Cohomological dimension of inverse semigroups

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An outline of homological algebra

- Projective modules are good.
- Free modules are better.
- R a ring, G a group: the group ring RG consists of all finite R-linear combinations $\sum_{g \in G} a_g g$ of the elements of G with obvious operations.

Resolutions

A *projective resolution* of a module M is a sequence of module maps

$$\cdots \to P_{k+1} \xrightarrow{\partial_{k+1}} P_k \xrightarrow{\partial_k} \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

such that:

- each P_j is projective,
- for each $k \geqslant 0$, $\ker \partial_k = \operatorname{im} \partial_{k+1}$,
- ε is surjective.

A resolution is an attempt to approximate M using projectives: it might involve non-zero terms for ever, or eventually become zero - we then say the resolution is of finite length.

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- As a trivial module over $R = \mathbb{Z}[t]/(t^2 - 1)$, the integral group ring of C_2, $\mathbb{Z}$ has an infinite free resolution

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  and none of finite length.
- If a group G acts freely on a contractible cell complex X then the cellular chain complex of X gives a free resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$–module.

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For a small category $\mathcal{C}$ a $\mathcal{C}$–module is

- a functor from $\mathcal{C}$ to abelian groups,
- equivalently, a collection of abelian groups indexed by the objects of $\mathcal{C}$, and for each arrow

$$x \xrightarrow{\alpha} y$$

of $\mathcal{C}$ a homomorphism $A_x \to A_y$ satisfying some obvious rules.

Inverse semigroups

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An *inverse semigroup* is

- a regular semigroup in which idempotents commute,
- a semigroup S in which, for each $s \in S$, there exists a **unique** $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$.

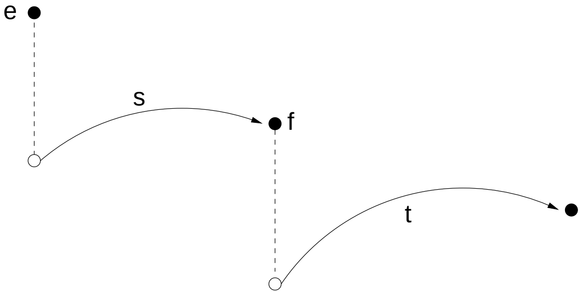
The *natural partial order* in S is given by

$$a \leqslant b \iff \text{there exists } e \in E(S) \text{ such that } a = eb.$$

Modules for inverse semigroups (I)

Loganathan's category $\mathfrak{L}(S)$:

- objects are idempotents in S,
- arrows are pairs (e, s) with $e \in E(S), s \in S$ such that $e \geqslant ss^{-1}$,
- (e, s) starts at e and ends at $s^{-1}s$,
- $(e, s)(f, t) = (e, st)$ when $s^{-1}s = f$.



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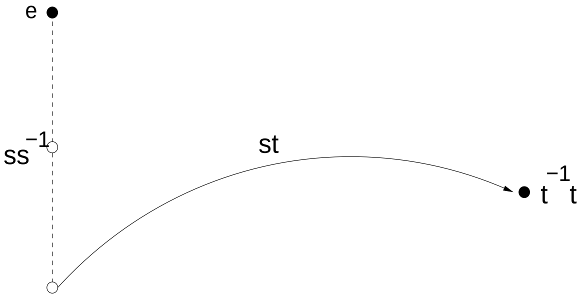


More about $\mathfrak{L}(S)$

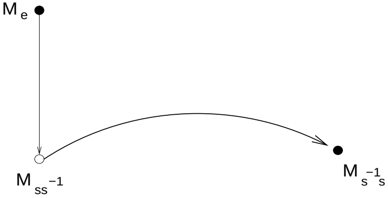
- $\mathfrak{L}(S)$ is left cancellative,
- arrow (e, s) uniquely decomposable as $(e, ss^{-1})(ss^{-1}, s)$
- $\mathfrak{L}(S)$ is a Zappa-Szép product of categories

$$\mathfrak{L}(S) = E(S) \bowtie S$$

Modules for inverse semigroups (II)

A module for S is now defined as a module for $\mathfrak{L}(S)$: Loganathan's 1981 recasting of ideas of Lausch (1975). So an $\mathfrak{L}(S)$– module $\mathcal{M}$ consists of:

- an abelian group M_e for each $e \in E(S)$,
- homomorphisms $M_e \to M_f$ whenever $e \geqslant f$,
- isomorphisms $M_{ss^{-1}} \to M_{s^{-1}s}$ (isoms since action by s on $M_{ss^{-1}}$ has inverse given by action by s^{-1}.



Cohomological dimension

The *projective dimension* of a module M is the smallest n such that M has a projective resolution of length n (so $P_n \neq 0$ but $P_k = 0$ for $k > n$).

The (integral) cohomological dimension $\operatorname{cd} G$ of a group G is the projective dimension of $\mathbb{Z}$ as a trivial G–module.

The cohomological dimension of an inverse semigroup S is the projective dimension of the module $\underline{\mathbb{Z}}$ in which $\underline{\mathbb{Z}}_e = \mathbb{Z}$ for all $e \in E(S)$ and all maps are identities.

The Gruenberg resolution

Resolution by relations, as Gruenberg's paper (1960) had it:

- G a group, F a free group mapping on to G, $N = \ker(F \xrightarrow{\theta} G)$,
- induced $\theta : \mathbb{Z}F \to \mathbb{Z}G$ with kernel $\mathfrak{r}$,
- *augmentation ideal* $\mathfrak{f}$ of F is $\ker(\mathbb{Z}F \xrightarrow{\varepsilon} \mathbb{Z})$ where $\varepsilon : w \mapsto 1$ for all $w \in F$.

Theorem (Gruenberg)

The complex of $\mathbb{Z}G$–modules

$$\ldots \to \mathfrak{r}^2/\mathfrak{r}^3 \to \mathfrak{f}\mathfrak{r}/\mathfrak{f}\mathfrak{r}^2 \to \mathfrak{r}/\mathfrak{r}^2 \to \mathfrak{f}/\mathfrak{f}\mathfrak{r} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

is a G–free resolution of $\mathbb{Z}$, and this construction gives a functor from the category of free presentations of G to the category of G–free resolutions of $\mathbb{Z}$.

“Yes... wonderful things.”

The Gruenberg resolution gives us:

- a free resolution from any free presentation of G,
- a module theory approach to the *relation module*: $\ker(\mathfrak{f}/\mathfrak{fr} \to \mathbb{Z}G) \cong N^{ab}$ as G–modules. We don't need abelianisation.
- generalised Hopf formulae for the homology of G:

$$H_{2q}(G) = \frac{\mathfrak{r}^q \cap \mathfrak{f}\mathfrak{r}^{q-1}\mathfrak{f}}{\mathfrak{f}\mathfrak{r}^q + \mathfrak{r}^q\mathfrak{f}} \qquad H_{2q+1}(G) = \frac{\mathfrak{f}\mathfrak{r}^q \cap \mathfrak{r}^q\mathfrak{f}}{\mathfrak{r}^{q+1} + \mathfrak{f}\mathfrak{r}^q\mathfrak{f}}.$$

- Gives Webb's approach to the relation module etc for categories (2011).

Let's do all this for inverse semigroups

Loganathan defines $\mathbb{Z}S$ as the $\mathfrak{L}(S)$–module with $(\mathbb{Z}S)_e$ free abelian on the $\mathcal{L}$–class of e in S:

$$(\mathbb{Z}S)_e = \textbf{free abelian group}\{s \in S : s^{-1}s = e\}.$$

$\mathbb{Z}S$ need not be free as a $\mathfrak{L}(S)$–module, but it is projective. So we want to construct a version of the Gruenberg resolution:

$$\ldots \to \mathfrak{r}^2/\mathfrak{r}^3 \to \mathfrak{f}\mathfrak{r}/\mathfrak{f}\mathfrak{r}^2 \to \mathfrak{r}/\mathfrak{r}^2 \to \mathfrak{f}/\mathfrak{f}\mathfrak{r} \to \mathbb{Z}S \to \underline{\mathbb{Z}} \to 0$$

using projective $\mathfrak{L}(S)$–modules.

The limits to ambition

- Want:

$$\dots \to \mathfrak{r}^2/\mathfrak{r}^3 \to \mathfrak{f}\mathfrak{r}/\mathfrak{f}\mathfrak{r}^2 \to \mathfrak{r}/\mathfrak{r}^2 \to \mathfrak{f}/\mathfrak{f}\mathfrak{r} \to \mathbb{Z}S \to \underline{\mathbb{Z}} \to 0$$

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- Get:

$$\mathcal{F} \to D \to \mathbb{Z}S \to \underline{\mathbb{Z}} \to 0$$

  Here D is an inverse semigroup version of Crowell's *derivation module* and/or Gruenberg's $\mathfrak{f}/\mathfrak{f}\mathfrak{r}$ and $\mathcal{F}$ is an $\mathfrak{L}(S)$–module 'free on the relations'.

The derivation module

- $\theta : T \to S$ a hom of inverse semigroups, $\mathcal{A} = \bigcup_{e \in E(S)} A_e$ an $\mathfrak{L}(S)$–module.
- $\eta : T \to \mathcal{A}$ is a θ–*derivation* if,
  - $a\eta \in A_{(a^{-1}a)\theta}$,
  - whenever $a, b \in T$ with $a^{-1}a \geqslant bb^{-1}$,

$$(ab)\eta = a\eta \lhd ((a^{-1}a)\theta, b\theta) + b\eta.$$

- the *derivation module* D_θ has $(D_\theta)_e$
  - generated by all $(a, s) \in T \times S$ with $(a^{-1}a)\theta \geqslant ss^{-1}$ and $s^{-1}s = e$
  - subject to relations

$$(ab, s) - (b, s) = (a, (b\theta)s)$$

  for all $a, b \in T$ with $a^{-1}a \geqslant bb^{-1}$.
- image of (a, s) in D_θ is written $\langle a, s \rangle$.

What is it good for? (I)

Derivation module D_θ converts θ–derivations to homomorphisms:

Theorem

There exists a canonical θ–derivation

$$\delta : T \longrightarrow D_\theta, \qquad \delta : a \mapsto \langle t, (t^{-1}t)\theta \rangle$$

such that, given any θ–derivation $\eta : T \to \mathcal{A}$ to an $\mathfrak{L}(S)$–module $\mathcal{A}$, there is a unique $\mathfrak{L}(S)$–map $\xi : D_\theta \to \mathcal{A}$ such that $\eta = \delta\xi$.

$$\begin{array}{ccc} T & \xrightarrow{\ \delta\ } & D_\theta \\ {\scriptstyle \eta}\downarrow & \swarrow {\scriptstyle \xi} & \\ \mathcal{A} & & \end{array}$$

What is it good for? (II)

Theorem

If S is an inverse monoid and F a free inverse monoid with $\theta : F \to S$ surjective then

- D_θ *is a projective* $\mathfrak{L}(S)$*–module,*
- $\partial_1 : D_\theta \to \mathbb{Z}S, \langle a, s\rangle \mapsto (a\theta)s - s$ *maps* D_θ *on to the augmentation module of* S*.*

The kernel of ∂_1, following Gruenberg, we define to be the *relation module* $\mathcal{M}_\theta$ of θ.

The relation module

Let $\langle X : \ell_1 = r_1, \ell_2 = r_2, \ldots \rangle$ be a presentation of the inverse monoid M, with F the free monoid generated by X.

Theorem

The relation module $\mathcal{M}_\theta$ is generated, as an $\mathfrak{L}(S)$–module, by all elements of the form $\langle \ell_i, e \rangle - \langle r_i, e \rangle$ where $e = (\ell_i^{-1} r_i)\theta$.

Cohomological dimension 0

A group G has cohom dim 0 if and only if it is trivial. [Exercise: $\mathbb{Z}$ is a projective G–module . . .]

Theorem

An inverse monoid has cohomological dimension 0 *if and only if it is a semilattice (so every element is an idempotent).*

Proof.

- (Leech 1975) Use Laudal's 1972 characterization of small categories of cd 0 and apply to $\mathfrak{L}(S)$.
- **or**: use the fact that $\underline{\mathbb{Z}}$ is a projective $\mathfrak{L}(S)$–module and generalise the argument for groups.

□

Arboreal inverse monoids

An *arboreal* inverse monoid M is one given by a presentation $\langle X : e_i = f_i \rangle$ where e_i, f_i are idempotents in F – that is, words whose freely reduced form is equal to 1.

Theorem (Margolis-Meakin 1993)

- *X–generated arboreal inverse monoids are the E–unitary quotients of F with maximum group image free on X,*
- *M is arboreal if and only if each of its Schützenberger graphs is a tree,*
- *finitely presented arboreal inverse monoids have decidable word problem.*

Cohomological dimension 1

Theorem

An arboreal inverse monoid has cohomological dimension 1.

Proof.

For presentations of the type that define arboreal inverse monoids, the relation module $\mathcal{M} = 0$ since

$$\begin{aligned}\langle e_i, (e_i)\theta\rangle &= \langle e_i e_i, (e_i)\theta\rangle \\ &= \langle e_i, (e_i)\theta(e_i)\theta\rangle + \langle e_i, (e_i)\theta\rangle \\ &= \langle e_i, (e_i)\theta\rangle + \langle e_i, (e_i)\theta\rangle.\end{aligned}$$

So we have a projective resolution

$$0 \to D \to \mathbb{Z}S \to \underline{\mathbb{Z}} \to 0.$$

□

Go on, go on ,go on

Conjecture: [Steinberg] An inverse monoid M acts freely on a presheaf of trees over $E(M)$ if and only if it is E–unitary, has free maximum group image, and cd $= 1$.

Arboreal inverse monoids act freely on their Schützenberger trees.

Stallings-Swan Theorem (1969). A group G has cd $G \leqslant 1$ if and only if it is free.

What is the right notion of 'free'? Surely, guided by results of Margolis-Meakin-Yamamura 1999:

'free' should be: M is arboreal.