Symmetric and dual symmetric inverse monoids

James East

University of Sydney

Study Semigroup
University of York
20th May 2010

Inverse semigroups

Definition

A semigroup S is inverse if

$$(\forall s \in S)(\exists! s' \in S) \quad ss's = s \quad \text{and} \quad s'ss' = s'.$$

Equivalently (and non-trivially), S is inverse if:

- S is regular: $(\forall s \in S)(\exists t \in S)\ sts = s$, and
- idempotents of S commute.

Inverse semigroups

Note that the above definition(s) are expressed in terms of quasi-identities. For example, "idempotents commute":

$$(\forall x, y \in S) \quad \big((xx = x) \wedge (yy = y)\big) \Rightarrow (xy = yx).$$

It is possible to define the class of inverse semigroups, as algebras $\langle S, \cdot, ' \rangle$ with a binary and unary operation, using only identities:

$$(xy)z = x(yz),$$
$$xx'x = x,\ x'xx' = x',$$
$$(xy)' = y'x',\ (x')' = x,$$
$$xx'y'y = y'yxx'.$$

In other words, the class of inverse semigroups is a variety: it is closed under taking subalgebras, homomorphic images, and arbitrary direct products.

Inverse semigroups

Example

Any group is an inverse semigroup: $g' = g^{-1}$.

Interesting fact: a semigroup S is a group if and only if it satisfies the quasi-identity:

$$(\forall s \in S)(\exists! s' \in S) \quad ss's = s.$$

Example

Any semilattice (semigroup of commuting idempotents) is an inverse semigroup: $e' = e$.

Symmetric inverse monoids

The symmetric inverse monoid on a set X is

$$\mathcal{I}_X = \{\text{bijections } A \to B \mid A, B \text{ subsets of } X\}.$$

The semigroup operation is "compose wherever possible" (composition as binary relations).

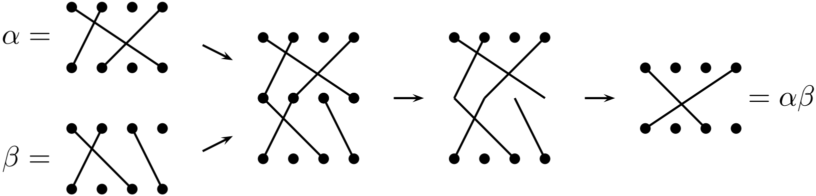
If $\alpha : A \to B$ and $\beta : C \to D$, then $\alpha\beta : E \to F$, where

- $E = (B \cap C)\alpha^{-1}$ and $F = (B \cap C)\beta$, and
- $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in E$.

The inverse semigroup structure is given by $\alpha' = \alpha^{-1}$ — the inverse function, with $\text{dom}(\alpha^{-1}) = \text{im}(\alpha)$ and $\text{im}(\alpha^{-1}) = \text{dom}(\alpha)$, etc.

Symmetric inverse monoids

Pitcures help in the finite case:



The Wagner-Preston Theorem

Theorem (Wagner-Preston)

If S is an inverse semigroup, then there is a faithful representation

$$\varphi : S \to \mathcal{I}_S : s \mapsto \varphi_s,$$

given by

$$\varphi_s : Ss^{-1} \to Ss : x \mapsto xs.$$

Note: One may check that

$$\varphi_s = \rho_s \cap (\rho_{s^{-1}})^{-1}$$

where $\rho : S \to \mathcal{T}_S : s \mapsto \rho_s$ is the Cayley representation, defined by

$$\rho_s : S \to S : x \mapsto xs.$$

Dual symmetric inverse monoids

The dual symmetric inverse monoid on a set X is the set of all block bijections on X:

$$\mathcal{I}_X^* = \{\text{bijections } \mathbf{A} \to \mathbf{B} \mid \mathbf{A}, \mathbf{B} \text{ quotients of } X\}.$$

The semigroup operation is defined dually (in a categorical sense).

If $\alpha : X/\varepsilon_1 \to X/\varepsilon_2$ and $\beta : X/\eta_1 \to X/\eta_2$, then

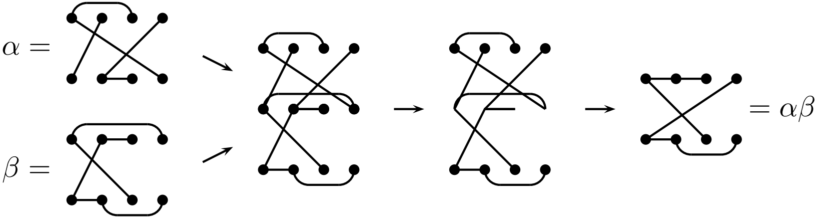
$$\alpha\beta : X/\zeta_1 \to X/\zeta_2, \qquad \text{where}$$

- $\zeta_1 = (\varepsilon_2 \vee \eta_1)\alpha^{-1}$ and $\zeta_2 = (\varepsilon_2 \vee \eta_1)\beta$.

A block of X/ζ_1 is a union of ε_1-classes — it is mapped to a ζ_2-class as appropriate.

Dual symmetric inverse monoids

Pitcures help in the finite case:



The inverse semigroup structure on $\mathcal{I}_X^*$ is given by turning pictures upside down.

The FitzGerald-Leech Theorem

Theorem (FitzGerald and Leech)

Let S be an inverse semigroup. For $s \in S$ define an equivalence

$$\varepsilon_s = \{(x, y) \in S \times S \mid xs = ys\}.$$

There is a faithful representation

$$\chi : S \to \mathcal{I}^*_S : s \mapsto \chi_s,$$

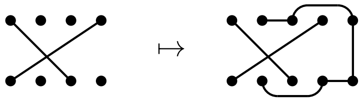
given by

$$\chi_s : S/\varepsilon_{s^{-1}} \to S/\varepsilon_s : [x]_{\varepsilon_{s^{-1}}} \mapsto [xs]_{\varepsilon_s}.$$

Note: One may check that $\chi_s = \left(\rho_s \cup (\rho_{s^{-1}})^{-1}\right)^+$, where $\rho : S \to \mathcal{T}_S : s \mapsto \rho_s$ is the Cayley representation, and α^+ denotes the least block bijection containing the relation α.

The FitzGerald-Leech Theorem

The Fitz-Gerald-Leech theorem is difficult. But there is a simple embedding $\mathcal{I}_X \to \mathcal{I}_Y^*$, where Y is a set with $|Y| = |X| + 1$. In pictures:



A block bijection is uniform if blocks in the domain are mapped to blocks of equal size in the image.

When X is finite, the image of the above embedding lies in the submonoid

$$\mathfrak{F}_Y^* = \{\alpha \in \mathcal{I}_Y^* \mid \alpha \text{ is uniform}\}.$$

The factorisable part of $\mathcal{I}_X^*$

The monoid

$$\mathfrak{F}_X^* = \left\{\alpha \in \mathcal{I}_X^* \,\middle|\, \alpha \text{ is uniform}\right\}$$

is the largest factorisable inverse monoid contained in $\mathcal{I}_X^*$.

An inverse monoid S is factorisable if $S = E(S)G(S)$, where

- $E(S) = \{e \in S \mid e^2 = e\}$, the semilattice of idempotents, and
- $G(S) = \{g \in S \mid gg^{-1} = g^{-1}g = 1\}$, the group of units.

Theorem

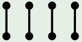
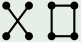
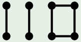
Any finite inverse semigroup S with $|S| = n$ embeds in $\mathfrak{F}_{n+1}^*$.

The factorisable part of $\mathcal{I}_X^*$

The "$n+1$" in the above theorem is sharp, even if S has an identity.

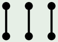
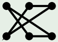
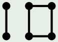
Example

Let $S = \{1, s, s^2\}$ where $s^3 = s$. The representation in $\mathfrak{F}_4^*$ from the theorem is given by:

$1 \mapsto$  , $s \mapsto$  , $s^2 \mapsto$  .

But S does not embed in $\mathfrak{F}_3^*$ — maximal subgroups of $\mathfrak{F}_3^*$ at non-identity idempotents are trivial, but $\{s, s^2\}$ is a subgroup of S.

However, S does of course embed in $\mathcal{I}_3^*$. The embedding is:

$1 \mapsto$  , $s \mapsto$  , $s^2 \mapsto$  .

The factorisable part of $\mathcal{I}_X$

There is a notion of "uniform" for elements of the symmetric inverse monoid $\mathcal{I}_X$.

Say $\alpha \in \mathcal{I}_X$ is uniform if

$$\big|X \setminus \mathrm{dom}(\alpha)\big| = \big|X \setminus \mathrm{im}(\alpha)\big|.$$

The monoid

$$\mathfrak{F}_X = \{\alpha \in \mathcal{I}_X \,|\, \alpha \text{ is uniform}\}$$

is the largest factorisable inverse monoid contained in $\mathcal{I}_X$. We have

$$\mathfrak{F}_X = \mathcal{I}_X \;\text{ iff }\; X \text{ is finite.}$$

So the Vagner-Preston Theorem implies that any finite inverse semigroup embeds in some $\mathfrak{F}_X$.

The factorisable part of $\mathcal{I}_X$

Suppose now X is infinite. Let X' be disjoint from X, and let

$$' : X \to X' : x \mapsto x'$$

be a bijection. Considering the elements of $\mathcal{I}_X$ as binary relations, we have an obvious embedding

$$\mathcal{I}_X \to \mathfrak{F}_{X \cup X'} : \alpha \mapsto \alpha.$$

So every inverse semigroup embeds in some $\mathfrak{F}_X$.

But if S is a monoid, this is not a *monoid embedding*.

The factorisable part of $\mathcal{I}_X$

Let M be an infinite monoid, and $\phi : M \to \mathcal{I}_X$ any monoid embedding. Define

$$\Phi : M \to \mathfrak{F}_{X \cup X'} : m \mapsto \begin{cases} m\phi \cup \mathrm{id}_{X'} & \text{if } m \in G(M) \\ m\phi & \text{if } m \in M \setminus G(M). \end{cases}$$

- This is a bijective map.
- It is a monoid homomorphism if and only if the set $M \setminus G(M)$ is an ideal of M.
- And $M \setminus G(M)$ is an ideal of M if and only if M contains no *bicyclic* submonoid.

The factorisable part of $\mathcal{I}_X$

Theorem

An infinite inverse monoid M embeds as a submonoid in $\mathfrak{F}_M$ if and only if M contains no bicyclic submonoid.

The earlier embedding $\mathcal{I}_X \to \mathcal{I}^*_{X+1}$ restricts to an embedding

$$\mathfrak{F}_X \to \mathfrak{F}^*_{X+1}.$$

Theorem

An infinite inverse monoid M embeds as a submonoid in $\mathfrak{F}^*_M$ if and only if M contains no bicyclic submonoid.