FREE IDEMPOTENT GENERATED SEMIGROUPS: SUBSEMIGROUPS, RETRACTS AND MAXIMAL SUBGROUPS

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ABSTRACT. Let S be a subsemigroup of a semigroup T and let $\mathrm{IG}(E)$ and $\mathrm{IG}(F)$ be the free idempotent generated semigroups over the biordered sets of idempotents of E of S and F of T, respectively. We examine the relationship between $\mathrm{IG}(E)$ and $\mathrm{IG}(F)$, including the case where S is a retract of T. We give sufficient conditions satisfied by T and S such that for any $e \in E$, the maximal subgroup of $\mathrm{IG}(E)$ with identity e is isomorphic to the corresponding maximal subgroup of $\mathrm{IG}(F)$. We then apply this result to some special cases and, in particular, to that of the partial endomorphism monoid $\mathrm{PEnd}\,\mathbf{A}$ and the endomorphism monoid $\mathrm{End}\,\mathbf{A}$ of an independence algebra $\mathbf{A}$ of finite rank. As a corollary, we obtain Dolinka's reduction result for the case where $\mathbf{A}$ is a finite set.

1. Introduction

Let S be a semigroup with set $E = E(S)$ of idempotents. It is shown in the seminal work of Nambooripad [24] that E carries a certain abstract structure, that of a *biordered set*. Conversely, Easdown [11] showed that, for any biordered set E, there exists a semigroup S whose set $E(S)$ of idempotents is biorder isomorphic to E.

Given a fixed biordered set E, which we may take to be $E(S)$ for an idempotent generated semigroup S, the set of all those idempotent generated semigroups whose idempotents carry the biorder structure of E forms a category, within which there is an initial object, called the *free idempotent generated semigroup* $\mathrm{IG}(E)$ over E, given by the following presentation:

$$\mathrm{IG}(E) = \langle\, \overline{E} : \overline{e}\,\overline{f} = \overline{ef},\, e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \,\rangle,$$

where here $\overline{E} = \{\overline{e} : e \in E\}$. The relations in the above presentation correspond to taking *basic products* in E, that is, products between $e, f \in E$ where $ef = e$, $fe = f$, $fe = e$ or $ef = f$. Such products may usefully be reformulated in terms of the quasi-orders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ defined on S. For any semigroup T and $a, b \in T$ we have

$$a \leq_{\mathcal{L}} b \Leftrightarrow T^1 a \subseteq T^1 b \text{ and } a \leq_{\mathcal{R}} b \Leftrightarrow aT^1 \subseteq bT^1$$

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where here T^1 is the semigroup T with an identity adjoined if necessary. The equivalence relations associated with $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are Green's relations $\mathcal{L}$ and $\mathcal{R}$. For further details of Green's relations we refer the reader to the standard text [20]. It is easy to see that for $e, f \in E$ the product ef is basic if and only if $e \leq_{\mathcal{L}} f$, $f \leq_{\mathcal{L}} e$, $e \leq_{\mathcal{R}} f$ or $f \leq_{\mathcal{R}} e$ and in this case, both ef and fe are idempotents. Clearly, $\mathrm{IG}(E)$ is idempotent generated, and there is a natural map $\boldsymbol{\phi} : \mathrm{IG}(E) \to S$, given by $\overline{e}\boldsymbol{\phi} = e$, such that $\boldsymbol{\phi}$ is a morphism with image S (given that S is idempotent generated). Finally, we have the following result taken from [11, 24], which exhibits the close relationship between the regular $\mathcal{D}$-classes of $\mathrm{IG}(E)$ and S.

Proposition 1.1. *Let E be a biordered set, let $S = \langle E \rangle$ be any idempotent generated semigroup with biordered set of idempotents $E = E(S)$, and let $\mathrm{IG}(E)$ and $\boldsymbol{\phi}$ be defined as above.*

(IG1) *The restriction of $\boldsymbol{\phi}$ to the set of idempotents of $\mathrm{IG}(E)$ is a bijection onto E (and an isomorphism of biordered sets).*

(IG2) *The morphism $\boldsymbol{\phi}$ induces a bijection between the set of all $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in the $\mathcal{D}$-class $D_{\overline{e}}$ of $\overline{e}$ in $\mathrm{IG}(E)$ and set of all $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in the $\mathcal{D}$-class D_e of e in S.*

(IG3) *The restriction of $\boldsymbol{\phi}$ to the maximal subgroup $H_{\overline{e}}$ of $\mathrm{IG}(E)$ is a morphism onto the maximal subgroup H_e of S.*

Given their universal nature, it is important to investigate semigroups of the form $\mathrm{IG}(E)$ if one is interested in understanding arbitrary idempotent generated semigroups. From **(IG1)-(IG3)**, it is clear that to understand the regular $\mathcal{D}$-classes of $\mathrm{IG}(E)$, the key is to understand the maximal subgroups, and this has been a major focus in recent years. The early work [22, 25, 27, 28] led to the (incorrect) conjecture that all maximal subgroups of $\mathrm{IG}(E)$ were free. Brittenham, Margolis and Meakin [2] gave the first counterexample by showing that the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$ can arise. An unpublished counterexample of McElwee from the 2010s was announced by Easdown [12] in 2011. Motivated by this significant discovery, Gray and Ruškuc [18] showed that *any* group occurs as a maximal subgroup of some $\mathrm{IG}(E)$. Alternative proofs can be found in [9, 15].

With the above established, interest turns to the structure of maximal subgroups of $\mathrm{IG}(E)$ where $E = E(S)$ for naturally arising semigroups S. Gray and Ruškuc [19] investigated the biordered set of idempotents of the full transformation monoid $\mathcal{T}_n$ on a finite n-element set; for any $\varepsilon \in E$ with rank r where $1 \leq r \leq n-2$, they show that $H_{\overline{\varepsilon}}$ is isomorphic to H_ε and hence to the symmetric group $\mathcal{S}_r$. Dolinka [7] proved that the same holds when $\mathcal{T}_n$ is replaced by $\mathcal{PT}_n$, where $\mathcal{PT}_n$ is the finite partial transformation monoid on n elements, by reducing the problem for $\mathcal{PT}_n$ to that for $\mathcal{T}_n$. Brittenham, Margolis and Meakin [3] studied the biordered set E of idempotents of the matrix monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D, where $n \geq 3$. It is shown that for any rank 1 idempotent $\varepsilon \in E$, $H_{\overline{\varepsilon}}$ is isomorphic to H_ε and hence to the multiplicative group D^* of D. Later, Dolinka and Gray [8] showed that for $\varepsilon \in E$ with rank r where $r < n/3$, $H_{\overline{\varepsilon}}$ is isomorphic to H_ε and hence to the general linear group $GL_r(D)$. Further, Dolinka, Gould and Yang [5] explored the biordered set E of the endomorphism monoid of a free

G-act $F_n(G)$ with $n \in \mathbb{N}, n \geq 3$. They showed that for any rank r idempotent $\varepsilon \in E$, with $1 \leq r \leq n-2$, we have $H_{\overline{\varepsilon}}$ is isomorphic to H_ε and hence to the wreath product $G \wr \mathcal{S}_r$. We note that in the cases above if rank ε is $n-1$ then $H_{\overline{\varepsilon}}$ is free and if rank ε is n or 0 then $H_{\overline{\varepsilon}}$ is trivial. Besides the above investigations into maximal subgroups, abundancy and weak abundancy of $\mathrm{IG}(E)$ were studied by Gould and Yang [6]. In [10] Dolinka, Gray and Ruškuc considered the word problem for $\mathrm{IG}(E)$ and gave an example of a band B such that the word problem for $\mathrm{IG}(B)$ is unsolvable, whereas the word problem for each of the maximal subgroups is solvable.

The aim of this paper is to continue the study of maximal subgroups of free idempotent generated semigroups, but from a somewhat different point of view. Let T be a semigroup with a set $F = E(T)$ of idempotents, and let S be a subsemigroup of T with a set $E = E(S)$ of idempotents, so that $E \subseteq F$. The reader should note that, to avoid over-defining our notation, in most of this article e represents an element of E, whereas $\overline{e}$ represents an element of $\overline{E}$, an element of $\overline{F}$, an element of $\mathrm{IG}(E)$ and an element of $\mathrm{IG}(F)$; the interpretation of $\overline{e}$ should be clear from the context. Given an idempotent $e \in E$, we would like to explore conditions such that the maximal subgroup of $\mathrm{IG}(E)$ with identity $\overline{e}$ and the corresponding maximal subgroup of $\mathrm{IG}(F)$ are isomorphic. This will enable us to use a reduction approach to determine maximal subgroups of $\mathrm{IG}(F)$ in terms of those of $\mathrm{IG}(E)$, inspired by that mentioned above of Dolinka [7] for $\mathcal{PT}_n$. We then apply our main result to several special cases, and in particular, as exhibited in Theorem 3.3, to the study of the endomorphism monoid End $\mathbf{A}$ and the partial endomorphism monoid PEnd $\mathbf{A}$ of an independence algebra $\mathbf{A}$ of finite rank. Putting $E = E(\mathrm{End}\,\mathbf{A})$ and $F = E(\mathrm{PEnd}\,\mathbf{A})$ we show that for any $\varepsilon \in E$, the maximal subgroup of $\overline{\varepsilon}$ in $\mathrm{IG}(E)$ is isomorphic to the corresponding maximal subgroup of $\mathrm{IG}(F)$. Note that our result is independent of the exact nature of the group concerned, which is still unknown in general. As a corollary, we obtain the main result of Dolinka [7].

2. A general presentation for maximal subgroups of $\mathrm{IG}(E)$

Given that the mathematical arguments in this work depend heavily on the general presentation of maximal subgroups of $\mathrm{IG}(E)$ over an arbitrary biordered set E obtained in [18], it is necessary for us to recall some details here.

Let E be a biordered set. From [11] we can assume that $E = E(S)$ for some idempotent generated semigroup S: we fix E and S for this section. An *E-square* is a sequence (e, f, g, h, e) of elements of E with $e \,\mathcal{R}\, f \,\mathcal{L}\, g \,\mathcal{R}\, h \,\mathcal{L}\, e$. We draw such an E-square as $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$. If, in addition, there exists $k \in E$ such that either:

$$\begin{cases} ek = e,\ fk = f,\ ke = h,\ kf = g \text{ or} \\ ke = e,\ kh = h,\ ek = f,\ hk = g, \end{cases}$$

then we call it a *singular square*. If the first condition holds then we may say it is an *up-down singular square* and is *up-down singularised* by k, if we want to be specific. If the second condition holds it is a *left-right singular square*. It follows from [2] that the idempotents within a singular E-square form a rectangular band, but the converse is not necessarily true. We say that a $\mathcal{D}$-class D of S is *singularisable* if every E-square within it is singular if and only if the idempotents within the E-square form a rectangular band. If S is a subsemigroup of T and we wish to emphasise that the E-squares are singularised by elements of S, we may say D is S*-singularisable*.

For any fixed $e \in E$, we let $\overline{H}$ be the maximal subgroup of $\overline{e}$ in $\mathrm{IG}(E)$, and so $\overline{H} = H_{\overline{e}}$, the (group) $\mathcal{H}$-class $H_{\overline{e}}$ of $\overline{e}$ in $\mathrm{IG}(E)$. We use I and Λ to denote the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes, respectively, in the $\mathcal{D}$-class $\overline{D} = D_{\overline{e}}$ of $\overline{e}$ in $\mathrm{IG}(E)$. In view of properties **(IG1)-(IG3)**, I and Λ also label the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes, respectively, in the $\mathcal{D}$-class $D = D_e$ of e in S. For every $i \in I$ and $\lambda \in \Lambda$, let $\overline{H}_{i\lambda}$ and $H_{i\lambda}$ denote, respectively, the $\mathcal{H}$-class corresponding to the intersection of the $\mathcal{R}$-class indexed by i and the $\mathcal{L}$-class indexed by λ in $\mathrm{IG}(E)$, respectively S, so that $\overline{H}_{i\lambda}$ and $H_{i\lambda}$ are $\mathcal{H}$-classes of $\overline{D}$ and D, respectively. Where $\overline{H}_{i\lambda}$ (equivalently, $H_{i\lambda}$) contains an idempotent, we denote it by $\overline{e}_{i\lambda}$ (respectively, $e_{i\lambda}$). Without loss of generality we assume $1 \in I \cap \Lambda$ and $\overline{e} = \overline{e}_{11} \in \overline{H}_{11} = \overline{H}$, so that $e = e_{11} \in H_{11} = H$. For each $\lambda \in \Lambda$, we abbreviate $\overline{H}_{1\lambda}$ by $\overline{H}_\lambda$, and $H_{1\lambda}$ by H_λ and so, $\overline{H}_1 = \overline{H}$ and $H_1 = H$.

Let $\overline{h}_\lambda$ be an element in $\overline{E}^*$ such that $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$, for each $\lambda \in \Lambda$. Our notation should be interpreted as follows: whereas $\overline{h}_\lambda$ lies in the free monoid on $\overline{E}$, by writing $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$ we mean that the image of $\overline{h}_\lambda$ under the natural map that takes $\overline{E}^*$ to (right translations in) the full transformation monoid on $\mathrm{IG}(E)$ yields $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$. In fact, it follows from **(IG1)-(IG3)**, that the action of any generator $\overline{f} \in \overline{E}$ on an $\mathcal{H}$-class contained in the $\mathcal{R}$-class of $\overline{e}$ in $\mathrm{IG}(E)$ is equivalent to the action of f on the corresponding $\mathcal{H}$-class in the original semigroup S (see [18, 10]). Thus $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$ in $\mathrm{IG}(E)$ is equivalent to the corresponding statement $H_1 h_\lambda = H_\lambda$ for S, where h_λ is the image of $\overline{h}_\lambda$ under the natural map to $\langle E \rangle$.

We say that $\{\overline{h}_\lambda \mid \lambda \in \Lambda\}$ forms a *Schreier system of representatives* if every prefix of $\overline{h}_\lambda$ (including the empty word) is equal to some $\overline{h}_\mu$, where $\mu \in \Lambda$. Notice that the condition on $\overline{h}_\lambda\overline{e}_{i\mu}$ that $\overline{h}_\lambda\overline{e}_{i\mu} = \overline{h}_\mu$ is equivalent to saying that $\overline{h}_\lambda\overline{e}_{i\mu}$ lies in the Schreier system.

Define $K = \{(i, \lambda) \in I \times \Lambda : H_{i\lambda} \text{ is a group } \mathcal{H}\text{-class}\}$. Since D_e is regular, for each $i \in I$ we can find and fix an element $\omega(i) \in \Lambda$ such that $(i, \omega(i)) \in K$, so that $\omega : I \to \Lambda$ is a function. Again, for convenience, we take $\omega(1) = 1$.

Theorem 2.1. [18] *Let the Schreier system $\{\overline{h}_\lambda \mid \lambda \in \Lambda\}$ and the function ω be chosen as above. The maximal subgroup $\overline{H}$ of $\overline{e}$ in $\mathrm{IG}(E)$ is defined by the presentation*

$$\mathcal{P} = \langle F : \Sigma \rangle$$

with generators:

$$F = \{f_{i,\lambda} : \ (i, \lambda) \in K\}$$

and defining relations Σ:

$(R1)$ $f_{i,\lambda} = f_{i,\mu} \quad (\overline{h}_\lambda\overline{e}_{i\mu} = \overline{h}_\mu)$;

$(R2)$ $f_{i,\omega(i)} = 1 \quad (i \in I)$;

$(R3)$ $f_{i,\lambda}^{-1}f_{i,\mu} = f_{k,\lambda}^{-1}f_{k,\mu} \quad \left(\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{k\lambda} & e_{k\mu} \end{bmatrix} \text{ is a singular square} \right)$.

In using the above result, it is often convenient to identify $\overline{H}$ with the free group $\widetilde{F}$ on F factored by the normal subgroup determined by the given relations of Σ. Note that if there are no non-trivial singular squares, then $\overline{H}$ is free. In the rest of this paper, we refer to a presentation chosen and fixed as above as being *standard*, within which we use lower case letters to denote individual generators of a generating set denoted by the corresponding capital letter.

3. Maximal subgroups of IG(E): semigroups and subsemigroups

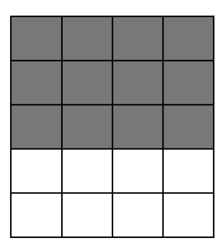
Throughout this section, we use T to denote a semigroup with set F of idempotents, and S to denote a subsemigroup of T with set E of idempotents. The $\mathcal{R}$-relations on T and S are denoted by $\mathcal{R}^T$ and $\mathcal{R}^S$, respectively; and for any $a \in S$, the $\mathcal{R}$-classes of a in T and S are denoted by R_a^T and R_a^S, respectively. Similar notations apply to relations $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$. Our aim in this section is to explore some sufficient conditions such that the maximal subgroup of $\mathrm{IG}(E)$ with identity $\overline{e}$ is isomorphic to the maximal subgroup of $\mathrm{IG}(F)$ with identity $\overline{e}$, where $e \in E$.

We say that S and T satisfy *Condition* (R) if for any $a \in S$ where a is regular in T, we have $R_a^S = R_a^T$ (so that a is also regular in S). Notice that if T is regular then S is a union of $\mathcal{R}$-classes of T.

Lemma 3.1. *Let S and T satisfy Condition (R). Then for any regular element $a \in S$, $D_a^S = D_a^T \cap S$.*

Proof. Clearly, we have $D_a^S \subseteq D_a^T \cap S$. For any $k \in D_a^T \cap S$, there exists $w \in T$ such that $a \mathrel{\mathcal{R}^T} w \mathrel{\mathcal{L}^T} k$, but $R_a^S = R_a^T$ by Condition (R), implying $w \in S$ and $a \mathrel{\mathcal{R}^S} w$, and so w is regular in S. Since $k \in S$ and k is regular in T, the comment preceding the lemma tells us that k is regular in S so that as $w \mathrel{\mathcal{L}^T} k$, we have $w \mathrel{\mathcal{L}^S} k$. So $k \in D_a^S$, and hence $D_a^S = D_a^T \cap S$, as required. $\square$

If S and T satisfy Condition (R), then the egg-box diagram of a typical $\mathcal{D}$-class D_e^T of $e \in E$ of T can be depicted as follows:



where the grey part denotes the egg-box diagram of the $\mathcal{D}$-class D_e^S of S. For notational convenience, we put $D = D_e^T$ and $D' = D_e^S (= D_e^T \cap S)$, where $e \in E$. Suppose further that

S and T are idempotent generated. In line with the convention of the previous section, let I index the $\mathcal{R}$-classes of D, and let I' be the subset of I indexing the $\mathcal{R}$-classes of D'. Let Λ index the $\mathcal{L}$-classes of D and D'. The $\mathcal{R}$-class in D indexed by $i \in I$ is denoted by R_i, while L_λ denotes the $\mathcal{L}$-class in D indexed by $\lambda \in \Lambda$, so that $H_{i\lambda}$ is the $\mathcal{H}$-class in D, which is the intersection of R_i and L_λ, and if $H_{i\lambda}$ is group, we use $e_{i\lambda}$ to denote its identity. Let $K = \{(i, \lambda) \in I \times \Lambda : \ H_{i\lambda} \text{ is a group}\}$ and $K' = K \cap (I' \times \Lambda)$.

With S and T as above, we say in addition that S and T satisfy *Condition* (P) if for every $\mathcal{D}$-class $D = D_e^T$ of $e \in E$, we have that for all $i \in I$, there exists $i' \in I'$ such that for all $j \in I$ and $\lambda, \mu \in \Lambda$, $e_{j\mu}e_{i\lambda} \in D$ implies that $e_{j\mu}e_{i'\lambda} \in D$ and $e_{j\mu}e_{i'\lambda} = e_{j\mu}e_{i\lambda}$. Under these circumstances, for each $i \in I$, we choose and fix $i' \in I$, and in particular, if $i \in I'$, we fix i' to be i. It is implicit in Condition (P) that for each $\lambda \in \Lambda$ if $e_{i\lambda}$ exists then $e_{i'\lambda}$ also exists. Let $\{\overline{h}_\lambda : \lambda \in \Lambda\}$ be a fixed Schreier system for D', where $\overline{h}_\lambda \in \overline{E}^*$. Then by Condition (R) and Lemma 3.1, we may also fix this as a Schreier system for D.

Lemma 3.2. *Let S be an idempotent generated subsemigroup of an idempotent generated semigroup T, with $F = E(T)$ and $E = E(S)$, satisfying Conditions (R) and (P). Using the above notation, let $\mathcal{P} = \langle U; \Sigma \rangle$ be the standard presentation of the maximal subgroup of* $\mathrm{IG}(F)$ *with identity $\overline{e}$, where $e \in D \cap E$ and D is T-singularisable. Then for all $(i, \lambda) \in K$, we have $(i', \lambda), (i', \omega(i)) \in K'$ and*

$$u_{i,\lambda} = u_{i',w(i)}^{-1} u_{i',\lambda}$$

is a consequence of the relations in $\mathcal{P}$.

Proof. Let $(i, \lambda) \in K$, and so $e_{i\lambda}$ exists, and also $e_{i\omega(i)}$ exists. Since S and T satisfy Condition (P), there exists $i' \in I$ such that both $e_{i'\lambda}$ and $e_{i'\omega(i)}$ exist, and hence we have an E-square

$$\begin{bmatrix} e_{i\lambda} & e_{i\omega(i)} \\ e_{i'\lambda} & e_{i'\omega(i)} \end{bmatrix}.$$

Further, $e_{i\lambda}e_{i\omega(i)} = e_{i\omega(i)} \in D$ implies that $e_{i\lambda}e_{i'\omega(i)} = e_{i\omega(i)}$, so that the idempotents within this E-square form a rectangular band by Lemma 2.5 of [15], and hence, since D is T-singularisable, a singular square. By $(R3)$,

$$u_{i,\lambda}^{-1} u_{i,\omega(i)} = u_{i',\lambda}^{-1} u_{i',\omega(i)}$$

but we know from $(R2)$ that $u_{i,\omega(i)} = 1$, giving $u_{i,\lambda} = u_{i',\omega(i)}^{-1} u_{i',\lambda}$ in $H_{\overline{e}}$, as required. ☐

Still with the same assumptions, let $\mathcal{Q} = \langle G; \Gamma \rangle$ be the standard presentation of the maximal subgroup of $\mathrm{IG}(E)$ with identity $\overline{e}$. We take the function $\omega' : I' \to \Lambda$ in $(R2)$ of $\mathcal{Q}$ to be the restriction to I' of the function $\omega : I \to \Lambda$ in $(R2)$ of $\mathcal{P}$.

Theorem 3.3. *Let S be an idempotent generated subsemigroup of an idempotent generated semigroup T, with $F = E(T)$ and $E = E(S)$, satisfying Conditions (R) and (P). Suppose that the regular $\mathcal{D}$-classes $D = D_e^T$ and $D' = D_e^S$ of $e \in E$ are T- and S- singularisable, respectively. Then the maximal subgroup $H_{\overline{e}}^E$ of* $\mathrm{IG}(E)$ *with identity $\overline{e}$ is isomorphic to the maximal subgroup of $H_{\overline{e}}^F$ of* $\mathrm{IG}(F)$ *with identity $\overline{e}$.*

Proof. We need show that $H_{\bar{e}}^{F}$ given by the presentation $\mathcal{P} = \langle U; \Sigma\rangle$ is isomorphic to $H_{\bar{e}}^{E}$ given by the presentation $\mathcal{Q} = \langle G; \Gamma\rangle$. Let $\widetilde{U}$ and $\widetilde{G}$ be the free groups on U and G, respectively. In view of our convention, define a mapping

$$\theta : \widetilde{G} \longrightarrow H_{\bar{e}}^{F},\ g_{i,\lambda} \mapsto u_{i,\lambda}$$

for all $(i,\lambda) \in K'(= K \cap (I' \times \Lambda))$. We show that $\Gamma \subseteq \ker\theta$. In $(R1)$, if $\overline{h}_\lambda \bar{e}_{i\mu} = \overline{h}_\mu$, then by the choice of Schreier system, $u_{i,\lambda} = u_{i,\mu}$ in $\mathcal{P}$, so that $g_{i,\lambda}\theta = g_{i,\mu}\theta$, and hence the pair $(g_{i,\lambda}, g_{i,\mu})$ lies in $\ker\theta$. In $(R2)$, since the function from I' to Λ is the restriction to I' of the function from I to Λ and $g_{i,\omega(i)} = 1$ in $\mathcal{Q}$, we deduce $u_{i,\omega(i)} = 1$ in $H_{\bar{e}}^{F}$ so that $g_{i,\omega(i)}\theta = 1\theta$, and hence the pair $(g_{i,\omega(i)}, 1)$ lies in $\ker\theta$. In $(R3)$, if $\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{k\lambda} & e_{k\mu}\end{bmatrix}$ is singular in D', then it must also singular in D, so that we have $u_{i,\lambda}^{-1}u_{i,\mu} = u_{k,\lambda}^{-1}u_{k,\mu}$ in $\mathcal{P}$, and so $(g_{i,\lambda}^{-1}g_{i,\mu}, g_{k,\lambda}^{-1}g_{k,\mu})$ lies in $\ker\theta$. Thus $\Gamma \subseteq \ker\theta$, and so there exists a morphism $\overline{\theta} : H_{\bar{e}}^{E} \longrightarrow H_{\bar{e}}^{F}$ given by $g_{i,\lambda}\overline{\theta} = u_{i,\lambda}$ for all $(i,\lambda) \in K'$.

Next, we define a mapping

$$\psi : \widetilde{U} \longrightarrow H_{\bar{e}}^{E},\ u_{i,\lambda} \mapsto g_{i',\omega(i)}^{-1} g_{i',\lambda}$$

for all $(i,\lambda) \in K$. Notice that ψ is well-defined, from the first part of Lemma 3.2. We show that $\Sigma \subseteq \ker\psi$. In $(R1)$, if $\overline{h}_\lambda \bar{e}_{i\mu} = \overline{h}_\mu$, then by the choice of Schreier system, $g_{i,\lambda} = g_{i,\mu}$ in $\mathcal{Q}$. Also, $i = i' \in I'$, and so

$$u_{i,\lambda}\psi = g_{i',\omega(i)}^{-1}g_{i',\lambda} = g_{i,\omega(i)}^{-1}g_{i,\lambda} = g_{i,\lambda}$$

and as similarly $u_{i,\mu}\psi = g_{i,\mu}$ we have $u_{i,\lambda}\psi = u_{i,\mu}\psi$. In $(R2)$, we have

$$u_{i,\omega(i)}\psi = g_{i',\omega(i)}^{-1}g_{i',\omega(i)} = 1 = 1\psi.$$

Hence the pair $(u_{i,\omega(i)}, 1)$ lies in $\ker\psi$. In $(R3)$, if $\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{k\lambda} & e_{k\mu}\end{bmatrix}$ is singular in D, then it follows from Condition (P) that $\begin{bmatrix} e_{i'\lambda} & e_{i'\mu} \\ e_{k'\lambda} & e_{k'\mu}\end{bmatrix}$ is an E-square in D'. We show it is singular in D'. First, since $e_{i\lambda}e_{k\mu} = e_{i\mu} \in D$, we have $e_{i\lambda}e_{k'\mu} = e_{i\mu}$ by Condition (P), so that

$$e_{i'\lambda}e_{k'\mu} = e_{i'\lambda}e_{i\lambda}e_{k'\mu} = e_{i'\lambda}e_{i\mu}.$$

Further, it is easy to see $e_{i'\lambda}e_{i\mu} \mathcal{L}\ e_{i\lambda}e_{i\mu} = e_{i\mu} \in D$ and so $e_{i'\lambda}e_{i\mu} \in D$, so that we have $e_{i'\lambda}e_{i\mu} = e_{i'\lambda}e_{i'\mu} = e_{i'\mu}$, giving $e_{i'\lambda}e_{k'\mu} = e_{i'\mu}$, and hence $\begin{bmatrix} e_{i'\lambda} & e_{i'\mu} \\ e_{k'\lambda} & e_{k'\mu}\end{bmatrix}$ is a rectangular band. Since D' is S-singularisable, we deduce that $\begin{bmatrix} e_{i'\lambda} & e_{i'\mu} \\ e_{k'\lambda} & e_{k'\mu}\end{bmatrix}$ is a singular square in D', implying $g_{i',\lambda}^{-1}g_{i',\mu} = g_{k',\lambda}^{-1}g_{k',\mu}$ in $H_{\bar{e}}^{E}$. Further

$$(u_{i,\lambda}^{-1}u_{i,\mu})\psi = (g_{i',\omega(i)}^{-1}g_{i',\lambda})^{-1}g_{i',\omega(i)}^{-1}g_{i',\mu} = g_{i',\lambda}^{-1}g_{i',\mu}$$

and as similarly $(u_{k,\lambda}^{-1}u_{k,\mu})\psi = g_{k',\lambda}^{-1}g_{k',\mu}$, we have $(u_{i,\lambda}^{-1}u_{i,\mu})\psi = (u_{k,\lambda}^{-1}u_{k,\mu})\psi$, so that the pair $(u_{i,\lambda}^{-1}u_{i,\mu}, u_{k,\lambda}^{-1}u_{k,\mu})$ lies in $\ker\psi$. Hence there exists a well defined morphism $\overline{\psi} : H_{\bar{e}}^{F} \longrightarrow H_{\bar{e}}^{E}$, given by $u_{i,\lambda}\overline{\psi} = g_{i',\omega(i)}^{-1}g_{i',\lambda}$ for all $(i,\lambda) \in K$.

Now we are left with showing that $\overline{\theta}$ and $\overline{\psi}$ are mutually inverse. For convenience we now consider $g_{i,\lambda}$ and $u_{i,\lambda}$ as being elements (indeed, the generators) of $H_{\bar{e}}^{F}$ and $H_{\bar{e}}^{E}$, respectively. On one hand,

$$g_{i,\lambda}\overline{\theta}\ \overline{\psi} = u_{i,\lambda}\overline{\psi} = g_{i',\omega(i)}^{-1}g_{i',\lambda} = g_{i,\omega(i)}^{-1}g_{i,\lambda} = g_{i,\lambda}$$

and note that the third equality is because that $i' \in I'$ so that $i' = i$. On the other hand,

$$u_{i,\lambda}\overline{\psi}\ \overline{\theta} = (g_{i',\omega(i)}^{-1}g_{i',\lambda})\overline{\theta} = u_{i',\omega(i)}^{-1}u_{i',\lambda} = u_{i,\lambda}$$

and note that the last equality follows from Lemma 3.2. This completes the proof. □

4. Applications of the main result to independence algebras

The aim of this section is to give some applications of Theorem 3.3. In particular, we study the case of the partial endomorphism monoid $\mathrm{PEnd}\,\mathbf{A}$ and the endomorphism monoid $\mathrm{End}\,\mathbf{A}$ of an independence algebra $\mathbf{A}$ of finite rank. *Independence algebras* [16] (also known as *v^*-algebras* [26]) include sets, vector spaces, and free G-acts, where G is a group. For basic ideas from universal algebra we refer the reader to [4, 17, 23]. We follow the convention of using bold face letters for algebras and corresponding non-bold letters for the underlying sets, where convenient.

Let $\mathbf{A}$ be a (universal) algebra. For any $a_1, \cdots, a_m \in A$, a term built from these elements may be written as $t(a_1, \cdots, a_m)$ where $t(x_1, \cdots, x_m) : A^m \to A$ is a term operation. For any subset $X \subseteq A$, we use $\langle X\rangle$ to denote the universe of the subalgebra generated by X, consisting of all $t(a_1, \cdots, a_m)$, where $m \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, $a_1, \cdots, a_m \in X$, and t is an m-ary term operation.

We say that an algebra $\mathbf{A}$ satisfies the *exchange property* (EP) if for every subset X of A and all elements $x, y \in A$:

$$y \in \langle X \cup \{x\}\rangle \text{ and } y \notin \langle X\rangle \text{ implies } x \in \langle X \cup \{y\}\rangle.$$

A subset X of A is called *independent* if for each $x \in X$ we have $x \notin \langle X\backslash\{x\}\rangle$. We say that a subset X of A is a *basis* of $\mathbf{A}$ if X generates A and is independent. As explained in [16], any algebra satisfying the exchange property or, indeed, any subalgebra of such, has a basis, and in such an algebra a subset X is a basis if and only if X is a minimal generating set if and only if X is a maximal independent set. All bases of such an algebra $\mathbf{A}$ have the same cardinality, called the *rank* of $\mathbf{A}$. Further, any independent subset X can be extended to be a basis of $\mathbf{A}$.

We say that a partial mapping θ from A into itself is a *partial endomorphism* of $\mathbf{A}$ if $\mathrm{dom}\,\theta$ is a subalgebra of $\mathbf{A}$ and for any m-ary term operation $t(x_1, \cdots, x_m)$ and any $a_1, \cdots, a_m \in \mathrm{dom}\,\theta$ we have

$$t(a_1, \cdots, a_m)\theta = t(a_1\theta, \cdots, a_m\theta).$$

Of course, if $\operatorname{dom} \mathbf{A} = A$, we call θ an *endomorphism* of $\mathbf{A}$. We denote the image and the kernel of a partial endomorphism θ of $\mathbf{A}$ by $\operatorname{im}\theta$ and $\ker\theta$, respectively, so that $\ker\theta$ is a congruence on $\operatorname{dom}\theta$. The *rank* of θ is defined as the cardinality of any basis of the subalgebra $\operatorname{im}\theta$.

An algebra $\mathbf{A}$ satisfying the exchange property is called an *independence algebra* if it satisfies the *free basis property*, by which we mean that any map from a basis of $\mathbf{A}$ to $\mathbf{A}$ can be extended to an endomorphism of $\mathbf{A}$.

For an algebra $\mathbf{A}$ we let $\operatorname{PEnd}\mathbf{A}$ be the subsemigroup of the semigroup of all partial transformations $\mathcal{PT}_A$ on the set A, consisting of all partial endomorphisms of $\mathbf{A}$, and let $\operatorname{End}\mathbf{A}$ be the subsemigroup of $\operatorname{PEnd}\mathbf{A}$ consisting of all endomorphisms of $\mathbf{A}$. The inverse subsemigroup of $\operatorname{PEnd}\mathbf{A}$ consisting of the one-one maps, that is, the local automorphisms of $\mathbf{A}$, are the subject of [21].

Lemma 4.1. *Let $\mathbf{A}$ be an independence algebra.*

(i) If X, Y are independent subsets and $\mu : X \longrightarrow Y$ is a bijection, then μ extends uniquely to an isomorphism $\gamma : \langle X\rangle \longrightarrow \langle Y\rangle$.

(ii) The monoids $\operatorname{End}\mathbf{A}$ and $\operatorname{PEnd}\mathbf{A}$ are regular.

Proof. (i) is essentially [16, Lemma 3.7] and that $\operatorname{End}\mathbf{A}$ is regular is [16, Proposition 4.7].

Let $\alpha \in \operatorname{PEnd}\mathbf{A}$ and let $\operatorname{dom}\alpha = B$. Choose a basis C for $\operatorname{im}\alpha$ and for each $c \in C$ pick $b_c \in B$ with $b_c\alpha = c$. Extend C to a basis $C \cup C'$ for A. Define $\gamma \in \operatorname{End}\mathbf{A}$ by $c\gamma = b_c$ for all $c \in C$ and $c'\gamma = c'$ for all $c' \in C'$. Now $\operatorname{dom}\alpha = \operatorname{dom}\alpha\gamma$ and $\operatorname{im}\alpha\gamma \subseteq \operatorname{dom}\alpha$, so that $\operatorname{dom}\alpha\gamma\alpha = \operatorname{dom}\alpha$. For any $a \in \operatorname{dom}\alpha$, $a\alpha = t(c_1, \cdots, c_k)$ for some $c_1, \cdots, c_k \in C$ and term function t. Then

$$\begin{aligned} a\alpha\gamma\alpha &= t(c_1, \cdots, c_k)\gamma\alpha = t(c_1\gamma, \cdots, c_k\gamma)\alpha = t(b_{c_1}, \cdots, b_{c_k})\alpha \\ &= t(b_{c_1}\alpha, \cdots, b_{c_k}\alpha) = t(c_1, \cdots, c_k) = a\alpha. \end{aligned}$$

Thus $\alpha = \alpha\gamma\alpha$. Notice that as $\gamma \in \operatorname{End}\mathbf{A}$, we have shown that both $\operatorname{End}\mathbf{A}$ and $\operatorname{PEnd}\mathbf{A}$ are regular. $\square$

Lemma 4.2. [16, Proposition 4.5] *For any $\alpha, \beta \in \operatorname{End}\mathbf{A}$, the following statements are true:*

(i) $\alpha \leq_{\mathcal{L}} \beta$ if and only if $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ so that $\alpha \mathrel{\mathcal{L}} \beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$;

(ii) $\alpha \leq_{\mathcal{R}} \beta$ if and only if $\ker\beta \subseteq \ker\alpha$ so that $\alpha \mathrel{\mathcal{R}} \beta$ if and only if $\ker\alpha = \ker\beta$;

(iii) $\alpha \mathrel{\mathcal{D}} \beta$ if and only $\operatorname{rank}\alpha = \operatorname{rank}\beta$;

(iv) $\alpha \leq_{\mathcal{J}} \beta$ if and only if $\operatorname{rank}\alpha \leq \operatorname{rank}\beta$;

(iv) $\mathcal{D} = \mathcal{J}$.

We set about showing the analogue of Lemma 4.2 for $\operatorname{PEnd}\mathbf{A}$. We remark that $\varepsilon \in \operatorname{PEnd}\mathbf{A}$ is idempotent if and only if $\operatorname{im}\varepsilon \subseteq \operatorname{dom}\varepsilon$ and $\varepsilon|_{\operatorname{im}\varepsilon} = I_{\operatorname{im}\varepsilon}$, where we use the notation I_Y to denote the identity map on any set Y. For $\alpha \in \operatorname{PEnd}\mathbf{A}$ let π_α be defined by

$$\pi_\alpha = \ker\alpha \cup \omega_{A \setminus \operatorname{dom}\alpha}$$

where ω_X is the universal relation on a set X. Notice that if $\ker\alpha = \ker\beta$ for $\alpha, \beta \in \operatorname{PEnd}\mathbf{A}$, then perforce $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Parts (i) and (ii) of the following may be deduced from the infinitary version of the results in [14] together with Lemma 4.1. However, we give a proof for completeness.

Lemma 4.3. *Let* $\alpha, \beta \in \mathrm{PEnd}\,\mathbf{A}$. *Then:*
(i) $\alpha \leq_{\mathcal{L}} \beta$ *if and only if* $\mathrm{im}\,\alpha \subseteq \mathrm{im}\,\beta$*;*
(ii) $\alpha \leq_{\mathcal{R}} \beta$ *if and only if* $\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\beta$ *and* $\pi_\beta \subseteq \pi_\alpha$*;*
(iii) if $\alpha \leq_{\mathcal{L}} \beta$ *then* $\mathrm{rank}\,\alpha \leq \mathrm{rank}\,\beta$*;*
(iv) if $\alpha \leq_{\mathcal{R}} \beta$ *then* $\mathrm{rank}\,\alpha \leq \mathrm{rank}\,\beta$*.*

Proof. (i) If $\alpha \leq_{\mathcal{L}} \beta$, then there exists $\gamma \in \mathrm{PEnd}\,\mathbf{A}$ such that $\alpha = \gamma\beta$, so that $\mathrm{im}\,\alpha = \mathrm{im}\,\gamma\beta \subseteq \beta$. Conversely, assume that $\mathrm{im}\,\alpha \subseteq \mathrm{im}\,\beta$ and X is a basis for the subalgebra $\mathrm{dom}\,\alpha$. Then for each $a \in X$, there exists $a' \in \mathrm{dom}\,\beta$ such that $a\alpha = a'\beta$. Define $\gamma \in \mathrm{PEnd}\,\mathbf{A}$ with $\mathrm{dom}\,\gamma = \langle X \rangle = \mathrm{dom}\,\alpha$ and $a\gamma = a'$, for all $a \in X$. Then $a\gamma\beta = a'\beta = a\alpha$ for all $a \in X$. Since $\mathrm{im}\,\gamma \subseteq \mathrm{dom}\,\beta$, $\mathrm{dom}\,\gamma\beta = \mathrm{dom}\,\gamma = \mathrm{dom}\,\alpha = \langle X \rangle$, and it follows that $\alpha = \gamma\beta$.

(ii) If $\alpha \leq_{\mathcal{R}} \beta$, then $\alpha = \beta\delta$ for some $\delta \in \mathrm{PEnd}\,\mathbf{A}$. Clearly $\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\beta$. Let $(x, y) \in \pi_\beta$. If $x, y \in A \setminus \mathrm{dom}\,\beta$, then $x, y \in A \setminus \mathrm{dom}\,\alpha$, so $(x, y) \in \pi_\alpha$. On the other hand, if $(x, y) \in \ker\beta$, then $x, y \in \mathrm{dom}\,\beta$ and $x\beta = y\beta$. If $x \in \mathrm{dom}\,\alpha$, then as

$$x\alpha = x\beta\delta = y\beta\delta = y\alpha$$

we have $y \in \mathrm{dom}\,\alpha$ and $(x, y) \in \ker\alpha \subseteq \pi_\alpha$. Otherwise, $(x, y) \in \omega_{A\setminus \mathrm{dom}\,\alpha} \subseteq \pi_\alpha$. Thus $\pi_\beta \subseteq \pi_\alpha$.

Conversely, suppose that $\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\beta$ and $\pi_\beta \subseteq \pi_\alpha$. Observe first that if $a \in \mathrm{dom}\,\alpha$ and $a\beta = b\beta$ for some $b \in \mathrm{dom}\,\beta$, then as $\pi_\beta \subseteq \pi_\alpha$, we have $(a, b) \in \pi_\alpha$. Since clearly $(a, b) \notin \omega_{A\setminus \mathrm{dom}\,\alpha}$, we must have $(a, b) \in \ker\alpha$ so that $b \in \mathrm{dom}\,\alpha$ and $a\alpha = b\alpha$.

We now define $\delta \in \mathrm{PEnd}\,\mathbf{A}$ by $\mathrm{dom}\,\delta = (\mathrm{dom}\,\alpha)\beta$ and for all $a \in \mathrm{dom}\,\alpha$, $(a\beta)\delta = a\alpha$. Notice if $a\beta = a'\beta$ for any $a' \in \mathrm{dom}\,\alpha$ then $a\alpha = a'\alpha$ as above. It is easy to check that δ is a morphism. Clearly $\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\beta\delta$. On the other hand, if $d \in \mathrm{dom}\,\beta\delta$, then $d \in \mathrm{dom}\,\beta$ and $d\beta = d'\beta$ for some $d' \in \mathrm{dom}\,\alpha$. The above shows that $d \in \mathrm{dom}\,\alpha$. Thus $\mathrm{dom}\,\alpha = \mathrm{dom}\,\beta\delta$ and it is then immediate that $\alpha = \beta\delta$ so $\alpha \leq_{\mathcal{R}} \beta$ as required.

(iii) This is an immediate consequence of (i).

(iv) Suppose that $\alpha \leq_{\mathcal{R}} \beta$ and choose γ with $\alpha = \beta\gamma$. Then $\alpha = \beta'\gamma'$, where β' is the restriction of β to $D = C\beta^{-1}$ where $C = \mathrm{im}\,\beta \cap \mathrm{dom}\,\gamma$, and γ' is the restriction of γ to C. If X is a basis for $D = \mathrm{dom}\,\alpha$ then as $D \subseteq \mathrm{dom}\,\beta$ we have $|X| \leq \mathrm{rank}\,\beta$ and $X\beta'\gamma'$ is a generating set for $\mathrm{im}(\beta\gamma) = \mathrm{im}\,\alpha$, giving $\mathrm{rank}\,\alpha \leq |X\beta'\gamma'| \leq |X|$ and hence the required result. □

The next lemma finishes the analogue of Lemma 4.2 for $\mathrm{PEnd}\,\mathbf{A}$.

Lemma 4.4. *For any* $\alpha, \beta \in \mathrm{PEnd}\,\mathbf{A}$:
(i) $\alpha\ \mathcal{L}\ \beta$ *if and only if* $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$;
(ii) $\alpha\ \mathcal{R}\ \beta$ *if and only if* $\ker\alpha = \ker\beta$;
(iii) $\alpha\ \mathcal{D}\ \beta$ *if and only if* $\mathrm{rank}\,\alpha = \mathrm{rank}\,\beta$*;*
(iv) $\alpha \leq_{\mathcal{J}} \beta$ *if and only if* $\mathrm{rank}\,\alpha \leq \mathrm{rank}\,\beta$*;*
(v) $\mathcal{D} = \mathcal{J}$*.*

Proof. (i) Follows immediately from Lemma 4.3 (i).

(ii) Notice that if $\ker\alpha = \ker\beta$, then by definition $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and so also $\pi_\alpha = \pi_\beta$ and so by Lemma 4.3 (ii) we have $\alpha\ \mathcal{R}\ \beta$. On the other hand, if $\alpha\ \mathcal{R}\ \beta$, then again from Lemma 4.3 (ii), $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\pi_\alpha = \pi_\beta$, and it follows that $\ker\alpha = \ker\beta$.

(iii) If $\alpha\ \mathcal{D}\ \beta$, then there exists $\gamma \in \mathrm{PEnd}\,\mathbf{A}$ such that $\alpha\ \mathcal{R}\ \gamma\ \mathcal{L}\ \beta$. From (i) and (ii), we have $\ker\alpha = \ker\gamma$ and $\operatorname{im}\gamma = \operatorname{im}\beta$. Clearly, $\operatorname{rank}\gamma = \operatorname{rank}\beta$. Further, $\ker\alpha = \ker\gamma$ (and so also $\operatorname{dom}\alpha = \operatorname{dom}\gamma$) implies that $\operatorname{im}\alpha \cong \operatorname{dom}\alpha/\ker\alpha = \operatorname{dom}\gamma/\ker\gamma \cong \operatorname{im}\gamma$, so that $\operatorname{rank}\alpha = \operatorname{rank}\gamma$. Hence $\operatorname{rank}\alpha = \operatorname{rank}\beta$.

Conversely, suppose that $\operatorname{rank}\alpha = \operatorname{rank}\beta$. Let X and Y be bases of $\operatorname{im}\alpha$ and $\operatorname{im}\beta$, respectively, so that $|X| = |Y|$. Let $\mu : X \longrightarrow Y$ be a bijection with inverse $\mu^{-1} : Y \longrightarrow X$. Extend μ to an isomorphism $\gamma : \operatorname{im}\alpha \longrightarrow \operatorname{im}\beta$. Then $\operatorname{dom}\alpha = \operatorname{dom}\alpha\gamma$ and as γ is an isomorphism, $\ker\alpha = \ker\alpha\gamma$ so $\alpha\ \mathcal{R}\ \alpha\gamma$. Clearly $\operatorname{im}\alpha\gamma = \operatorname{im}\beta$ so that $\alpha\gamma\ \mathcal{L}\ \beta$ and hence $\alpha\ \mathcal{D}\ \beta$ as desired.

(iv) Let $\alpha \leq_{\mathcal{J}} \beta$ with $\alpha = \gamma\beta\delta$ for $\gamma, \delta \in \mathrm{PEnd}\,\mathbf{A}$. Then $\alpha \leq_{\mathcal{L}} \beta\delta \leq_{\mathcal{R}} \beta$. From Lemma 4.3 we have that $\operatorname{rank}\alpha \leq \operatorname{rank}\beta$.

Conversely, if $\operatorname{rank}\alpha \leq \operatorname{rank}\beta$, then let X be a basis of $\operatorname{im}\beta$ and pick a subset X' of X with $|X'| = \operatorname{rank}\alpha$. Then $\operatorname{rank}\beta I_{\langle X'\rangle} = \operatorname{rank}\langle X'\rangle = \operatorname{rank}\alpha$ so that by (iii), $\alpha\,\mathcal{D}\,\beta I_{\langle X'\rangle}$. Since $\mathcal{D} \subseteq \mathcal{J}$ we have that $\alpha\,\mathcal{J}\,\beta I_{\langle X'\rangle}$ and hence $\alpha \leq_{\mathcal{J}} \beta$.

(v) This is an immediate consequence of (iii) and (iv). $\square$

The first part of the next result is from [13].

Proposition 4.5. *Let* $\mathbf{A}$ *be an independence algebra of finite rank* n. *Then*

(i) $\mathrm{End}\,\mathbf{A} \setminus \mathrm{Aut}\,\mathbf{A}$ *is an ideal of* $\mathrm{End}\,\mathbf{A}$ *and is idempotent generated;*

(ii) $\mathrm{PEnd}\,\mathbf{A} \setminus \mathrm{Aut}\,\mathbf{A}$ *is an ideal of* $\mathrm{PEnd}\,\mathbf{A}$ *and is idempotent generated.*

Proof. Let $\alpha \in \mathrm{PEnd}\,\mathbf{A}$ and suppose that α lies in the group of units, that is, the $\mathcal{H}$-class of I_A. From Lemma 4.4 we deduce that $\operatorname{dom}\alpha = A$ so that $\alpha \in \mathrm{End}\,\mathbf{A}$. Thus $\mathrm{PEnd}\,\mathbf{A}$ and $\mathrm{End}\,\mathbf{A}$ share the same group of units. From [16, Proposition 3.12] we have that $\mathrm{Aut}\,\mathbf{A} = \{\alpha \in \mathrm{End}\,\mathbf{A} : \operatorname{rank}\alpha = n\}$. Immediately from Lemma 4.4 we deduce that $\mathrm{End}\,\mathbf{A} \setminus \mathrm{Aut}\,\mathbf{A}$ and $\mathrm{PEnd}\,\mathbf{A} \setminus \mathrm{Aut}\,\mathbf{A}$ are ideals of $\mathrm{End}\,\mathbf{A}$ and $\mathrm{PEnd}\,\mathbf{A}$, respectively.

That $\mathrm{End}\,\mathbf{A} \setminus \mathrm{Aut}\,\mathbf{A}$ is idempotent generated is contained in Theorem 2.1 [13].

Suppose now that $\beta \in \mathrm{PEnd}\,\mathbf{A}$ with $\operatorname{rank}\beta \leq n-1$, that is, $\beta \notin \mathrm{Aut}\,\mathbf{A}$. Let X be a basis for $\operatorname{dom}\beta$ and extend X to a basis $X \cup Y$ for A. Define $\beta' \in \mathrm{End}\,\mathbf{A}$ by $x\beta' = x\beta$ for all $x \in X$ and $y\beta' = a_0$ for all $y \in Y$, for some fixed $a_0 \in \operatorname{im}\beta$. Then $\operatorname{rank}\beta = \operatorname{rank}\beta' \leq n-1$, so that β' is a product of idempotents of $\mathrm{End}\,\mathbf{A}\setminus\mathrm{Aut}\,\mathbf{A}$. Now observe that $\beta = I_{\operatorname{dom}\beta}\beta'$. $\square$

It follows from Lemmas 4.2 and 4.4 that Condition (R) holds for $\mathrm{PEnd}\,\mathbf{A}$ and $\mathrm{End}\,\mathbf{A}$. Consistent with our earlier notation, let $E = E(\mathrm{End}\,\mathbf{A})$ and let $F = E(\mathrm{PEnd}\,\mathbf{A})$.

We now take a rank r idempotent $\varepsilon \in E$, where $0 \leq r < n$. The $\mathcal{D}$-classes of ε in $\mathrm{PEnd}\,\mathbf{A}$ and $\mathrm{End}\,\mathbf{A}$, denoted by D and D', respectively, are given by

$$D = \{\alpha \in \mathrm{PEnd}\,\mathbf{A} :\ \operatorname{rank}\alpha = r\}, D' = \{\alpha \in \mathrm{End}\,\mathbf{A} :\ \operatorname{rank}\alpha = r\}.$$

Our next aim is to show that the maximal subgroup of $\mathrm{IG}(E)$ containing $\overline{\varepsilon}$ is isomorphic to the maximal subgroup of $\mathrm{IG}(F)$ containing $\overline{\varepsilon}$ by using Theorem 3.3. It remains to show that D and D' are singularisable and Condition (P) holds.

Lemma 4.6. *The $\mathcal{D}$-classes D of* $\mathrm{PEnd}\,\mathbf{A}$ *and D' of* $\mathrm{End}\,\mathbf{A}$ *are up-down singularisable.*

Proof. Consider an E-square $\begin{bmatrix}\alpha & \beta\\ \delta & \gamma\end{bmatrix}$ of $\mathrm{PEnd}\,\mathbf{A}$. If it is singularisable (in $\mathrm{PEnd}\,\mathbf{A}$ or $\mathrm{End}\,\mathbf{A}$), then $\{\alpha,\beta,\gamma,\delta\}$ is a rectangular band by [2].

Conversely, suppose that $\{\alpha,\beta,\gamma,\delta\}$ is a rectangular band. Let U be a basis for $B = \langle \mathrm{im}\,\alpha \cup \mathrm{im}\,\beta\rangle$. Notice $\mathrm{im}\,\alpha = \mathrm{im}\,\delta \subseteq \mathrm{dom}\,\delta = \mathrm{dom}\,\gamma$ and $\mathrm{im}\,\beta = \mathrm{im}\,\gamma \subseteq \mathrm{dom}\,\gamma$, so that $B \subseteq \mathrm{dom}\,\gamma$. Extend U by V to form a basis $U \cup V$ for $\mathrm{dom}\,\gamma$. Define σ in $\mathrm{PEnd}\,\mathbf{A}$ by $\mathrm{dom}\,\sigma = \mathrm{dom}\,\gamma$ and $u\sigma = u$ for all $u \in U$, $v\sigma = v\gamma$ for all $v \in V$. Since $\mathrm{im}\,\sigma = B = \langle U\rangle$, we see that σ is idempotent. Clearly $\alpha\sigma = \alpha$ and $\beta\sigma = \beta$.

We know $\mathrm{im}\,\alpha \subseteq \mathrm{dom}\,\alpha$ and $\mathrm{im}\,\beta \subseteq \mathrm{dom}\,\beta = \mathrm{dom}\,\alpha$, so that $\mathrm{im}\,\sigma = B \subseteq \mathrm{dom}\,\alpha$, giving

$$\mathrm{dom}\,\sigma\alpha = \mathrm{dom}\,\sigma = \mathrm{dom}\,\gamma = \mathrm{dom}\,\delta = \mathrm{dom}\,\sigma\beta.$$

Let $a \in \mathrm{im}\,\alpha$ and $b \in \mathrm{im}\,\beta$. Since $\{\alpha,\beta,\gamma,\delta\}$ is a rectangular band, we have $\alpha = \beta\delta$ and as $\mathrm{im}\,\alpha = \mathrm{im}\,\delta$ we see

$$a\sigma\alpha = a\alpha = a = a\delta \text{ and } b\sigma\alpha = b\alpha = b\beta\delta = b\delta.$$

It follows that $u\sigma\alpha = u\delta$ for all $u \in U$. For $v \in V$ we have $v\sigma\alpha = v\gamma\alpha = v\delta$. Thus $\sigma\alpha = \delta$. Similarly, $u\sigma\beta = u\gamma$ for all $u \in U$ and for $v \in V$ we have $v\sigma\beta = v\gamma\beta = v\gamma$, as $\gamma\ \mathcal{L}\ \beta$. Thus $\sigma\beta = \gamma$ and σ singularises our given E-square.

The above shows that D is $\mathrm{PEnd}\,\mathbf{A}$-singularisable. If $\alpha,\beta,\gamma,\delta \in \mathrm{End}\,\mathbf{A}$, then $\mathrm{dom}\,\gamma = A$ so that $\sigma \in \mathrm{End}\,\mathbf{A}$ also and D' is $\mathrm{End}\,\mathbf{A}$-singularisable. $\square$

Lemma 4.7. *The semigroups* $\mathrm{PEnd}\,\mathbf{A}$ *and* $\mathrm{End}\,\mathbf{A}$ *satisfy Condition (P).*

Proof. Let D be the $\mathcal{D}$-class of $\mathrm{PEnd}\,\mathbf{A}$ consisting of the elements of rank r. Let I index the kernels of elements in D and let I' be a subset of I indexing the kernels of elements in $D' = D \cap \mathrm{End}\,\mathbf{A}$. Note that $D' \neq \emptyset$. Let $i \in I$ and let B be the domain corresponding to i. If $B = A$, then we take $i' = i$. If $B \subsetneq A$, then we choose a basis $\{x_1, \cdots, x_m\}$ for B. Note that here we must have $m \geq r$. We now extend this basis to a basis $\{x_1, \cdots, x_m, x_{m+1}, \cdots, x_n\}$ for $\mathbf{A}$. For any $\alpha \in R_i$ we define $\alpha' \in \mathrm{End}\,\mathbf{A}$ by

$$x_i\alpha' = x_i\alpha \text{ for } 1 \leq i \leq m; x_j\alpha' = x_1\alpha \text{ for } m+1 \leq j \leq n.$$

Then $\mathrm{im}\,\alpha = \mathrm{im}\,\alpha'$ and $\alpha'|_B = \alpha$. Let i' index the kernel of α'. It is easy to check that i' is independent of the choice of α and if α is idempotent then α' is also idempotent, from which it follows that $\varepsilon'_{i\lambda} = \varepsilon_{i'\lambda}$. Let $j \in I$ and let $\lambda, \mu \in \Lambda$ be such that $\varepsilon_{j\lambda}\varepsilon_{i\mu} \in D$. Then we must have $\lambda \subseteq B$. For if there were fewer than r independent elements in $\lambda \cap B$, then $\mathrm{rank}(\varepsilon_{i\lambda}\varepsilon_{i\mu}) < r$, a contradiction. Thus there are r independent elements in $\lambda \cap B$ and it follows that as λ is generated by r independent elements, $\lambda = \lambda \cap B$ and so $\lambda \subseteq B$. Note that $\varepsilon_{i'\mu}$ exists and has kernel indexed by i'. Clearly, $\varepsilon_{j\lambda}\varepsilon_{i'\mu} = \varepsilon_{j\lambda}\varepsilon_{i\mu}$, as required. $\square$

As a direct application of Theorem 3.3, we have the following result.

Theorem 4.8. *Let* $\mathbf{A}$ *be an independence algebra of finite rank* n*, let* $\operatorname{End}\mathbf{A}$ *be the endomorphism monoid of* $\mathbf{A}$ *with biordered set* E *and let* $\operatorname{PEnd}\mathbf{A}$ *be the partial endomorphism monoid of* $\mathbf{A}$ *with biordered set* F*. Then for any* $\varepsilon \in E$*, the maximal subgroup of* $\operatorname{IG}(E)$ *containing* $\overline{\varepsilon}$ *is isomorphic to the corresponding maximal subgroup of* $\operatorname{IG}(F)$ *containing* $\overline{\varepsilon}$*.*

Let E be a biordered set such that $S = \langle E \rangle$. We say that $e \in E$ is *good* if $H_{\overline{e}} \cong H_e$, where $H_{\overline{e}}$ (H_e) is the maximal subgroup of $\operatorname{IG}(E)$ (resp., S) having identity $\overline{e}$ (resp., e). We can immediately deduce the goodness of some idempotents in semigroups of partial maps, calling on the existing results for total maps.

Corollary 4.9. *The following idempotents are good.*

(1) Any idempotent $e \in \operatorname{PEnd} V$ *with* $n \geq 3$ *and* $\operatorname{rank} e < n/3$*, where* V *is an* n*-dimensional vector space over a division ring* D*.*

(2) Any idempotent $e \in \operatorname{PEnd} F_n(G)$ *with* $n \geq 3$ *and* $\operatorname{rank} e \leq n-2$*, where* $F_n(G)$ *is a free* G*-act of rank* n *over a group* G*.*

(3) Any idempotent $e \in \mathcal{PT}_n$ *with* $n \geq 3$ *and* $\operatorname{rank} e \leq n-2$*.*

Proof. (1) follows from Theorem 4.8 and [8]. (2) follows from Theorem 4.8 and [5]. (3) has already been observed in [7] and is a special case of (2). □

5. RETRACTS AND OTHER APPLICATIONS

We now state our second application of Theorem 3.3 with regard to the notion of retract. We say that a subsemigroup S of a semigroup T is a *retract* of T (via θ) if there exists an epimorphism θ from T onto S such that $\theta|_S = \mathrm{I}_S$. Note that for an arbitrary independence algebra $\mathbf{A}$, $\operatorname{End}\mathbf{A}$ is not a retract of $\operatorname{PEnd}\mathbf{A}$, as the latter always has a zero but the former need not.

Proposition 5.1. *Let* S *be a subsemigroup of a semigroup* T *with* $E = E(S)$ *and* $F = E(T)$*.*

(i) There is a natural homomorphism from $\operatorname{IG}(E)$ *to* $\operatorname{IG}(F)$*.*

(ii) If S *is a retract of* T*, then* $\operatorname{IG}(E)$ *embeds in* $\operatorname{IG}(F)$*.*

Proof. For the purposes of this result we let $\operatorname{IG}(E)$ be generated by $\overline{E}$ and $\operatorname{IG}(F)$ by $\overline{\overline{F}}$, with obvious conventions.

(i) For all $(e,f) \in E \times E$, we have (e,f) is basic in E if and only if it is basic in F. Thus $\psi : \operatorname{IG}(E) \longrightarrow \operatorname{IG}(F)$ given by $\overline{e}\psi = \overline{\overline{e}}$ is a homomorphism.

(ii) Suppose now that S is a retract of T via the epimorphism θ. Define $\theta' : \overline{\overline{F}}^{+} \longrightarrow \operatorname{IG}(E)$ by $\overline{\overline{f}}\theta' = \overline{f\theta}$. If (e,f) is a basic pair in F, then it is easy to see that $(e\theta, f\theta)$ is a basic pair in E, and it follows that θ' induces a homomorphism $\overline{\theta} : \operatorname{IG}(F) \longrightarrow \operatorname{IG}(E)$ where $\overline{\overline{f}}\,\overline{\theta} = \overline{f\theta}$.

Consider now $\overline{e_1}\cdots\overline{e_m}, \overline{f_1}\cdots\overline{f_n} \in \operatorname{IG}(E)$ with $(\overline{e_1}\cdots\overline{e_m})\psi = (\overline{f_1}\cdots\overline{f_n})\psi$. Then

$$\overline{\overline{e_1}}\cdots\overline{\overline{e_m}} = \overline{\overline{f_1}}\cdots\overline{\overline{f_n}}$$

so that by applying $\overline{\theta}$ we have

$$\overline{e_1}\cdots\overline{e_m} = \overline{e_1\theta}\cdots\overline{e_m\theta} = (\overline{\overline{e_1}}\cdots\overline{\overline{e_m}})\overline{\theta} = (\overline{\overline{f_1}}\cdots\overline{\overline{f_n}})\overline{\theta} = \overline{f_1\theta}\cdots\overline{f_n\theta} = \overline{f_1}\cdots\overline{f_n}$$

so that ψ is an injection and $\mathrm{IG}(E) \cong \langle\overline{\overline{E}}\rangle \subseteq \mathrm{IG}(F)$. Further, $\mathrm{IG}(E)$ is clearly a retract of $\mathrm{IG}(F)$ via $\overline{\theta}$, or, more precisely, $\langle\overline{\overline{E}}\rangle$ is a retract of $\mathrm{IG}(F)$ via $\overline{\theta}\psi$. □

Corollary 5.2. *Let S be a retract of T via θ, with $E = E(S)$ and $F = E(T)$. Then, regarding $\mathrm{IG}(E)$ as a subsemigroup of $\mathrm{IG}(F)$, for any $e \in E$ there is an epimorphism from the maximal subgroup of $\mathrm{IG}(F)$ containing $\overline{e}$, to the corresponding maximal subgroup in $\mathrm{IG}(E)$.*

In the case where S is a retract of T, we now establish some sufficient conditions that will allow us to apply Theorem 3.3. We will say that a $\mathcal{D}$-class D of a semigroup T is *stable* if for all $a, b \in D$ we have $ab \in D$ if and only if $a \mathrel{\mathcal{R}} ab \mathrel{\mathcal{L}} b$; if T is finite certainly every $\mathcal{D}$-class of T is stable. If T is stable in the sense of [1] or [29] then in view of [1, Corollary 1.1], certainly each $\mathcal{D}$-class of T is stable in our sense.

Lemma 5.3. *Let S be a retract of T via θ, with $E = E(S)$ and $F = E(T)$. Let $e \in E$ and put $D = D_e^T$ and $D' = D_e^S$. Suppose that S and T are idempotent generated, Condition (R) holds, D is stable and for each $f \in D \cap F$ we have $f \mathrel{\mathcal{L}^T} f\theta$. Then Condition (P) holds.*

Proof. Let $f, g \in D \cap F$ with $fg \in D$. As $f \mathrel{\mathcal{L}^T} f\theta$ we have $fg \mathrel{\mathcal{L}^T} (f\theta)g$. Now D is stable so $(f\theta)g \mathrel{\mathcal{R}^T} f\theta$ and so as $f\theta \in S$, Condition (R) gives that $(f\theta)g \in S$. Then

$$fg = f(f\theta)g = f((f\theta)g)\theta = f(f\theta)(g\theta) = f(g\theta),$$

where certainly $g\theta \in D$.

Let I index the $\mathcal{R}$-classes of D and let I' be the subset of I indexing the $\mathcal{R}$-classes of D'. Let Λ index the $\mathcal{L}$-classes of D and D'. Let $i \in I$ and pick $\lambda \in \Lambda$ such that $e_{i\lambda}$ exists. Then $e_{i\lambda} \mathrel{\mathcal{L}^T} e_{i\lambda}\theta$; let i' index the $\mathcal{R}$-class of $e_{i\lambda}\theta$. Notice that $e_{i\lambda}\theta = e_{i'\lambda}$. Of course, if $e_{i\lambda} \in S$ then $i' = i$. Note that i' does not depend upon the choice of λ, as if $\mu \in \Lambda$ is such that $e_{i\mu}$ exists, then $e_{i\lambda}\theta \mathrel{\mathcal{R}^T} e_{i\mu}\theta$. Moreover, as $e_{i\mu}\theta \mathrel{\mathcal{L}^T} e_{i\mu}$ we have $e_{i\mu}\theta = e_{i'\mu}$. Let $j \in I$ and $\kappa, \tau \in \Lambda$ such that $e_{j\kappa}e_{i\tau} \in D$. By the above, $e_{j\kappa}e_{i\tau} = e_{j\kappa}(e_{i\tau}\theta) = e_{j\kappa}e_{i'\tau}$. □

We finish this work by giving an example where S is a retract of T and the conditions of Lemma 5.3 hold.

Let S be a semigroup, let L be a left zero band and let $T = S \times L$. For a fixed $u \in L$, it is easy to see $S \cong S' = S \times \{u\}$, and S' is a subsemigroup of T. For ease of notation, we identify S with S'. Notice that S is a retract of T via $\theta : T \longrightarrow S$ given by $(a, l)\theta = (a, u)$ for all $(a, l) \in T$. Clearly, if S is idempotent generated, then so is T.

Lemma 5.4. *Let S be an idempotent generated semigroup, let L be a left zero band, and let $T = S \times L$. Regard S as a subsemigroup of T by choosing u as above, let $E = E(S)$ and $F = E(T)$. Suppose that $e \in E$ and $D' = D_e^S$ is stable and singularisable via up-down singular squares. Let $D = D_e^T$. Then*

(i) S and T satisfy Condition (R);

(ii) for all $(f, k) \in D \cap F$ we have $(f, k) \mathrel{\mathcal{L}^T} (f, k)\theta$;

(iii) D is stable and singularisable.

Proof. (i) Let $a = (a,u) \in S$ be regular and suppose $(a,u)\ \mathcal{R}^T\ (b,k)$. Then $(b,k) = (a,u)(c,l)$ for some $(c,l) \in T^1$ and it follows that $k = u$, so $(b,k) = (b,u) \in S$, and we can take $(c,l) \in S^1$. Also, $(a,u) = (b,k)(d,m)$ for some $(d,m) \in T^1$ and again we can take $(d,m) \in S^1$, so that $(a,u)\ \mathcal{R}^S\ (b,k)$. Thus Condition (R) holds.

(ii) Note that an element $(a,l) \in F$ if and only if $(a,u) \in E$. If $(f,k) \in D \cap F$, then $(f,k)\theta = (f,u)$ and $(f,k)(f,u) = (f,k)$ and $(f,u)(f,k) = (f,u)$, so that $(f,k)\theta\ \mathcal{L}\ (f,k)$.

(iii) Now consider the structure of the $\mathcal{D}$-class D. Notice that, for any $(a,l),(b,k) \in T$, $(a,l)\ \mathcal{L}^T\ (b,k)$ if and only if $a\ \mathcal{L}^S\ b$; $(a,l)\ \mathcal{R}^T\ (b,k)$ if and only if $l = k$ and $a\ \mathcal{R}^S\ b$. We know from Section 3 that D' is a union of $\mathcal{R}^T$-classes of D and $D' = D \cap S$. Notice further that $D \backslash D' = \{(a,l) : l \neq u, (a,u) \in D'\}$. To see this, let $(a,l) \in D\backslash D'$. Then $(a,l)\ \mathcal{L}^T\ (b,k)\ \mathcal{R}^T\ (e,u)$ for some $(b,k) \in T$, giving $a\ \mathcal{L}^S\ b\ \mathcal{R}^S\ e$, and so $(a,u) \in D'$. Conversely, if $(b,u) \in D'$, then $(b,k)\ \mathcal{L}^T\ (b,u)$, and so $(b,k) \in D\backslash D'$.

To see that D is stable, let $(a,l),(b,k) \in D$ so $(a,u),(b,u) \in D'$. If $(a,l)(b,k) = (ab,l) \in D$, then $(ab,u) \in D'$ so $a\ \mathcal{R}^S\ ab\ \mathcal{L}^S\ b$ and hence $(a,l)\ \mathcal{R}^T\ (ab,l)\ \mathcal{L}^T\ (b,k)$.

We now show that D is singularisable. Let $e,f,g,h \in S$. Clearly any rectangular band $\{(e,u),(f,u),(g,u),(h,u)\}$ with $(e,u)\,\mathcal{R}^T\,(f,u)\,\mathcal{L}^T\,(g,u)\,\mathcal{R}^T\,(h,u)\,\mathcal{L}^T\,(e,u)$ is singularisable. Consider now a rectangular band $\{(e,l),(f,l),(g,k),(h,k)\}$ in D where $\begin{bmatrix}(e,l) & (f,l)\\ (h,k) & (g,k)\end{bmatrix}$ is an E-square in T. Then $\begin{bmatrix}(e,u) & (f,u)\\ (h,u) & (g,u)\end{bmatrix}$ is an E-square and a rectangular band in S. Thus it is up-down singularisable by some $(p,u) \in E$. Then

$$(e,u)(p,u) = (e,u), (f,u)(p,u) = (f,u), (p,u)(e,u) = (h,u) \text{ and } (p,u)(f,u) = (g,u).$$

It follows that

$$(e,l)(p,k) = (e,l), (f,l)(p,k) = (f,l), (p,k)(e,l) = (h,k) \text{ and } (p,k)(f,l) = (g,k)$$

so that $\begin{bmatrix}(e,l) & (f,l)\\ (h,k) & (g,k)\end{bmatrix}$ is singularisable by (p,k). ☐

We now put together the preceding results in this section.

Theorem 5.5. *Let S be an idempotent generated semigroup, let L be a left zero band, and let $T = S \times L$. Regard S as a subsemigroup of T by choosing u as above, let $E = E(S)$ and $F = E(T)$. Suppose that $e \in E$ and $D' = D^S_e$ is stable and singularisable via up-down singular squares. Then the maximal subgroup of $\overline{e}$ in $\mathrm{IG}(E)$ is isomorphic to the corresponding maximal subgroup of $\overline{e} = \overline{(e,u)}$ in $\mathrm{IG}(F)$, and hence to that of any $\overline{(e,k)}$.*

Proof. From Lemma 5.4, S and T satisfy Condition (R), $D = D^T_e$ is stable and singularisable and for each $(f,k) \in D \cap F$ we have $(f,k)\ \mathcal{L}^T\ (f,k)\theta$. By Lemma 5.3, Condition (P) holds. Thus Theorem 3.3 proves the first claim. For the second, observe that $(e,k)\ \mathcal{L}^T\ (e,u) = e$ so that $\overline{(e,k)}\ \mathcal{L}\ \overline{(e,u)}$ in $\mathrm{IG}(F)$. ☐

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