Structure theorems for weakly B-abundant semigroups

Y. H. Wang

Regular semigroups

Definition

- An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$.
- A semigroup S is called *regular* if all its elements are regular.
- A regular semigroup is called an *orthodox semigroup* if the set of all its idempotents forms a band.
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Definition

- Given a semigroup S, J.A. Green defined the following relations on S in 1951. For any $a, b \in S$

$$a \ \mathcal{L} \ b \Leftrightarrow S^1a = S^1b,$$
$$a \ \mathcal{R} \ b \Leftrightarrow aS^1 = bS^1,$$
$$a \ \mathcal{J} \ b \Leftrightarrow S^1aS^1 = S^1bS^1,$$
$$\mathcal{H} = \mathcal{L} \wedge \mathcal{R}, \qquad \mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

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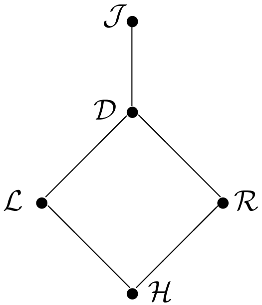

Green's star equivalences

Definition

- Given a semigroup S, for any $a, b \in S$,
  $a\,\mathcal{L}^*\,b$ if $a\,\mathcal{L}\,b$ in a semigroup T such that $S \subseteq T$.
  Dually, The relation $\mathcal{R}^*$on S is defined.
  $\mathcal{D}^*$ denotes the join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$.
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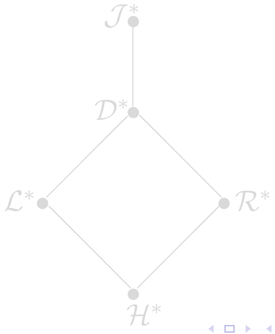
- Also, we have the following diagram.



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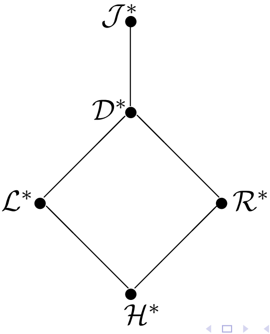

Abundant semigroups

Definition

- A semigroup is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent.
- Let S be an abundant semigroup and E be its set of idempotents. if a is an element of S, then a^* and $a^\dagger$ denote typical idempotents in L_a^* and R_a^*, respectively. S is said to be *idempotent-connected* (IC) if for each element a in S and for some $a^\dagger$, a^*, there exists a bijection $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^\dagger \rangle$, where for any $e \in E$, $\langle e \rangle$ is the principal order ideal generated by e.
- An abundant semigroup S is called a *type W semigroup* if S satisfies the condition (IC) and its set of idempotents forms a band.
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$\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ equivalences

Definition

- Given a semigroup S, M.V. Lawson studied the subset of all its idempotents U instead of the whole idempotent set $E(S)$ and further more generalized Green's star relations. For any $a, b \in S$,

$$a\,\widetilde{\mathcal{L}}_U\, b \Leftrightarrow (\forall e \in U)(ae = a \text{ if and only if } be = b),$$
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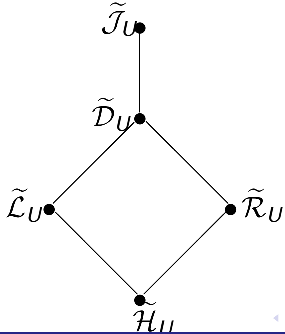

Definition

- A semigroup S with subset of all its idempotents U is said to be *weakly U-abundant* if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U.
- A weakly U-abundant semigroup S satisfies the *congruence condition* (C) if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.

Notation

- B :=a band
  $a^{\dagger}$:= a typical idempotent in $\widetilde{R}_a \cap B$
  a^{*} := a typical idempotent in $\widetilde{L}_a \cap B$
  $\langle e \rangle$:= the principal order ideal generated by $e \in B$
  $= \{x \in B : x \leq e\}$
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- S satisfies the *condition (IC)* (with respect to B) if for any $a \in S$ and some a^*, $a^\dagger$, there exists an order isomorphism $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ such that $xa = a(x\alpha)$.
- S satisfies the *condition (WIC)* (with respect to B) if for any $a \in S$ and some a^*, $a^\dagger$, if $x \in \langle a^\dagger \rangle$ then there exists $y \in B$ with $xa = ay$, and dually, if $z \in \langle a^* \rangle$ then there exists $t \in B$ with $ta = az$.

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Structure theorems for weakly B-abundant semigroups

- A regular semigroup S is an orthodox semigroup if and only if it is the spined product of the Hall semigroup W_B with an inverse semigroup, where the Hall semigroup W_B is a fundamental subsemigroup of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$.
- In 1981, A. El-Qallali and J. Fountain showed that an abundant semigroup is a type W semigroup if and only if it is the spined product of the Hall semigroup W_B with a type A semigroup.
- In 2008, A. El-Qallali, J. Fountain and V.A.R.Gould showed that a weakly B-abundant semigroup S with (C) and (WIC)(res. (IC)) if and only if it is the spined product of U_B (res. V_B) with a weakly $B/\mathcal{D}$-ample semigroup which is the analogue for weakly B-abundant semigroup of inverse semigroups, where $W_B \subseteq V_B \subseteq U_B$.
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- In 2008, G.M.S. Gomes and V.A.R. Gould constructed S_B from a band B, which is a $\bar{B}$-fundamental weakly $\bar{B}$-abundant semigroup with (C), where $\bar{B} = \{(\rho_e, \lambda_e) : e \in B\}$.

Lemma (Gomes and Gould) Let S be a weakly B-abundant semigroup with (C). Then $\theta : S \to S_B$ given by

$$a\theta = (\alpha_a, \beta_a),$$

where for all $x \in B^1$, $L_x\alpha_a = L_{(xa)^*}$ and $R_x\beta_a = R_{(ax)^\dagger}$, is an admissible homomorphism with kernel μ_B. Moreover, $\theta|_B : B \to \bar{B}$ is an isomorphism.

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Theorem 1 Let S be a weakly B-abundant semigroup with (C) and δ be the smallest admissible Ehresmann congruence on S. The mapping $\phi : a \mapsto (a\theta, a\delta)$ is an isomorphism from a weakly B-abundant semigroup S with (C) to the spined product $S_B * S/\delta$ of S_B and S/δ with respect to S_B/δ_1, $\delta_1^\natural$ and ψ, where $\psi : S/\delta \to S_B/\delta_1$ defined by $s\delta\psi = s\theta\delta_1$ for any $s \in S$ is an admissible homomorphism and $\psi\mid_{B/\delta}: B/\delta \to \bar{B}/\delta_1$ is an isomorphism, if and only if
(i) for any $a, b \in S$, $a \,\delta\, b$ implies $a\theta \,\delta_1\, b\theta$ and $e \,\delta\, f$ if and only if $e\theta \,\delta_1\, f\theta$ for any $e, f \in B$;
(ii) $\delta \cap \mu_B = \iota$, where $\mu_B = ker\theta$;
(iii) if $x \in S_B$ and $(x, s\delta) \in S_B * S/\delta$ for some $s \in S$ then there exists $t \in S$ such that $x = t\theta$ and $t\delta = s\delta$.

Definition A weakly B-abundant semigroup S is said to be a *weakly B-abundant semigroup with (C) and (N)* if S satisfies the congruence condition (C) and the set of distinguished idempotents is a normal band.

Lemma Let S be a weakly B-abundant semigroup with (C) and (N). Then the relation δ on S defined by the rule

$$a\,\delta\,b \Leftrightarrow a = a^{\dagger}ba^{*} \text{ and } b = b^{\dagger}ab^{*},$$

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Lemma Let S be a weakly B-abundant semigroup with (C) and (N). Then the relation δ defined in the above Lemma satisfies these conditions $\mathrm{(i)}, \mathrm{(ii)}, \mathrm{(iii)}$ in Theorem 1.

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Structure theorems for weakly B-abundant semigroups

Theorem 2 A weakly B-abundant semigroup S with (C) and (N) is isomorphic to the spined product $S_B * S/\delta$ of S_B and S/δ with respect to S_B/δ_1, $\delta_1^\natural$ and ψ, where $\psi : S/\delta \to S_B/\delta_1$ defined by $s\delta\psi = s\theta\delta_1$ for any $s \in S$ is an admissible homomorphism and $\psi\mid_{B/\delta}: B/\delta \to \bar{B}/\delta_1$ is an isomorphism.

Structure theorems for weakly B-abundant semigroups

Definition A weakly B-abundant semigroup S is said to a *weakly B-superabundant semigroup* if every $\widetilde{\mathcal{H}}_B$-class contains a distinguished idempotent.

Lemma Let S be a weakly B-abundant semigroup. For any $e, f \in B$, $e\,\widetilde{\mathcal{D}}_B\, f \Leftrightarrow e\,\mathcal{D}^B\, f$ if and only if S is a weakly B-superabundant semigroup.

Lemma Let S be a weakly B-superabundant semigroup with (C). Then the relation δ on S defined by the rule

$$a\,\delta\,b \Leftrightarrow a = a^{\dagger}ba^{*} \text{ and } b = b^{\dagger}ab^{*},$$

for some $a^{\dagger}, a^{*}, b^{\dagger}, b^{*} \in B$ with $a^{\dagger}\,\widetilde{\mathcal{R}}_B\, a\,\widetilde{\mathcal{L}}_B\, a^{*}$, and $b^{\dagger}\,\widetilde{\mathcal{R}}_B\, b\,\widetilde{\mathcal{L}}_B\, b^{*}$,is the smallest admissible Ehresmann congruence on S.

Definition A weakly B-abundant semigroup S is said to a *weakly B-superabundant semigroup* if every $\widetilde{\mathcal{H}}_B$-class contains a distinguished idempotent.

Lemma Let S be a weakly B-abundant semigroup. For any $e, f \in B$, $e\,\widetilde{\mathcal{D}}_B\,f \Leftrightarrow e\,\mathcal{D}^B\,f$ if and only if S is a weakly B-superabundant semigroup.

Lemma Let S be a weakly B-superabundant semigroup with (C). Then the relation δ on S defined by the rule

$$a\,\delta\,b \Leftrightarrow a = a^{\dagger}ba^{*} \text{ and } b = b^{\dagger}ab^{*},$$

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Lemma Let S be a weakly B-superabundant semigroup with (C) . Then the relation δ defined in the above Lemma satisfies these conditions $\mathrm{(i)}, \mathrm{(ii)}, \mathrm{(iii)}$ in Theorem 1.

Theorem 3 A weakly B-superabundant semigroup S with (C) is isomorphic to the spined product $S_B * S/\delta$ of S_B and S/δ with respect to S_B/δ_1, $\delta_1^\natural$ and ψ, where $\psi : S/\delta \to S_B/\delta_1$ defined by $s\delta\psi = s\theta\delta_1$ for any $s \in S$ is an admissible homomorphism and $\psi\mid_{B/\delta}: B/\delta \to \bar{B}/\delta_1$ is an isomorphism.

Examples

Example 1 consider this normal band $B = \{e, f, 0\}$ with table

$*$	1	f	0
e	e	f	0
f	e	f	0
0	0	0	0

S_B does not have (WIC).

Examples

Example 2 Let $\langle a \rangle$ be a monogenic monoid generated by a and $X = \{x_i : i \in N\}$ be a right zero semigroup. Set $S = \langle a \rangle \bigcup X$. We define the operation $*$ as the following table:

$*$	1	a	a^n	x_i
1	1	a	a^n	x_i
a	a	a^2	a^{n+1}	x_i
a^m	a^m	a^{m+1}	a^{m+n}	x_i
x_j	x_j	x_{j+1}	x_{j+n}	x_i

Then we can check that S is a weakly B-superabundant semigroup with the distinguished band $\{1\} \bigcup X$. Moreover we can find S satisfies the congruence condition. But we yet find that for any $x_i \in \langle 1 \rangle$ there doesn't exist $x_j \in X$ such that $x_i a^n \neq a^n x_j$, where $n \geq 1$, so S fails to have (WIC).