An Introduction to Thompson's Group V

John Fountain

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Properties

1. V contains every finite group
2. V is simple
3. V is finitely presented
4. V has type FP_∞
5. V has solvable word problem
6. V has solvable conjugacy problem
7. V has a subgroup isomorphic to $F_2 \times F_2$
8. The generalised word problem for V is undecidable

Descriptions

1. Thompson's original
2. V is isomorphic to $\mathrm{Aut}_{\mathscr{A}}(A)$ where $\mathscr{A}$ is the (strict) symmetric monoidal category freely generated by an idempotent object (A, α)
3. V is a subgroup of the group of unitary elements of the Cuntz C^*-algebra $\mathcal{O}_2$
4. V is a group of homeomorphisms of the Cantor set
5. V is the group of automorphisms of the Jónsson-Tarski algebra

Strict Symmetric Monoidal Category

A category $\mathscr{C}$ with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a distinguished object I (the unit object) such that

- $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for all objects A, B, C;
- $I \otimes A = A \otimes I$ for all objects A;

and for all objects A, B, an isomorphism $s_{A,B} : A \otimes B \to B \otimes A$ such that

$$s_{A \otimes B, C} = (s_{A,C} \otimes I_B) \circ (I_A \otimes s_{B,C})$$

An idempotent object (C, γ) in $\mathscr{C}$ is an object C with an isomorphism $\gamma : C \otimes C \to C$.
$\mathscr{A}$ is the strict symmetric monoidal category freely generated by an idempotent object (A, α) if for any strict symmetric monoidal category $\mathscr{C}$ and idempotent object (C, γ) in $\mathscr{C}$, there is a unique monoidal functor $F : \mathscr{A} \to \mathscr{C}$ such that $F(A) = C$ and $F(\alpha) = \gamma$.
F monoidal means it preserves $\otimes$ and maps unit object to unit object.

Cuntz algebra $\mathcal{O}_2$

- H: a separable infinite dimensional Hilbert space.
- $\mathcal{B}(H)$: algebra of bounded operators on H.
- $\mathcal{O}_2$ is the C^*-subalgebra of $\mathcal{B}(H)$ generated by two isometries S_1, S_2 on H satisfying $S_1 S_1^* + S_2 S_2^* = I$
- An operator $A : H \to H$ is unitary if $AA^* = I = A^*A$
- V is (isomorphic to) the subgroup of the unitary group consisting of unitaries that are sums of products of the S_i and S_i^*.

Cantor set C

$$\{[0,1]\}$$
$$\{[0,\tfrac{1}{3}]\ ,\ [\tfrac{2}{3},1]\}$$
$$\{[0,\tfrac{1}{9}]\ ,\ [\tfrac{2}{9},\tfrac{1}{3}]\ ,\ [\tfrac{2}{3},\tfrac{7}{9}]\ ,\ [\tfrac{8}{9},1]\}$$
$$\{[0,\tfrac{1}{27}]\ ,\ [\tfrac{2}{27},\tfrac{1}{9}]\ ,\ [\tfrac{2}{9},\tfrac{7}{27}]\ ,\ [\tfrac{8}{27},\tfrac{1}{3}]\ ,\ [\tfrac{2}{3},\tfrac{19}{27}]\ ,\ [\tfrac{20}{27},\tfrac{7}{9}]\ ,\ [\tfrac{8}{9},\tfrac{25}{27}]\ ,\ [\tfrac{26}{27},1]\}$$
$$\vdots$$

We define homeomorphisms of C using covers of C by pairwise disjoint intervals chosen from the above.

Higman algebras

- An algebra S with k ($k \geqslant 2$) unary operations $\alpha_1, \ldots, \alpha_k$ and one k-ary operation λ satisfying

$$(a\alpha_1, \ldots, a\alpha_k)\lambda = a$$
$$(a_1, \ldots, a_k)\lambda\alpha_i = a_i \qquad (i = 1, \ldots, k)$$

  for all $a, a_1, \ldots, a_k \in S$
- Let $\mathcal{F}_k(1)$ be the free Higman k-algebra on a one element set.
- When $k = 2$ this is the Jónsson-Tarski algebra
- $V_{k,1}$:= the automorphism group of $\mathcal{F}_k(1)$; $V = V_{2,1}$
- $\mathcal{F}_k(1)$ has bases of every finite non-zero cardinality.

Scott 1984, Birget 2004

- Let $A = \{a_1, \ldots, a_k\}$ and $u, v \in A^*$; u is a prefix of v if $v = uw$ for some $w \in A^*$.
- Prefix code P over A: $P \subseteq A^*$ and no element of P is a prefix of any other.
- P is a maximal prefix code over A if it is not a proper subset of any other prefix code over A.
- If R a right ideal of A^*, then $R = PA^*$ for a uniquely determined prefix code P; P is the unique minimal set of generators for R.
- R is essential if $R \cap I \neq \emptyset$ for every right ideal I of A^*.
- $R = PA^*$ is essential if and only if P is a maximal prefix code.

Scott 1984, Birget 2004, A^*-isomorphisms

- R_1, R_2 right ideals of A^*. A bijection $\varphi : R_1 \to R_2$ is an A^*-isomorphism if $\varphi(uv) = \varphi(u)v$ for all $u \in R_1$, $v \in A^*$.
- An A^*-isomorphism $\varphi : P_1A^* \to P_2A^*$ (P_1, P_2 prefix codes) restricts to a bijection from P_1 to P_2.
- An extension of an A^*-isomorphism $\varphi : R_1 \to R_2$ is an A^*-isomorphism $\psi : I_1 \to I_2$ of right ideals I_1, I_2 where $R_i \subseteq I_i$ $(i = 1, 2)$ and $\psi(u) = \varphi(u)$ for all $u \in R_1$. φ is maximal if it has no proper extension.
- An isomorphism φ between essential right ideals of A^* has a **unique** maximal extension, $\mathbf{max}(\varphi)$
- $V_{k,1}$ is the group consisting of maximal isomorphisms between finitely generated essential right ideals of A^* with multiplication:

$$\varphi\psi = \mathbf{max}(\varphi \circ \psi)$$

where $\circ$ is composition of partial functions. $V = V_{2,1}$

F-inverse monoids

Let S be an inverse semigroup.

- The natural partial order on S: $a \leqslant b$ if and only if $a = eb$ for some idempotent e.
- The minimum group congruence on S: $a\sigma b$ if and only if $ea = eb$ for some idempotent e.
- S is F-inverse if every σ-class has a maximum element (under the natural partial order). If $a \in S$, let $\mathbf{max}(a)$ denote the maximum element in $a\sigma$.
- If S is F-inverse, it is necessarily a monoid; $1 = \mathbf{max}(e)$ for any idempotent e.
- If S is F-inverse, then $S/\sigma \cong G$ where $G = \{\mathbf{max}(a) : a \in S\}$ with multiplication $\bullet$ given by

$$\mathbf{max}(a) \bullet \mathbf{max}(b) = \mathbf{max}(\mathbf{max}(a)\,\mathbf{max}(b)).$$

Lawson 2007, Birget 2010; the monoid R_k^e

- R_k^e is the set of all isomorphisms between finitely generated essential right ideals of A^* ($A = \{a_1, \ldots, a_k\}$) with multiplication composition of partial functions.

- R_k^e is F-inverse:
  $\varphi \leqslant \theta$ if and only if θ is an extension of φ;
  $\theta \sigma \psi$ if and only if $\exists \varphi \in R_k^e$ such that $\varphi \leqslant \theta, \psi$;
  hence the maximum element in the σ-class of θ is $\mathbf{max}(\theta)$.

- $R_k^e / \sigma \cong V_{k,1}$.

Inverse hulls

C right cancellative. For $a \in C$, the mapping ρ_a defined by
$$r\rho_a = ra.$$
is one-one with domain C. $IH(C) = \mathrm{Inv}\langle \rho_a : a \in C \rangle$ is the inverse hull of C.

$$IH^0(C) = \begin{cases} IH(C) & \text{if } 0 \in IH(C) \\ IH(C) \cup \{0\} & \text{otherwise.} \end{cases}$$

If $C = A^*$ where $A = \{a_1, \ldots, a_k\}$, then $IH^0(C)$ is the polycyclic monoid

$$P_k = \langle A \cup A^{-1} \mid aa^{-1} = 1; ab^{-1} = 0 \text{ if } a \neq b\ (a, b \in A) \rangle.$$

Orthogonal Completions 1

S inverse semigroup with zero. $a, b \in S$ are orthogonal ($a \perp b$) if

$$a^{-1}b = 0 = ab^{-1}.$$

Clearly, $a \perp b$ iff $aa^{-1} \perp bb^{-1}$ and $a^{-1}a \perp b^{-1}b$.
$A \subseteq S$ is orthogonal if $a \perp b$ for all distinct $a, b \in A$.

S is orthogonally complete if it satisfies:

1. $\{a_1, \ldots, a_n\}$ orthogonal implies $a_1 \vee \cdots \vee a_n$ exists (natural po), and
2. multiplication distributes over joins of finite orthogonal sets.

Examples

1. Symmetric inverse monoids.
2. $IH^0(FC(X))$ where $FC(X)$ is the free commutative monoid on X.

Orthogonal Completions 2

S inverse semigroup with zero.

$$D(S) = \{A \subseteq S : 0 \in A, |A| < \infty, A \text{ is orthogonal}\}.$$

Theorem (Lawson)

1. *$D(S)$ is an inverse subsemigroup of 2^S; it is a monoid if S is a monoid.*
2. *$\iota : S \to D(S)$ given by $a \mapsto \{0, a\}$ embeds S in $D(S)$*
3. *$D(S)$ is orthogonally complete.*
4. *If $\theta : S \to T$ is a homomorphism to an orthogonally complete inverse semigroup T, then there is a unique join preserving homomorphism $\varphi : D(S) \to T$ such that $\iota\varphi = \theta$.*

Say $D(S)$ is the orthogonal completion of S.

An alternative view of R_k^e

Let R_k denote the inverse monoid of all A^*-isomorphisms between finitely generated right ideals of A^* where $A = \{a_1, \ldots, a_k\}$.

Theorem (Lawson)

$D(P_k) \cong R_k$.

S inverse monoid with zero.

$S^e = \{a \in S : Saa^{-1}$ and $Sa^{-1}a$ are essential$\}$ is an inverse submonoid of S.

Theorem (Lawson)

$D^e(P_k) \cong R_k^e$.