Applications of order-preserving partial permutations

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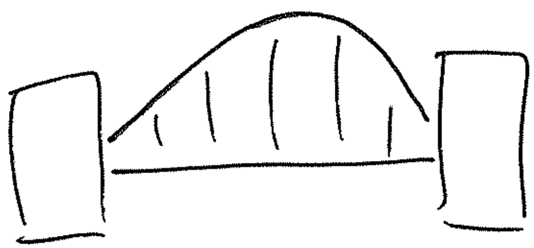

Motivation / Objects of Study

Let:
- $\underline{n} = \{1, 2, \ldots, n\}$
- $S_n = \{\text{permutations on } \underline{n}\}$ — the symmetric group

Cayley's Theorem (for groups)

Every finite group G embeds in some S_n.

– proved with $n = |G|$.

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Let:

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- $\mathcal{I}_n = \{\text{partial permutations on } \underline{n}\}$ — the symmetric inverse semigroup

Vagner – Preston Theorem

Any inverse semigroup S embeds in some $\mathcal{I}_n$.

— again, $n = |S|$.

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- $\mathcal{I}_n = \{\text{partial permutations on } \underline{n}\}$ — the symmetric inverse semigroup
- $\mathcal{I}_n \setminus S_n = \{\text{strictly partial permutations on } \underline{n}\}$ — the singular part of $\mathcal{I}_n$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\\ \text{Any finite non-unital inverse semigroup } S \text{ embeds in some } \mathcal{I}_n \setminus S_n.\\ \quad — \; n = |S|.\end{array}}$$

Motivation / Objects of Study

Let:

- $\underline{n} = \{1, 2, \ldots, n\}$
- $\mathcal{S}_n = \{$ permutations on $\underline{n}\}$ — the symmetric group
- $\mathcal{I}_n = \{$ partial permutations on $\underline{n}\}$ — the symmetric inverse semigroup
- $\mathcal{I}_n \setminus \mathcal{S}_n = \{$ strictly partial permutations on $\underline{n}\}$ — the singular part of $\mathcal{I}_n$
- $\mathcal{O}_n = \{$ order-preserving partial permutations on $\underline{n}\}$

Theorem

Any finite aperiodic inverse semigroup S embeds in some $\mathcal{O}_n$.

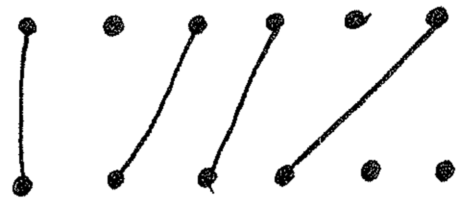
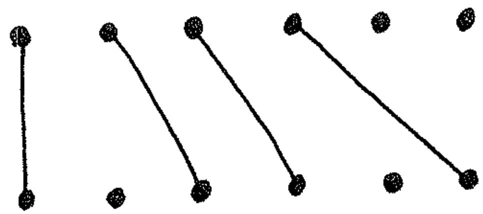
— $n = |S|$.

The Monoids $\mathcal{L}_n$ and $\mathcal{R}_n$

Let $A \subseteq \underline{n}$ with $|A| = k$. Define:

- $\lambda_A \in \mathcal{O}_n$ by $\mathrm{dom}(\lambda_A) = A$ & $\mathrm{im}(\lambda_A) = \underline{k}$,
- $\rho_A = \lambda_A^{-1} \in \mathcal{O}_n$ — $\mathrm{dom}(\rho_A) = \underline{k}$ & $\mathrm{im}(\rho_A) = A$.

If $A = \{1,3,4,6\} \subseteq \underline{6}$ then

$\lambda_A =$  & $\rho_A =$ 

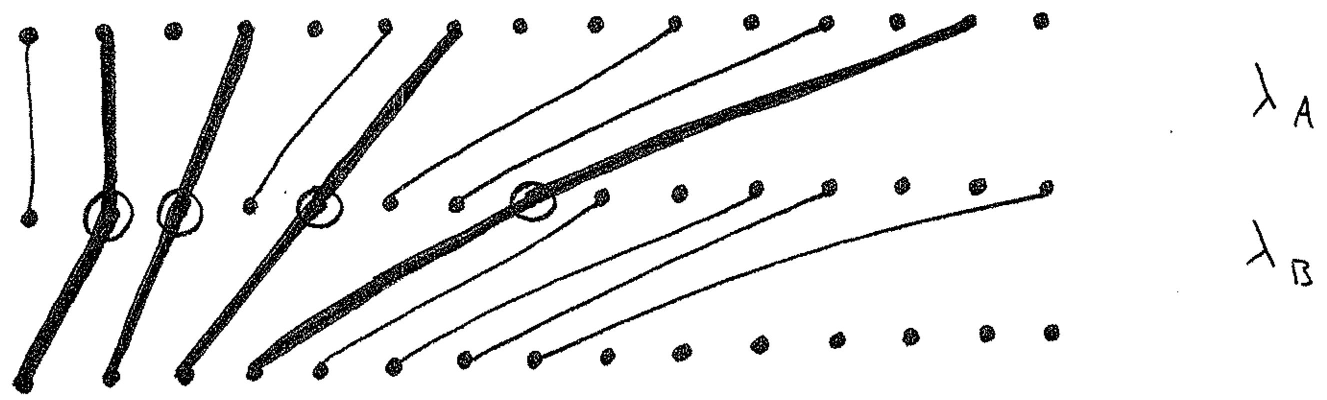
Proposition

Let

- $\mathcal{L}_n = \{ \lambda_A \mid A \subseteq \underline{n} \}$
- $R_n = \{ \rho_A \mid A \subseteq \underline{n} \}$.

Then $\mathcal{L}_n$ and R_n are (anti-isomorphic) submonoids of $\mathcal{O}_n$.

Proof:



$= \lambda_C$ where $C = \{2, 4, 7, 14\} \subseteq A$. $\square$

For $i \in \underline{n}$ put

$$\lambda_i = \lambda_{\{i\}^c} \quad \& \quad \rho_i = \rho_{\{i\}^c} :$$

$\lambda_3 \in \mathcal{L}_6$ $\qquad$ $\rho_3 \in \mathcal{R}_6$

Lemma

Let $A \subseteq \underline{n}$ with $A^c = \{ i_1 < \cdots < i_k \}$. Then

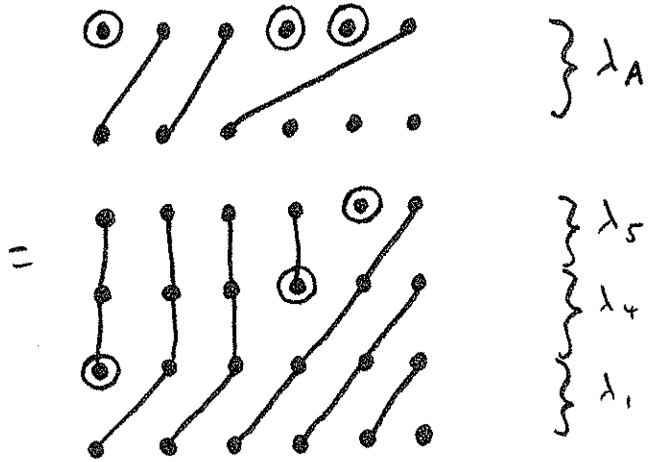
- $\lambda_A = \lambda_{i_k} \cdots \lambda_{i_1}$,
- $\rho_A = \rho_{i_1} \cdots \rho_{i_k}$.

So $\mathcal{L}_n = \langle L \rangle$ & $\mathcal{R}_n = \langle R \rangle$, where

$L = \{ \lambda_1, \ldots, \lambda_n \}$ & $R = \{ \rho_1, \ldots, \rho_n \}$.

Proof: Let $A = \{2,3,6\} \subseteq \underline{6}$.

Then $A^c = \{1,4,5\}$, and

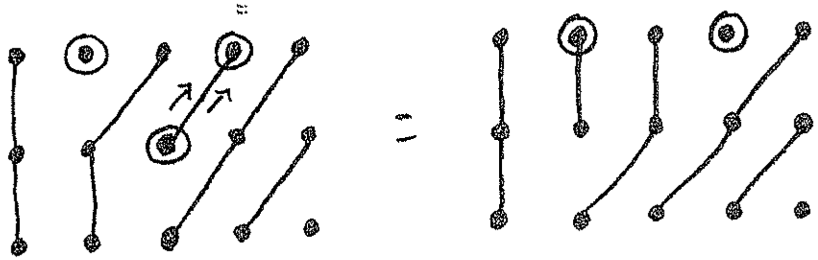


For defining relations, we need to be able to transform any word

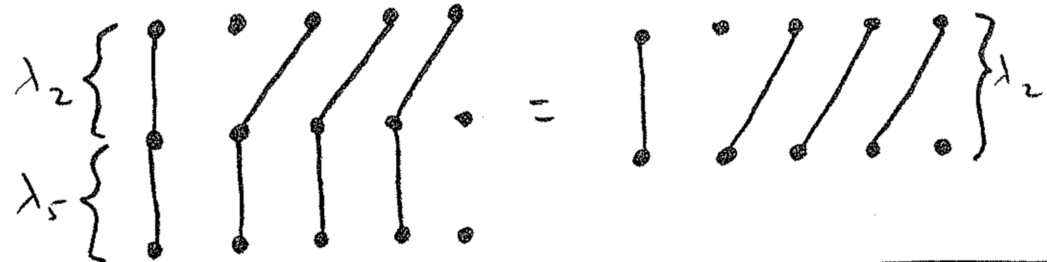
$$\lambda_a \lambda_b \lambda_c \cdots \lambda_z$$

into a descending word.

- eg $\lambda_2 \lambda_3 = \lambda_4 \lambda_2$



$$\boxed{\lambda_i \lambda_j = \lambda_{j+1} \lambda_i \quad \text{if } i \le j < n.} \qquad (L1)$$



$$\boxed{\lambda_i \lambda_n = \lambda_i \quad \forall i} \qquad (L2)$$

- $\rho_j \rho_i = \rho_i \rho_{j+1}$ if $i \le j < n$ (R1)
- $\rho_n \rho_i = \rho_i$ $\forall i$ (R2)

Theorem

$$\mathcal{L}_n = \langle L \mid (L1\text{-}L2) \rangle \ \& \ \mathcal{R}_n = \langle R \mid (R1\text{-}R2) \rangle.$$

The monoid $\mathcal{O}_n$

Recall $\mathcal{O}_n = \{$ order-preserving partial permutations on $\underline{n}\}$.

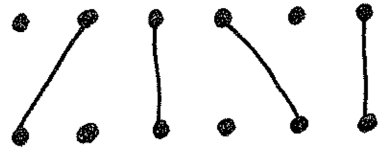
Lemma

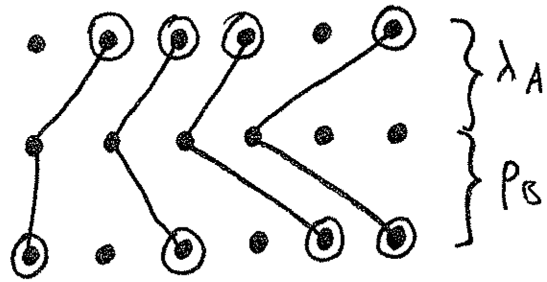
Let $\alpha \in \mathcal{O}_n$. Then

$$\alpha = \lambda_A \rho_B$$

where $A = \mathrm{dom}(\alpha)$ & $B = \mathrm{im}(\alpha)$.

So $\mathcal{O}_n = \mathcal{L}_n \mathcal{R}_n = \langle L \cup R \rangle$.

Proof: $\alpha =$ 

$=$ 

$\square$

We need relations that let us transform any word

$$\lambda \rho \rho \lambda \rho \lambda \lambda \lambda \rho \cdots$$

into one of the form

$$\lambda \lambda \lambda \lambda \rho \rho \rho \rho \rho .$$

$$\boxed{\rho_i \lambda_j = \begin{cases} \lambda_n \lambda_{j-1} \rho_i & \text{if } i < j \quad \text{(RL1)} \\ \lambda_n = \rho_n & \text{if } i = j \quad \text{(RL2)} \\ \lambda_n \lambda_j \rho_{i-1} & \text{if } i > j \quad \text{(RL3)} \end{cases}}$$

These also allow us to get a word of the form

$$\underbrace{\lambda \lambda \lambda \lambda \lambda}\ \underbrace{\rho \rho \rho \rho \rho}$$

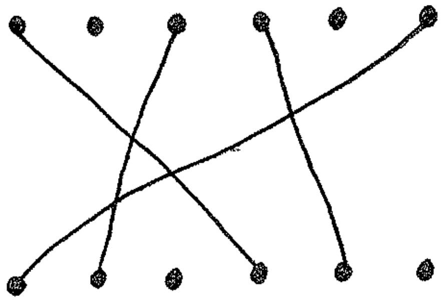
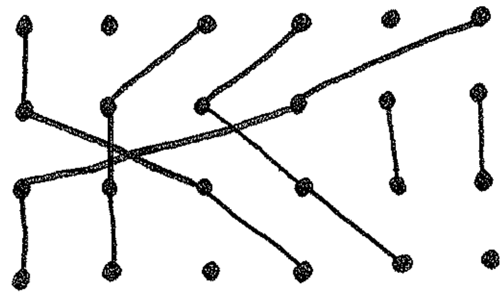
same length!

Theorem $\mathcal{O}_n = \langle L \cup R \mid \text{(L1–L2), (R1–R2), (RL1 – RL3)} \rangle.$

Applications

① Symmetric inverse monoid

$\mathcal{I}_n = \{$ partial permutations on $\underline{n} \}$

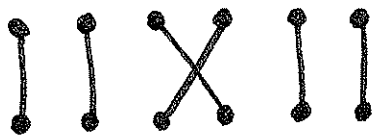
$\alpha =$  $=$ 

$\lambda_A \in \mathcal{L}_n$

$\pi \in \mathcal{S}_n$

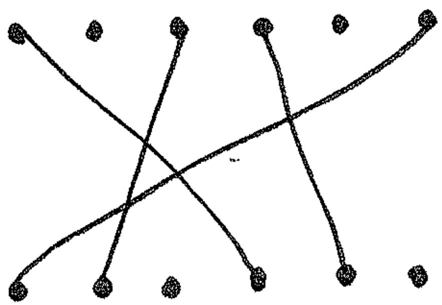
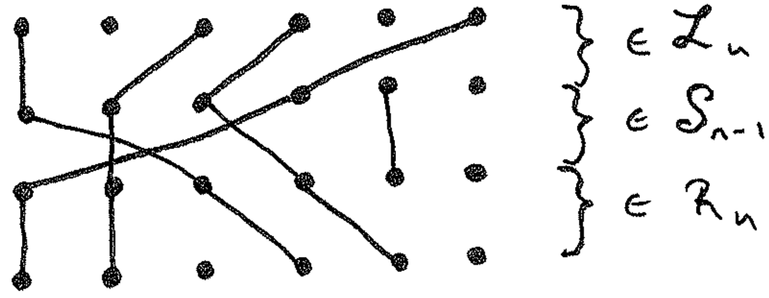
$\rho_B \in \mathcal{R}_n$

Theorem

- $\mathcal{I}_n = \mathcal{L}_n \mathcal{S}_n \mathcal{R}_n = \langle L \cup S \cup R \mid \text{relations} \rangle$
- $S = \{ s_1, \dots, s_{n-1} \}$
- $s_3 =$  $\in \mathcal{S}_6$

② Singular part of $\mathcal{I}_n$

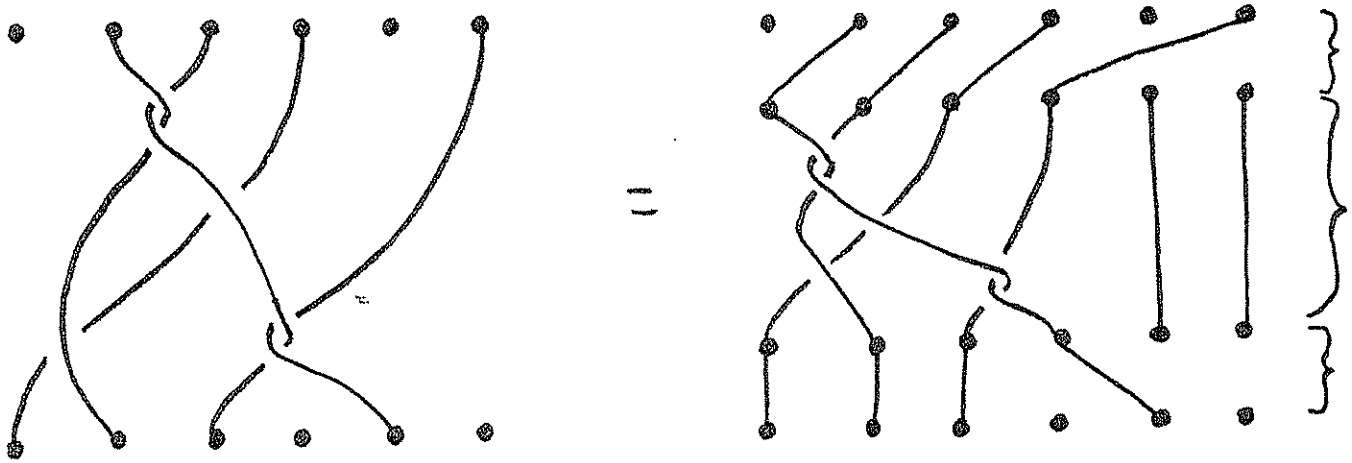
- $\mathcal{I}_n \setminus \mathcal{S}_n = \{$ strictly partial permutations on $\underline{n}\}$

$\alpha =$  $=$ 

Theorem

- $\mathcal{I}_n \setminus \mathcal{S}_n = \mathcal{L}_n \mathcal{S}_{n-1} \mathcal{R}_n = \langle L \cup T \cup R \mid \text{relations} \rangle$
- $T = \{t_1, \dots, t_{n-2}\}$ • $t_3 =$  $\in \mathcal{I}_6 \setminus \mathcal{S}_6$

③ The inverse braid monoid $\mathcal{I}\mathcal{B}_n = \{$ partial braids on $\underline{n}\}$



$\lambda_A \in \mathcal{L}_n$

$\beta \in \mathcal{B}_n$ — the braid group

$\rho_B \in \mathcal{R}_n$

Theorem

- $\mathcal{I}\mathcal{B}_n = \mathcal{L}_n \mathcal{B}_n \mathcal{R}_n = \langle L \cup S \cup R \mid \text{relations} \rangle$
- $S = \{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$
- $\sigma_3 =$  $\in \mathcal{B}_6$
- $\sigma_3^{-1} =$  $\in \mathcal{B}_6$

④ The singular part of $\mathcal{I}\mathcal{B}_n$

$$\mathcal{I}\mathcal{B}_n \setminus \mathcal{B}_n = \{ \text{strictly partial braids on } \underline{n} \}$$

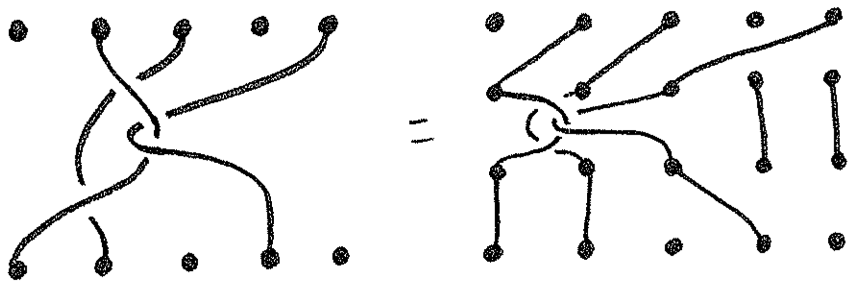
$$= \mathcal{L}_n \mathcal{B}_{n-1} \mathcal{R}_n$$

$$= \langle L \cup T \cup R \mid \text{relations} \rangle$$

- $T = \{ \tau_1^{\pm 1}, \dots, \tau_{n-2}^{\pm 1} \}$
  - $\tau_3 =$ 
  - $\tau_3^{-1} =$ 

$\in \mathcal{I}\mathcal{B}_6 \setminus \mathcal{B}_6$

(5) $\mathcal{POIB}_n = \{$ order-preserving partial braids on $\underline{n} \}$



$\} \in P_n$ — the pure braid group

$= $ kernel of projection $\mathcal{B}_n \to S_n$

Theorem

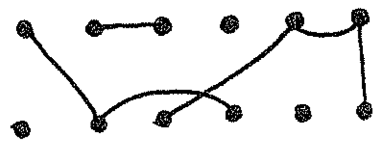
- $\mathcal{POIB}_n = \mathcal{L}_n P_n R_n = \langle L \cup A \cup R \mid \text{relations} \rangle$
- $A = \{ \alpha_{ij}^{\pm 1} \mid 1 \leq i < j \leq n \}$
- $\alpha_{35} =$  $\in P_6$

(6) $\mathcal{POIB}_n \setminus P_n = \{$ order-preserving strictly partial braids on $\underline{n} \}$

$= \mathcal{L}_n P_{n-1} R_n$ (etc...)

Applications of idea

- $T_n = \{\text{functions } \underline{n} \to \underline{n}\}$
- $T_n \setminus S_n = \{\text{non-invertible functions } \underline{n} \to \underline{n}\}$
- $PT_n = \{\text{partial functions } \underline{n} \to \underline{n}\}$
- $PT_n \setminus S_n = \{\text{non-invertible partial functions } \underline{n} \to \underline{n}\}$
- $PT_n \setminus T_n = \{\text{strictly partial functions } \underline{n} \to \underline{n}\}$
- $P_n = \{\text{partitions on } \underline{n}\}$



- $P_n \setminus S_n = \{\text{non-invertible partitions on } \underline{n}\}$

Thanks for listening!