We also have that $ua^\sharp bc = vc^2$ and $ua^\sharp = h^\sharp k$ for some $h \in \mathcal{S}(S)$ and $k \in S$ with $h\mathcal{R}'k$. Since $u \leq_r a$ and u is good we have by (Fv) that $ua\mathcal{R}'u$, so that

$$u\mathcal{R}^Q ua\mathcal{R}^Q ua^\sharp \mathcal{R}^Q h\mathcal{R}^Q k$$

and u, v, h and k are $\mathcal{R}'$–related. Also,

$$Qk = Qh^\sharp k = Qua^\sharp \subseteq Qa^\sharp = Qa.$$

From $ua^\sharp = h^\sharp k$ we have $hua = ka^2$ and from $h^\sharp kbc = vc^2$ we have $kbc = hvc^2$.

Conversely, we suppose that S satisfies conditions (Gi),...,(Giv) with respect to $\mathcal{P}$. Our aim is to construct via equivalence classes of ordered pairs of elements of S a semigroup Q in which S is embedded as a straight left order. This we do using a series of lemmas.

Let

$$\sum = \{(a, b) \in S \times S : a \in \mathcal{S}(S), a\mathcal{R}'b\}.$$

We note that by (Eiii), $\sum$ is a non–empty set. Define a relation $\sim$ on $\sum$ by

$$(a, b) \sim (c, d) \text{ if and only if } a\mathcal{R}'c \text{ and there exist } h, k \in H'_a \text{ with}$$

$$ha^2 = kca, hb = kd.$$

If $(a, b) \sim (c, d)$ then by (Fii) and (Fiv) we have that $b\mathcal{H}'hb = kd\mathcal{H}'d$.

Lemma 4.2 *The relation $\sim$ is an equivalence relation on $\sum$.*

Proof Let $(a, b) \in \sum$. Putting $h = k = a$ we see that $(a, b) \sim (a, b)$.

Suppose now that $(a, b) \sim (c, d)$ so that $a\mathcal{R}'c$ and $ha^2 = kca, hb = kd$ for some $h, k \in H'_a$. We have that $ac\mathcal{H}'c\mathcal{H}'c^2$ so that by (Gii), $uac = vc^2$ for some $u, v \in H'_c$. Then $uac, hac \in H'_c$ so that $suac = thac$ for some $s, t \in H'_c$. Since $\mathcal{R}' \subseteq \mathcal{R}^*$ we have $hac = kc^2$ so that from

$$svc^2 = suac = thac = tkc^2,$$

$c^2\mathcal{R}'d$ and $ac\mathcal{R}'b$ we have $svd = tkd$ and $sub = thb$. Thus

$$sub = thb = tkd = svd$$

and as $ub\mathcal{R}'vd\mathcal{R}'s$, condition (Evi) gives that $ub = vd$ and so $(c, d) \sim (a, b)$.

It remains to show that $\sim$ is transitive. Consider $(a, b), (c, d), (m, n) \in \sum$ where $(a, b) \sim (c, d)$ and $(c, d) \sim (m, n)$. Then $a\mathcal{R}'b\mathcal{R}'c\mathcal{R}'d\mathcal{R}'m\mathcal{R}'n$ and there are elements $h, k \in H'_a, u, v \in H'_c$ with

$$ha^2 = kca, hb = kd, uc^2 = vmc, ud = vn.$$

Since H'_c is right reversible there are elements $u', v' \in H'_c$ with $u'c^2 = v'amc$ and then one may pick $s, t \in H'_c$ with $su = tu'$. Thus

$$svmc = suc^2 = tu'c^2 = tv'amc$$

and as $mc\mathcal{R}'n, svn = tv'an$. Also,

$$tu'd = sud = svn = tv'an$$

gives by (Evi) that $u'd = v'an$. Now pick $p, q \in H'_c$ with $pkc^2 = qu'c^2$. Then

$$(ph)a^2 = pkca = qu'ca = (qv'a)ma$$

and

$$(ph)b = pkd = qu'd = (qv'a)n$$

using the fact that $a\mathcal{R}^*c\mathcal{R}^*c^2\mathcal{R}^*d$. But $ph, qv'a \in H'_a$ and so $\sim$ is transitive.

Put $Q = \sum / \sim$ and denote the $\sim$–equivalence class of $(a, b) \in \sum$ by $[a, b]$. Define a multiplication on Q by

$$[a, b][c, d] = [u, vd]$$

where there are elements $u, h \in \mathcal{S}(S), v, k \in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, au\mathcal{R}'bc, v\leq_\ell c, u\leq_r a, k\leq_\ell a, hua = ka^2$ and $hvc^2 = kbc$. Certainly by (Giv) such elements always exist. By (Ev),$vd\mathcal{R}'vc\mathcal{R}'v\mathcal{R}'u$ so that $(u, vd) \in\sim$.

Lemma 4.3 *The given multiplication on Q is well-defined.*

Proof Suppose that $[a,b]=[s,t]$ and $[c,d]=[x,y]$ in Q. Then $a\mathcal{R}'b\mathcal{R}'s\mathcal{R}'t$ and there exist $m,n\in H'_a$ with

$$ma^2=nsa, mb=nt. \tag{1}$$

Also,$c\mathcal{R}'d\mathcal{R}'x\mathcal{R}'y$ and there exist $p,q\in H'_c$ with

$$pc^2=qxc, pd=qy. \tag{2}$$

By the definition of multiplication,

$$[a,b][c,d]=[u,vd]$$

where there are elements $u,h\in\mathcal{S}(S)$, $v,k\in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v$, $au\mathcal{R}'bc$, $v\leq_\ell c$, $u\leq_r a$, $k\leq_\ell a$ and

$$hua=ka^2, hvc^2=kbc \tag{3}$$

and

$$[s,t][x,y]=[u',v'y]$$

where there are elements $u',h'\in\mathcal{S}(S)$, $v',k'\in S$ with $h'\mathcal{R}'k'\mathcal{R}'u'\mathcal{R}'v'$, $su'\mathcal{R}'tx$, $v'\leq_\ell x$, $u'\leq_r s$, $k'\leq_\ell s$ and

$$h'u's=k's^2, h'v'x^2=k'tx. \tag{4}$$

To show that multiplication is well–defined we must prove $(u,vd)\sim(u',v'y)$. Since $u\leq_r a$ and $ma^2=nsa$ we know that $mau=nsu$. Then

$$nsu\mathcal{R}'mbc=ntc\mathcal{R}'ntx\mathcal{R}'nsu'.$$

Certainly $ns\in\mathcal{S}(S)$ and $u\leq_r a\mathcal{R}'ns$, $u'\leq_r s\mathcal{R}'ns$ so that by (Evii), $u\mathcal{R}'u'$. Then as H'_u is right reversible, there are elements $e,f\in H'_u$ with

$$eu^2=fu'u. \tag{5}$$

The elements $hu^2, h'u'u$ lie in H'_u so there are elements g,i in H'_u with

$$ghu^2=ih'u'u. \tag{6}$$

We can then pick $j,\ell\in H'_u$ with

$$jeu^2=\ell ghu^2.$$

Now $u\leq_r ma$ and $ma\in\mathcal{S}(S)$ so by (Ev), $mau\mathcal{L}'u\mathcal{H}'\ell ghu^2$. From (Giii) there are elements $o\in\mathcal{S}(S)$ and $r\in S$ with $o\mathcal{R}'r$, $\ell ghu^2\leq_r o$ and $o\ell ghu^2=rmau$. This gives that

$$rnsu=rmau=o\ell ghu^2=o\ell gkau \tag{7}$$

using (3) and $u \leq_r a$. But $au\mathcal{R}^*bc$ so that $rmbc = o\ell gkbc$. Then using (1) and (3),

$$rntc = rmbc = o\ell gkbc = o\ell ghvc^2.$$

Since $c\mathcal{R}^*x$ this gives that $rntx = o\ell ghvcx$. We use (7) again to obtain

$$rnsu = o\ell ghu^2 = o\ell ih'u'u = o\ell ik'su,$$

from (4) and the fact that $u\mathcal{R}'u' \leq_r s$. But $su\mathcal{R}'su'\mathcal{R}'tx$ and so

$$o\ell ghvcx = rntx = o\ell ik'tx.$$

From $\ell\mathcal{R}'\ell ghu^2 \leq_r o$ and $o \in \mathcal{S}(S)$,(Eii) and (Eiv) give that $\ell ghvcx = \ell ik'tx$. The same two conditions then yield $ghvcx = ik'tx$ and so by (4) we have that $ghvcx = ih'v'x^2$. Now from $jeu^2 = \ell ghu^2$ we have $jev = \ell ghv$ and so $jevcx = \ell ih'v'x^2$. From (6),

$$jfu'u = jeu^2 = \ell ghu^2 = \ell ih'u'u,$$

so that $jfv' = \ell ih'v'$. Thus $jevcx = jfv'x^2$ and so $evcx = fv'x^2$. Since $evcx \leq_\ell x\mathcal{H}'pcx$ there are elements $w \in \mathcal{S}(S)$ and $z \in S$ with $w\mathcal{R}'z, evcx \leq_r w$ and $wevcx = zpcx$. This gives by (2) that

$$wfv'x^2 = wevcx = zpcx = zqx^2$$

and as $cx\mathcal{R}'d$ and $x^2\mathcal{R}'y$ we have that

$$wevd = zpd = zqy = wfv'y.$$

But $evd\mathcal{R}'evcx \leq_r w$ and

$$fv'y\mathcal{R}'fv'x\mathcal{R}'fv'\mathcal{R}'v'\mathcal{R}'ev\mathcal{R}'evc\mathcal{R}'evd$$

so by (Evi), $evd = fv'y$. Thus $(u, vd) \sim (u', v'y)$ as required.

Lemma 4.4 *If $[a, b] \in Q$ and $x \in \mathcal{S}(S)$ where $x\mathcal{R}'a$, then $[xa, xb] \in Q$ and $[a, b] = [xa, xb]$.*

Proof Since $xa\mathcal{H}'a$ and $xb\mathcal{H}'b$, certainly $[xa, xb] \in Q$. By the right reversibility of H'_a, there are elements $p, q \in H'_a$ with $pa^2 = qxa^2 = q(xa)a$, and as $a^2\mathcal{R}^*b$ one has $pb = qxb$.

Lemma 4.5 *The given multiplication on Q is associative.*

Proof Given $[a,b],[c,d],[h,k]\in Q$ let
$$X=([a,b][c,d])[h,k]$$
and
$$Y=[a,b]([c,d][h,k]).$$
Then $X=[u_1,v_1d][h,k]$ where there are elements $u_1,h_1\in\mathcal{S}(S),v_1,k_1\in S$ with $h_1\mathcal{R}'k_1\mathcal{R}'u_1\mathcal{R}'v_1, au_1\mathcal{R}'bc, v_1\leq_\ell c, u_1\leq_r a, k_1\leq_\ell a$ and
$$h_1u_1a=k_1a^2, h_1v_1c^2=k_1bc. \tag{8}$$
Now by Lemmas 4.3 and 4.4,
$$X=[h_1u_1,h_1v_1d][h,k]=[u_2,v_2k]$$
where there are elements $u_2,h_2\in\mathcal{S}(S),v_2,k_2\in S$ with $h_2\mathcal{R}'k_2\mathcal{R}'u_2\mathcal{R}'v_2$, $h_1u_1u_2\mathcal{R}'h_1v_1dh, v_2\leq_\ell h, u_2\leq_r h_1u_1, k_2\leq_\ell h_1u_1$ and
$$h_2u_2h_1u_1=k_2(h_1u_1)^2, h_2v_2h^2=k_2h_1v_1dh. \tag{9}$$
Considering Y, we have that
$$Y=[a,b][u_3,v_3k]$$
where there are elements $h_3,u_3\in\mathcal{S}(S),k_3,v_3\in S$ with $h_3\mathcal{R}'k_3\mathcal{R}'u_3\mathcal{R}'v_3$, $cu_3\mathcal{R}'dh, v_3\leq_\ell h, u_3\leq_r c, k_3\leq_\ell c$and
$$h_3u_3c=k_3c^2, h_3v_3h^2=k_3dh. \tag{10}$$
Again by Lemmas 4.3 and 4.4,
$$Y=[a,b][h_3u_3,h_3v_3k]=[u_4,v_4h_3v_3k]$$
where there are elements $u_4,h_4\in\mathcal{S}(S),v_4,k_4\in S$ with $h_4\mathcal{R}'k_4\mathcal{R}'u_4\mathcal{R}'v_4$, $au_4\mathcal{R}'bh_3u_3, v_4\leq_\ell h_3u_3, u_4\leq_r a, k_4\leq_\ell a$ and
$$h_4u_4a=k_4a^2, h_4v_4(h_3u_3)^2=k_4bh_3u_3. \tag{11}$$
Using Lemma 4.4 once more gives that $X=[h_2u_2,h_2v_2k]$ and $Y=[h_4u_4,h_4v_4h_3v_3k]$. Since $h_3u_3\mathcal{H}'u_3\leq_r c$ we have from (8) that $h_1v_1ch_3u_3=k_1bh_3u_3$ and so as $u_4\leq_r a$,
$$h_1u_1u_4=k_1au_4\mathcal{R}'k_1bh_3u_3=h_1v_1ch_3u_3\mathcal{R}'h_1v_1cu_3\mathcal{R}'h_1v_1dh\mathcal{R}'h_1u_1u_2$$
so that by (Evii),$u_1u_4\mathcal{R}'u_1u_2$. Now $au_4\mathcal{R}'bh_3u_3\mathcal{R}'bu_3\leq_r bc\mathcal{R}'au_1$ so as $u_4,u_1\leq_r a$, (Fi) gives that $u_4\leq_r u_1$. In addition, $u_2\leq_r h_1u_1\mathcal{R}'u_1$ so that by (Evii), $u_4\mathcal{R}'u_2$. Then $h_2u_2\mathcal{R}'h_4u_4$ and we can pick $m,n\in H'_{h_2u_2}$ with
$$mh_2u_2h_2u_2=nh_4u_4h_2u_2.$$

We can now make the following sequence of deductions. From $h_2u_2\mathcal{R}'u_2$,

$$mh_2u_2^2 = nh_4u_4u_2$$

and so as $u^2 \leq_r h_1u_1$ and $u^2 \leq_r a$ we have from (9) and (11) that

$$mk_2h_1u_1u_2 = mh_2u_2^2 = nh_4u_4u_2 = nk_4au_2.$$

Now using (8) and $u_2 \leq_r a$,

$$mk_2k_1au_2 = mk_2h_1u_1u_2 = nk_4au_2$$

and so as $u_2\mathcal{R}'u_4$,

$$mk_2k_1au_4 = nk_4au_4.$$

But $au_4\mathcal{R}'bh_3u_3\mathcal{R}'bu_3$ and so

$$mk_2k_1bu_3 = nk_4bu_3.$$

We now use the fact that $u_3 \leq_r c$, $h_3u_3\mathcal{R}'u_3$,(8),(10) and (11) to obtain

$$mk_2h_1v_1cu_3 = nk_4bu_3 = nh_4v_4h_3u_3^2 = nh_4v_4k_3cu_3.$$

We know that $cu_3\mathcal{R}'dh$ and so

$$mk_2h_1v_1dh = nh_4v_4k_3dh$$

from which (9) and (10) give

$$mh_2v_2h^2 = nh_4v_4h_3v_3h^2.$$

Finally as $h^2\mathcal{R}^*k$ we have

$$mh_2v_2k = nh_4v_4h_3v_3k.$$

We have constructed a semigroup Q, which we now show is the semigroup of left quotients for which we are looking. First we show that S is embedded in Q.

Let $s \in S$: from (Eiii) we know that $s\mathcal{R}'a$ for some $a \in \mathcal{S}(S)$. Define $\phi : S \to Q$ by

$$s\phi = [a, as] \text{ where } a \in \mathcal{S}(S), a\mathcal{R}'s.$$

Since $as\mathcal{H}'s\mathcal{R}'a$ this definition makes sense. Moreover, if $b \in \mathcal{S}(S)$ and $b\mathcal{R}'s$, then $b\mathcal{R}'a$ and there are elements $h, k \in H'_a$ with $ha^2 = kba$. But then $has = kbs$ and so $[a, as] = [b, bs]$, showing that ϕ is well–defined.

Lemma 4.6 *The function ϕ embeds S in Q.*

Proof Suppose that $s, t \in S$ and $s\phi = t\phi$, so that $(a, as) \sim (b, bt)$ where $a, b \in \mathcal{S}(S), a\mathcal{R}'s$ and $b\mathcal{R}'t$. Thus $a\mathcal{R}'b$ and there are elements $h, k \in H'_a$ with $ha^2 = kba$ and $has = kbt$. Since $a\mathcal{R}'t$,

$$hat = kbt = has$$

so by (Evi), $s = t$ and ϕ is one–one.

To show that ϕ is a homomorphism, again consider $s, t \in S$ where $s\mathcal{R}'a, t\mathcal{R}'b$ and $a, b \in \mathcal{S}(S)$. Then

$$s\phi t\phi = [a, as][b, bt] = [u, vbt]$$

where there are elements $u, h \in \mathcal{S}(S), v, k \in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, au\mathcal{R}'asb$, $v \leq_\ell b, u \leq_r a, k \leq_\ell a, hua = ka^2$ and $hvb^2 = kasb$.

Choose $c \in \mathcal{S}(S)$ with $st\mathcal{R}'c$, so that $(st)\phi = [c, cst]$. Now $u \leq_r a, sb \leq_r s\mathcal{R}'a$ and so from $au\mathcal{R}'asb$, (Evii) gives $u\mathcal{R}'sb$. Then $u\mathcal{R}'sb\mathcal{R}'st\mathcal{R}'c$ and we may choose $p, q \in H'_u$ with $pu^2 = qcu$.

From $hua = ka^2$ we have $hus = kas$ so that

$$hvb^2 = kasb = husb$$

and so by (Evi), $vb^2 = usb$. But $b\mathcal{R}'t$ and so

$$pvbt = pust = qcst$$

as required.

In view of the preceding lemma we may where convenient identify an element s of S with its image under ϕ in Q, and S with $S\phi$.

Lemma 4.7 *Let $x \in \mathcal{S}(S)$ and $[a, b] \in Q$. Then*
(i)if $a \leq_r x$ then $[x, x][a, b] = [a, b]$,
(ii)if $b \leq_\ell x$ then $[a, b][x, x] = [a, b]$.

Proof (i) By definition of multiplication,

$$[x, x][a, b] = [u, vb]$$

where there are elements $h, u \in \mathcal{S}(S), k, v \in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, xu\mathcal{R}'xa$, $v \leq_\ell a$, $u \leq_r x, k \leq_\ell x, hux = kx^2$ and $hva^2 = kxa$. Hence $u\mathcal{R}'a$ and there are elements $p, q \in H'_a$ with $pa^2 = qua$. From $hux = kx^2$ and $a \leq_r x$ we have $hua = kxa = hva^2$ and so $ua = va^2$. Thus $pa^2 = qva^2$ and as $a^2\mathcal{R}'b, pb = qvb.$.

(ii) By definition of multiplication,

$$[a,b][x,x]=[u,vx]$$

where there are elements $h,u\in\mathcal{S}(S),k,v\in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, au\mathcal{R}'bx$, $v\leq_\ell x,\ u\leq_r a, k\leq_\ell a, hua=ka^2$ and $hvx^2=kbx$. Since $b\leq_\ell x$ by assumption, we have that $au\mathcal{R}'bx\mathcal{R}'b\mathcal{R}'ab$ and so $u\mathcal{R}'b\mathcal{R}'a$. Thus we may pick $p,q\in H'_a$ with $pa^2=qua$.

From $hvx^2=kbx$ and $hvx, kb\leq_\ell x$ we have $hvx=kb$. We now pick $m,n\in H'_u$ with $mhu=nqu$, from which we obtain

$$npa^2=nqua=mhua=mka^2.$$

Then

$$npb=mkb=mhvx=nqvx$$

since $ua\mathcal{R}'v\mathcal{R}'vx$, and then by (Evi), $pb=qvx$ as required.

Lemma 4.8 *Let $b,c,d\in H'_a$ where $a\in\mathcal{S}(S)$. Then $[b,c][c,b]=[d,d]$.*

Proof Certainly $[x,y]\in Q$ for any $x,y\in H'_a$. Now $[b,c][c,b]=[u,vb]$ where there are elements $h,u\in\mathcal{S}(S),k,v\in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, bu\mathcal{R}'c^2, v\leq_\ell c, u\leq_r b$, $k\leq_\ell b, hub=kb^2$ and $hvc^2=kc^2$.

Since $bu\mathcal{R}'c^2\mathcal{R}'bd$ and $u\leq_r b$, we have $u\mathcal{R}'d$ and so there are elements $p,q\in H'_d$ with $pd^2=qud$. Now $hub=kb^2=hvb^2$ and so $ub=vb^2$. Thus $pd^2=qvbd$ and so $pd=qvb$ as required.

The following lemma is an easy consequence of Lemmas 4.7 and 4.8.

Lemma 4.9 *Every square–cancellable element of S lies in a subgroup of Q. If $a\in\mathcal{S}(S)$ then the group $\mathcal{H}$–class H_a of Q has identity $[a,a]$ and the inverse of $a\phi=[a,a^2]$ in H_a is $[a^2,a]$. Moreover, if $b\in H'_a$ then $[a,a]=[b,b]$.*

Lemma 4.10 *Let $q=[a,b]\in Q$. Then $q=a^\sharp b$.*

Proof Given $q=[a,b]$ by definition we have that $a\in\mathcal{S}(S)$ and $a\mathcal{R}'b$. Thus a lies in a subgroup of Q with group inverse $a^\sharp=[a^2,a]$ and $b\phi=[a,ab]$. Then

$$a^\sharp b=[a^2,a][a,ab]=[u,vab]$$

where there are elements $h,u\in\mathcal{S}(S),k,v\in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, a^2u\mathcal{R}'a^2$, $v\leq_\ell a, u\leq_r a^2, k\leq_\ell a^2, hua^2=ka^4$ and $hva^2=ka^2$. From $a^2u\mathcal{R}'a^2\mathcal{R}'a^3$ and $u\leq_r a^2$ we have $u\mathcal{R}'a$ and so there are elements $p,q\in H'_a$ with $pa^2=qua$. Since $v\leq_\ell a^2$ and $k\leq_\ell a^2$ we have from $hva^2=ka^2$ that $hv=k$. Thus $hua^2=hva^4$

and so $ua^2 = va^4$, giving $ua = va^3$. Now $pa^2 = qva^3$ so that $pb = qvab$ and $[a, b] = [u, vab] = a^\sharp b$ as required.

From Lemmas 4.9 and 4.10 we have

Corollary 4.11 *The semigroup S is a left order in Q.*

We now proceed to show that $\mathcal{P} = \mathcal{P}(Q)$.

Lemma 4.12 *Let $b \in S$ and as in (Fiii) let $s \in S$ where*

$$bs\mathcal{R}'b\mathcal{L}'sb\mathcal{R}'s\mathcal{L}'bs$$

so that $bs, sb \in \mathcal{S}(S), bsb\mathcal{H}'b$ and $sbs\mathcal{H}'s$. Then $b\phi = [bs, bsb]$ and has (semigroup) inverse $[sb, s]$ in Q. Further,

$$[bs, bsb][sb, s] = [bs, bs]$$

and

$$[sb, s][bs, bsb] = [sb, sb].$$

Proof By definition of multiplication,

$$[bs, bsb][sb, s] = [u, vs]$$

where there are elements $h, u \in \mathcal{S}(S), k, v \in S$ with $h\mathcal{R}'k\mathcal{R}'u\mathcal{R}'v, bsu\mathcal{R}'bsbsb,$ $v \leq_\ell sb, u \leq_r bs, k \leq_\ell bs, hubs = kbsbs$ and $hvsbsb = kbsbsb$. Since $u \leq_r bs$ and $bsb\mathcal{H}'b\mathcal{R}'bs$ we have from $bsu\mathcal{R}'bsbsb$ that $u\mathcal{R}'bsb\mathcal{R}'bs$. Thus there are elements $p, q \in H'_{bs}$ with $pbsbs = qubs$. We have

$$hvsbsb = kbsbsb = hubsb$$

and so $vsbsb = ubsb$ and $vsb = ub$. Then $pbsbs = qvsbs$ so that $pbs = qvs$ and $[bs, bsb][sb, s] = [bs, bs]$.

Computing now $[sb, s][bs, bsb]$ we have that it equals $[u', v'bsb]$ where there are elements $h', u' \in \mathcal{S}(S), k', v' \in S$ with $h'\mathcal{R}'k'\mathcal{R}'u'\mathcal{R}'v', sbu'\mathcal{R}'sbs, v' \leq_\ell bs,$ $u' \leq_r sb, k' \leq_\ell sb, h'u'sb = k'sbsb$ and $h'v'bsbs = k'sbs$. From $u', s \leq_r sb$ and $sbu'\mathcal{R}'sbs$ we have $u'\mathcal{R}'s\mathcal{R}'sb$ and so there are elements $p', q' \in H'_{sb}$ with $p'sbsb = q'u'sb$. We have

$$h'u'sb = k'sbsb = h'v'bsbsb$$

giving that $u'sb = v'bsbsb$ and $u's = v'bsbs$. Then $p'sbsb = q'v'bsbsb$ from which we deduce $p'sb = q'v'bsb$ so that $[sb, s][bs, bsb] = [sb, sb]$.

To see that $[sb, s]$ is the inverse of b is now a straightforward application of Lemma 4.7.

Lemma 4.13 *The embeddable *–pair $\mathcal{P}$ is induced by Q.*

Proof Let $b, c \in S$ and suppose that $s, t \in S$ are chosen so that $b\mathcal{R}'bs\mathcal{L}'s\mathcal{R}'sb\mathcal{L}'b$ and $c\mathcal{R}'ct\mathcal{L}'t\mathcal{R}'tc\mathcal{L}'c$. Put $b' = [sb, s]$ and $c' = [tc, t]$ so that b' and c' are inverses in Q of b, c respectively.

Assume first that $b \leq_r c$. Then $bs\mathcal{R}'b \leq_r c\mathcal{R}'ct$ so that $cc'b = [ct, ct][bs, bsb] = [bs, bsb] = b$ by Lemmas 4.7 and 4.12, so that $bQ^1 \subseteq cQ^1$. Conversely, if $bQ^1 \subseteq cQ^1$ then either $b = c$ (and certainly $b \leq_r c$) or $b = cq$ for some $q = [m, n]$ in Q. Now $cq = [ct, ctc][m, n] = [u, vn]$ where in particular $u \in \mathcal{S}(S), v \in S$ and $u \leq_r ct\mathcal{R}'c$. Then from $[bs, bsb] = [u, vn]$ we have $b\mathcal{R}'bs\mathcal{R}'u \leq_r c$. The proof that $b \leq_\ell c$ if and only if $Q^1 b \subseteq Q^1 c$ is similar and so we deduce that $\mathcal{P} = \mathcal{P}(Q)$.

Lemma 4.14 *The semigroup S is a straight left order in Q.*

Proof If $q \in Q$ then by Lemma 4.10, $q = [a, b] = a^\sharp b$ where $a\mathcal{R}'b$ in S. But $\mathcal{P} = \mathcal{P}(Q)$ so that $a\mathcal{R}b$ in Q and S is straight in Q.

5 Fully stratified left orders

If S is a straight left order in a semigroup Q and $\mathcal{P}(Q) = (\leq_\ell, \leq_r)$, then S is *stratified* if $\mathcal{L}' = \mathcal{L}^*$ and $\mathcal{R}' = \mathcal{R}^*$, and *fully stratified* if $\leq_\ell = \leq_{\mathcal{L}^*}$ and $\leq_r = \leq_{\mathcal{R}^*}$. Clearly a fully stratified left order is stratified.

Stratified left orders occur naturally in the theory of semigroups of quotients. It is known , for example, that every left order in a bisimple inverse ω–semigroup is stratified [G2] and it is easy to see they are indeed fully stratified. Moreover from Lemma 1.7 of [FG] it is immediate that a ring R that is a straight order in a ring Q is also fully stratified in Q. However, not every stratified left order is fully stratified, as is shown in the following example.

Example 5.1 .

Let S be a right but not left reversible,cancellative semigroup. Thus there are elements a, b in S with $aS \cap bS = \emptyset$. Pick $s \in S$ arbitrarily and let P be the 2×2 matrix over S given by $p_{11} = a, p_{12} = b, p_{21} = s, p_{22} = 0$. Put $T = \mathcal{M}^0(2, S, 2; P)$ and $Q = \mathcal{M}^0(2, G, 2; P)$, where G is the group of left quotients of S. As shown in [G3], T is a stratified left order in Q. We claim that T is not fully stratified.

Let $\alpha = (1, s, 2)$ and $\beta = (1, s, 1)$. If $x, y \in T^1$ and $\beta x = \beta y$ then either $x = y = 0$ and $\alpha x = \alpha y$ or $x \neq 0$ and $y \neq 0$. Clearly if $x, y \neq 0$ and $x\mathcal{R}y$ in Q then $x = y$ and $\alpha x = \alpha y$. But we have now covered all possibilities for x and y, for if $(1, s, 1)(1, t, j) = (1, s, 1)(2, u, j)$ then $sat = sbu$, giving $at = bu$ and $aS \cap bS \neq \emptyset$, a contradiction. Further, if $(1, s, 1)(1, t, j) = (1, s, 1)$ then $sat = s$ gives $satb^2 = sb^2$ and $atb^2 = b^2$, again a contradiction : similarly, $(1, s, 1)(2, t, j) = (1, s, 1)$ gives a contradiction. Thus $\alpha \leq_{\mathcal{L}^*} \beta$ in T, but $Q\alpha$ is not contained in $Q\beta$.

We now use Theorem 4.1 to describe fully stratified left orders.

Theorem 5.2 *A semigroup S is a fully stratified left order in some semigroup Q if and only if the $*$–pair $\mathcal{P} = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ satisfies conditions (Ei),(Eii)(r),(Eiii),(Evi),(Evii)(r) and (Gii),(Giii) and (Giv).*

Proof If S is a fully stratified left order in Q then the $*$–pair $\mathcal{P} = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*}) = \mathcal{P}(Q)$ is an embeddable $*$–pair. Thus by Lemma 3.3 and Theorem 4.1 $\mathcal{P}$ satisfies the given conditions.

Conversely we suppose that $\mathcal{P} = (\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ satisfies the stated conditions. Clearly (Gi) holds and so by Theorem 4.1 it is enough to show that $\mathcal{P}$ is an embeddable $*$–pair.

Suppose that $b \in S$ and $a \in \mathcal{S}(S)$ where $b \leq_{\mathcal{L}^*} a$. Then if $x, y \in S^1$ and $xba = yba$ we have by (Evi)(ℓ) that $xb = yb$, for it is clear from the definition of $\leq_{\mathcal{L}^*}$ that $xb, yb \leq_{\mathcal{L}^*} b$. This gives that $b\mathcal{R}^*ba$ and so (Ev)(ℓ) holds : dually, (Ev)(r) holds. Considering now (Eii)(ℓ), suppose that $b, c \in S$ and $b \leq_{\mathcal{L}^*} c$. Then from (Giii) $hb = kc$ for some $h \in \mathcal{S}(S)$ and $k \in S$ with $b \leq_{\mathcal{R}^*} h$. But then by (Ev)(r) $b\mathcal{L}^*hb = kc$ and it follows that (Eii)(ℓ) holds. By the comment at the beginning of the proof of Theorem 4.1 we also have that $\mathcal{P}$ is an embeddable $*$–pair as required.

6 Left orders in regular $\mathcal{H}$–semigroups

We recall from the introduction that an $\mathcal{H}$*–semigroup* is a semigroup on which Green's relation $\mathcal{H}$ is a congruence. Any left order in a regular $\mathcal{H}$–semigroup is straight [G1] . The aim of this final section is to specialise Theorem 4.1 to obtain Theorem 3.1 of [G4], which describes those semigroups that are left orders in regular $\mathcal{H}$–semigroups.

Lemma 6.1 *Let S be a straight left order in a semigroup Q. Then Q is an $\mathcal{H}$–semigroup if and only if $\mathcal{H}^Q \cap (S \times S)$ is a congruence on S.*

Proof It is clear that if Q is an $\mathcal{H}$–semigroup then $\mathcal{H}^Q \cap (S \times S)$ is a congruence on S. The proof of the converse is just as in Lemma 3.20 of [G4].

The approach of [G4] is via consideration of suitable pairs of equivalence relations on a semigroup. If $\mathcal{O} = (\mathcal{L}', \mathcal{R}')$ where $\mathcal{L}', \mathcal{R}'$ are equivalence relations on a semigroup S , then $\mathcal{O}$ is a *suitable pair* if $\mathcal{L}'$ is a right congruence contained in $\mathcal{L}^*$, $\mathcal{R}'$ is a left congruence contained in $\mathcal{R}^*$ and for any $a \in S, a \in \mathcal{S}(S)$ if and only if $a\mathcal{H}'a^2$, where $\mathcal{H}' = \mathcal{L}' \cap \mathcal{R}'$.

The following corollary makes use of two lemmas from the proof of Theorem 3.1 of [G4]. However, our object is to avoid the 'constructive' part of the proof of that theorem.

Corollary 6.2 *Let S be a semigroup and let $\mathcal{O} = (\mathcal{L}', \mathcal{R}')$ be a suitable pair for S. The following conditions are equivalent :*
(i)S is a left order in a regular $\mathcal{H}$–semigroup Q such that $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$ and $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$;
(ii)S satisfies conditions (A),(B),(C),(D) and (E) with respect to $\mathcal{O}$:
(A)$\mathcal{H}'$ is a congruence on S and $S/\mathcal{H}'$ is regular,
(B)if $a \in \mathcal{S}(S)$ then H'_a is right reversible,
(C)(ℓ)if $a \in \mathcal{S}(S), b, c \in S, a\mathcal{L}'b\mathcal{L}'c$ and $ba = ca$, then $b = c$,
(D)(ℓ)if $a, b \in \mathcal{S}(S)$ and $a\mathcal{L}'b$ then $ba\mathcal{H}'b$,
(E)(ℓ)if $a, b, c \in S$ and $a\mathcal{L}'cba$, then $a\mathcal{L}'ba$.

Proof (i)$\Rightarrow$(ii)Certainly S is a straight left order in Q and so by Theorem 4.1 $\mathcal{P}(Q) = \mathcal{P} = (\leq_\ell, \leq_r)$ is an embeddable *–pair satisfying (Gi),(Gii),(Giii) and (Giv). Note that there is no ambiguity in the notation $\mathcal{L}', \mathcal{R}', \mathcal{H}'$. From Proposition 2.6 of [G4], $\mathcal{H}'$ is a congruence on S and $S/\mathcal{H}' \cong Q/\mathcal{H}$ so is regular. Condition (B) is just (Gii), (C) follows from (Evi) and (D) from (Fii). To see that (E) holds, let $a, b, c \in S$ where $a\mathcal{L}'cba$. Then $Qa = Qcba \subseteq Qba \subseteq Qa$ so that $a\mathcal{L}'ba$; dually one can show that (E)(r) holds.

(ii)$\Rightarrow$(i) Let $T = S/\mathcal{H}'$ and let $\phi : S \to T$ be an onto homomorphism with kernel $\mathcal{H}'$. From Lemma 3.3 of [G4],

$$a\phi\mathcal{L}b\phi \text{ in } T \text{ if and only if } a\mathcal{L}'b \text{ in } S$$

and

$$a\phi\mathcal{R}b\phi \text{ in } T \text{ if and only if } a\mathcal{R}'b \text{ in } S,$$

for any $a, b \in S$, from which it follows that T is $\mathcal{H}$–trivial.

Define relations $\leq_\ell, \leq_r$ on S by

$$a \leq_\ell b \text{ if and only if } Ta\phi \subseteq Tb\phi$$

and

$$a \leq_r b \text{ if and only if } a\phi T \subseteq b\phi T$$

where $a, b \in S$. Then $\leq_\ell, \leq_r$ are preorders on S with associated equivalence relations $\mathcal{L}', \mathcal{R}'$ respectively. Let $\mathcal{P} = (\leq_\ell, \leq_r)$: we show that $\mathcal{P}$ is an embeddable *–pair satisfying (Gi),(Gii),(Giii) and (Giv). Note that it is built into the definition of a suitable pair that an element a of S is square–cancellable if and only if $a\mathcal{H}'a^2$ and we remark that this is also equivalent to $a\phi$ being idempotent.

It is easy to see that $\leq_\ell$ is right compatible with multiplication and $\leq_r$ is left compatible. Given $b, c \in S$ with $b \leq_\ell c$, then $b\phi = d\phi c\phi = (dc)\phi$ for some $d \in S$, so that $b\mathcal{H}'dc$ and certainly $b\mathcal{L}'dc$. Thus $b\mathcal{L}^* dc \leq_{\mathcal{L}^*} c$ by the assumption that $\mathcal{L}' \subseteq \mathcal{L}^*$. So $\leq_\ell \subseteq \leq_{\mathcal{L}^*}$ and dually, $\leq_r \subseteq \leq_{\mathcal{R}^*}$. We now have that $\mathcal{P}$ is a *–pair for S. Again, there is no ambiguity in using the notation $\mathcal{L}', \mathcal{R}', \mathcal{H}'$.

Using the above paragraph it is immediate that (Eii) holds. Moreover, it is straightforward to show that (Ei), (Eiii),(Ev) and (Evii) hold. Considering (Evi)(ℓ), suppose that $b, c \in S, a \in \mathcal{S}(S), b, c \leq_\ell a$ and $ba = ca$. Now from (Evii)(ℓ) we certainly have that $b\mathcal{L}'c$ and so $b\mathcal{L}'c\mathcal{L}'d$ for some $d \in \mathcal{S}(S)$. In T, $Td\phi \subseteq Ta\phi$ so that $Td\phi \subseteq Ta\phi d\phi \subseteq Td\phi$ and $d\mathcal{L}'ad$. Further, $adad\mathcal{H}'ad^2\mathcal{H}'ad$ so that $ad \in \mathcal{S}(S), bad = cad$ and $b\mathcal{L}'c\mathcal{L}'ad$. Condition (C)($\ell$) gives that $b = c$. The dual argument shows that (Evi)(r) holds. Thus $\mathcal{P}$ is an embeddable *–pair satisfying (Gi) and (Gii).

Let $b, c \in S$ where $b \leq_\ell c$. Then $b\mathcal{H}'dc$ for some $c \in S$. Let s be the element of S associated with b whose existence is guaranteed by (Fiii). Since $\mathcal{H}'$ is a congruence on S, $sb\mathcal{H}'sdc$ and since H'_{sb} is right reversible there are elements p, q in H'_{sb} with $psb = qsdc$ so that $bpsb = bqsdc$. It is routine to show that $b\mathcal{R}'bps\mathcal{R}'bqsd$ and $bps \in \mathcal{S}(S)$ so that putting $h = bps$ and $k = bqsd, h$ and k are the elements requied for (Giii). Conversely , given elements h and k as in (Giii), then $b\mathcal{L}'hb = kc \leq_\ell c$.

To apply Theorem 4.1 it remains to show that (Giv) holds. Let $a, c \in \mathcal{S}(S)$ and $b \in S$, where $a\mathcal{R}'b$. We use Lemma 3.4 of [G4] to find elements $s \in \mathcal{S}(S), t \in S$ with $s\mathcal{R}'t\mathcal{H}'abc^2$ and $sabc^2 = tc^4$. From (Evi)(ℓ) we have that $sab = tc^3$. Put $h = s, k = s^2a, u = sa^2$ and $v = tc$. We claim that h, k, u and v are the elements required by (Giv). For from $s\mathcal{R}'abc^2$ we have $s \leq_r a$ and as $s, a \in \mathcal{S}(S)$ it follows that $as\mathcal{H}'s\mathcal{R}'sa\mathcal{H}'u\mathcal{R}'k$ and $u \in \mathcal{S}(S)$. Also,

$$v = tc\mathcal{H}'abc^3\mathcal{H}'abc^2\mathcal{H}'t\mathcal{R}'s$$

so that h, k, u and v are $\mathcal{R}'$–related. Since $s \leq_r a$ we have

$$au = asa^2\mathcal{H}'sa^2 = u\mathcal{R}'abc^2\mathcal{H}'bc.$$

Clearly $v \leq_\ell c, u \leq_r a$ and $k \leq_\ell a$. Finally,

$$hua = s(sa^2)a = (s^2a)a^2 = ka^2$$

and

$$hvc^2 = s(tc)c^2 = (s^2a)bc = kbc.$$

By Theorem 4.1, S is a straight left order in a semigroup Q such that $\mathcal{P} = \mathcal{P}(Q)$ and Lemma 2.3 gives that Q is regular. In addition, $\mathcal{H}' = \mathcal{H}^Q \cap (S \times S)$ is a congruence on S so that by Lemma 4.1, Q is an $\mathcal{H}$–semigroup.

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