A tour of ideas behind restriction and related semigroups

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Outline

- What are restriction semigroups?
- Related semigroups? (some of which came first...)
- Proper restriction semigroups
- Proper covers
- Structure of proper restriction semigroups

Restriction semigroups: what are they?

Restriction semigroups may be obtained as/from:

- Varieties of algebras
- Representation by (partial) mappings
- Generalised Green's relations
- Inductive categories and inductive constellations

Let $S = (S, \cdot, ^+)$ be a semigroup equipped with a unary operation $^+$ (*that is, S is a* **unary semigroup**).

Definition S is **left restriction** if the following identities hold:

$$x^+x = x,\; x^+y^+ = y^+x^+,\; (x^+y)^+ = x^+y^+,\; xy^+ = (xy)^+x.$$

Restriction semigroups: varieties

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x.$$

Let S be left restriction and put

$$E = \{a^+ : a \in S\}.$$

For any $a^+ \in E$,

$$a^+ = (a^+a)^+ = a^+a^+$$

so that we see E is a **semilattice**, i.e. commutative semigroup of idempotents.
E is the **distinguished semilattice** of S
Also with $a^{++} = (a^+)^+$,

$$a^+ = a^{++}a^+ = a^+a^{++} = (a^+a^+)^+ = a^{++}.$$

Restriction semigroups: varieties

- Left restriction semigroups form a **variety** of unary semigroups.
- Dually, right restriction semigroups form a variety of unary semigroups, with unary operation denoted by *, satisfying the left/right duals of the axioms above.
- A bi-unary semigroup $S = (S, \cdot, ^+, ^*)$ is **restriction** if and only if satisfies the identities for left and right restriction semigroups together with

$$(x^*)^+ = x^* \text{ and } (x^+)^* = x^+.$$

Restriction semigroups: examples
Monoids

Let M be a monoid and define $a^+ = 1 = a^*$ for all $a \in M$. Then $M = (M, \cdot, ^+, ^*)$ is restriction.

We need to check the identities

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x,$$

their duals and

$$(x^*)^+ = x^* \text{ and } (x^+)^* = x^+.$$

Restriction semigroups: examples
Monoids

Definition A (left) restriction semigroup is **reduced** if $|E| = 1$.

We have seen a monoid is a reduced (left) restriction semigroup (in a different signature).

Conversely, let S be a reduced left restriction semigroup. Then let $E = \{u\}$, so that $u = u^+ = a^+$ for all $a \in S$. Since $a^+a = a$ for all $a \in S$, we have $ua = a$.
Also,

$$au = au^+ = (au)^+a = a$$

so that u is an identity for S.

$\mathcal{R}$ and $\mathcal{L}$

- For any $a, b \in S$ we have

$$\begin{aligned} a \,\mathcal{R}\, b &\Leftrightarrow aS^1 = bS^1 \\ &\Leftrightarrow \exists s, t \in S^1 \text{ with } a = bs \text{ and } b = at. \end{aligned}$$

- For any $a, b \in S$ we have

$$\begin{aligned} a \,\mathcal{L}\, b &\Leftrightarrow S^1 a = S^1 b \\ &\Leftrightarrow \exists s, t \in S^1 \text{ with } a = sb \text{ and } b = ta. \end{aligned}$$

- $\mathcal{R}$ ($\mathcal{L}$) is a left (right) congruence
- $\mathcal{R}$ and $\mathcal{L}$ are the universal relation on any group
- $\mathcal{R}$ and $\mathcal{L}$ are the trivial relation on any semilattice

Inverse semigroups

Definition S is *regular* if for all $a \in S$ there exists $x \in S$ with $a = axa$.

Notice that if $a = axa$, then $ax, xa \in E(S)$ and

$$ax \, \mathcal{R} \, a \, \mathcal{L} \, xa.$$

Fact S is regular if and only if every $\mathcal{R}$-class (or $\mathcal{L}$-class) contains an idempotent.

Definition S is inverse if S is regular and $E(S)$ is a semilattice.

Fact S is inverse if and only every element has a unique inverse, i.e. for all $a \in S$ there exists a unique a' in S such that

$$a = aa'a \text{ and } a' = a'aa'.$$

Fact S is inverse if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class contains a unique idempotent.

Restriction semigroups: examples
Inverse semigroups

Let S be an inverse semigroup. Recall that $E(S)$ is a semilattice. Put

$$a^+ = aa' \text{ and } a^* = a'a.$$

Then $S = (S, \cdot, ^+, ^*)$ is restriction with distinguished semilattice $E(S)$.

We need to check the identities

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x,$$

their duals and

$$(x^*)^+ = x^* \text{ and } (x^+)^* = x^+.$$

Let $a, b \in S$. Then

$$(a^+b)^+ = (aa'b)^+ = aa'b(aa'b)' = aa'bb'aa' = aa'bb' = a^+b^+.$$

Also

$$(ab)^+a = (ab)(ab)'a = ab(b'a')a = a(bb')(a'a) = a(a'a)(bb') = ab^+.$$

Restriction semigroups: representations

- Every semigroup S embeds in a **full transformation semigroup** $\mathcal{T}_X$
- Every group embeds in a **symmetric group** $\mathcal{S}_X$
- Every inverse semigroup S embeds (as an inverse semigroup) in the **symmetric inverse semigroup** $\mathcal{I}_X$
- $\mathcal{T}_X, \mathcal{S}_X$ and $\mathcal{I}_X$ are all subsemigroups of the semigroup $\mathcal{PT}_X$ of all partial mappings of X.
- Define $^+$ on $\mathcal{PT}_X$ by

$$\alpha^+ = I_{\text{dom }\alpha}.$$

  Then $\mathcal{PT}_X$ is left restriction with distinguished semilattice

$$E = \{I_Y : Y \subseteq X\}.$$

- S is left restriction if and only if it embeds in some $\mathcal{PT}_X$ (Trokhimenko).

Restriction semigroups: free algebras

- Since (left) restriction semigroups form varieties, free algebras exist.
- The free (left) restriction semigroup $\mathcal{FR}(X)$ ($\mathcal{FLR}(X)$) on any set X embeds into the free inverse semigroup $\mathcal{FI}(X)$ on X (Fountain, Gomes, G).
- The determination of the structure of $\mathcal{FI}(X)$ by Munn, Schein and Scheiblich is a classical early result of Semigroup Theory.
- The structure of $\mathcal{FLR}(X)$ and $\mathcal{FR}(X)$ is particularly nice: they are both **proper**.

Restriction semigroups: the relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$

Let $E \subseteq E(S)$

- The relation $\widetilde{\mathcal{R}}_E$ is defined by $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if

$$ea = a \Leftrightarrow eb = b$$

  for all $e \in E$.
- Note if $a \,\widetilde{\mathcal{R}}_E\, e \in E$, then as $ee = e$ we have $ea = a$.
- For a left restriction semigroup with distinguished semilattice E,

$$a \,\widetilde{\mathcal{R}}_E\, b \text{ if and only if } a^+ = b^+.$$

- The relation $\widetilde{\mathcal{L}}_E$ is defined dually.
- $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalence relations.
- If M is a monoid and $E = \{1\}$, then $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are universal.
- These relations were introduced by El-Qallali in his 1980 thesis (under Fountain) in case $E = E(S)$, later generalised by Lawson.

The relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ - connection to $\mathcal{R}$ and $\mathcal{L}$

Fact For any semigroup S and any E

$$\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E.$$

Proof Let $a \,\mathcal{R}\, b$. Then $a = bs$ and $b = at$ for some $s, t \in S^1$.

Hence

$$ea = a \Rightarrow eat = at \Rightarrow eb = b \Rightarrow ebs = bs \Rightarrow ea = a.$$

Fact If S is regular and $E = E(S)$, then $\widetilde{\mathcal{R}}_E = \mathcal{R}$.

Proof If $a \,\widetilde{\mathcal{R}}_{E(S)}\, b$ and $a = axa, b = byb$, then $b = axb$ and $a = bya$.

Restriction semigroups: definition using $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$

Fact A semigroup S is **left restriction** with **distinguished semilattice** E iff:

- E is a semilattice;
- every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E; it is then easy to see that for every $a \in S$ the $\widetilde{\mathcal{R}}_E$-class of a contains a unique element of E, which we call a^+;
- the relation $\widetilde{\mathcal{R}}_E$ is a left congruence and
- the **left ample** condition (AL) holds:

$$\text{for all } a \in S \text{ and } e \in E,\ ae = (ae)^+a.$$

Similarly for (right) restriction semigroups.

A bit of history

Different schools arrived at (left) restriction semigroups via different directions from 1960s onwards:

- Schweizer, Sklar, Schein, Trokhimenko: **function systems**
  Let T be a subsemigroup of $\mathcal{PT}_X$ or $\mathcal{B}_X$ (semigroup of binary relations on X).
  T may be equipped with additional operations such as $^+$, $\cap$, $(f,g) \mapsto f^+g$ etc.
  Can such T be axiomatised by first order formulae? By identities?
- Lawson: **Ehresmann semigroups**
  Lawson found a correspondence between Ehresmann semigroups and certain categories equipped with two orderings. As a special case, restriction semigroups correspond to inductive categories.

A bit more history

- Jackson and Stokes: **closure operators**
  Introduced 'twisted C-semigroups', with an axiomatisation equivalent to the one given here.
- Manes, Cockett, Lack: **category theory, computer science** Gave the axioms above. Also interested in restriction *categories*.
- Fountain: **generalisations of regular and inverse semigroups**
- Jones: **P-restriction semigroups** obtained from *regular* $*$*-semigroups*

Summary to date

- We have seen how to define (left) restriction semigroups as
  - varieties
  - by their representations as subalgebras of $\mathcal{PT}_X$
  - by using $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$.
- We have mentioned there is a connection between (left) restriction semigroups and ordered structures
- The approach using $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ is part of the **York** approach to studying semigroups via $\mathcal{R}^*, \mathcal{L}^*, \widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E$ and properties of idempotents.
- We introduced ample, abundant, weakly *U*-abundant semigroups, congruence and ample conditions.

Inverse semigroups: groups and semilattices

There are three major approaches to structure of inverse semigroups:

- The Ehresmann-Schein-Nambooripad association of **inductive groupoids** to inverse semigroups.
- Munn's construction of a **fundamental** inverse semigroup T_E from any semilattice E; if S is inverse then $T_{E(S)}$ contains a morphic image S' of S such that $E(S) \cong E(S')$.
- McAlister's approach using **proper covers**: if S is inverse then it has a proper preimage $\widehat{S}$ such that $E(\widehat{S}) \cong E(S)$ and such that the structure of $\widehat{S}$ is known.

We are going to develop the McAlister approach for (left) restriction semigroups.

Proper restriction semigroups: definition

Let S be (left) restriction.

- Recall that S is **reduced** if $|E| = 1$.
- σ_E is the least congruence **identifying all the idempotents of** E.
- The (left) restriction semigroup S/σ_E is reduced.
- A left restriction semigroup S is **proper** if

$$\left(a^+ = b^+,\, a\,\sigma_E\, b\right) \Rightarrow a = b.$$

- A restriction semigroup S is **proper** if

$$\left(a^+ = b^+,\, a\,\sigma_E\, b\right) \Rightarrow a = b \textbf{ and } \left(a^* = b^*,\, a\,\sigma_E\, b\right) \Rightarrow a = b.$$

- Monoids and semilattices are both proper restriction.
- If S is proper left restriction, then $\theta : S \to E \times S/\sigma_E$ given by

$$s\theta = (s^+, s\sigma_E)$$

is an injection.

Proper restriction semigroups: semidirect products

Let M be a monoid and Y a set. Then M **acts on the left of** Y if there is a map

$$M \times Y \to Y; \quad (m, y) \mapsto m \cdot y,$$

such that

$$1 \cdot y = y \text{ and } (mn) \cdot y = m \cdot (n \cdot y).$$

Suppose now that Y is a semigroup. Then M **acts by morphisms** if, in addition,

$$m \cdot (yz) = (m \cdot y)(m \cdot z).$$

In this case, define a product on $Y \times M$ by

$$(y, m)(z, n) = (y(m \cdot z), mn).$$

This product is associative, yielding the **semidirect product** $Y * M$.

Proper restriction semigroups: semidirect products

Let M be a monoid and let Y be a semilattice.

- $Y * M$ is proper left restriction with $(e, m)^+ = (e, 1)$.
- If M is a group, then $Y * M$ is proper inverse.
- Suppose that Y has a greatest element 1_Y. We say that M acts **doubly** on Y if M acts by morphisms on the left and right of Y and the compatibility conditions hold, that is

  $$(t \cdot e) \circ t = (1_Y \circ t)e \text{ and } t \cdot (e \circ t) = e(t \cdot 1_Y)$$

  for all $t \in M, e \in Y$.
- In this case

  $$Y *_m M = \{(e, t) : e \leq t \cdot 1_Y\} \subseteq Y * M$$

  is a proper restriction monoid with identity $(1_Y, 1)$ such that

  $$(e, t)^+ = (e, 1) \text{ and } (e, t)^* = (e \circ t, 1).$$

Proper restriction semigroups: W-products

Let M be a monoid acting by morphisms on the right of a semilattice Y such that

- $a \circ t = b \circ t \Rightarrow a = b$;
- $a \leq b \circ t \Rightarrow a = c \circ t$ for some c.

Let

$$W = \{(t, a \circ t) : t \in M, a \in Y\} \subseteq M * Y.$$

Then W is a proper restriction subsemigroup of the (reverse) semidirect product $M * Y$ where

$$(t, a \circ t)^+ = (1, a) \text{ and } (t, a \circ t)^* = (1, a \circ t).$$

(Construction due to Fountain, Gomes and Szendrei).

Proper covers

Let S be (left) restriction.

A **proper cover** of S is a proper (left) restriction semigroup $\widehat{S}$ and an onto morphism $\theta : \widehat{S} \twoheadrightarrow S$ such that θ separates distinguished idempotents.

Theorem Every (left) restriction semigroup has a proper cover (Branco, Fountain, Gomes, G).

Theorem Let S be restriction. Then S has a proper cover that is embeddable into a W-semigroup (Szendrei).

Proper covers: sketch of existence

- Let S be a restriction monoid with distinguished semilattice E. Define

$$s \cdot e = (se)^+ \text{ and } e \circ s = (es)^*.$$

Then these are actions of S on E by morphisms, satisfying the compatibility conditions.

- Let $s \in S$ and $e, f \in E$. From the identities $(x^+y)^+ = x^+y^+$ and $xy^+ = (xy)^+x$,

$$s \cdot ef = (sef)^+ = \left((se)^+sf\right)^+ = (se)^+(sf)^+ = (s \cdot e)(s \cdot f).$$

- Consequently, $E *_m S = \{(e, s) : e \leq s^+\} \subseteq E * S$ is proper restriction.
- Define $\theta : E *_m S \to S$ by $(e, s)\theta = es$. Then θ is a covering morphism.
- Make adaptations for semigroup/one-sided case.

Proper left restriction semigroups: structure

Let T be a monoid acting on the left of a semilattice $\mathcal{X}$ via morphisms. Suppose that $\mathcal{X}$ has subsemilattice $\mathcal{Y}$ with upper bound ε such that
(a) for all $t \in T$ there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$
(b) if $e \leq t \cdot \varepsilon$ then for all $f \in \mathcal{Y}$, $e \wedge t \cdot f \in \mathcal{Y}$.
Then $(T, \mathcal{X}, \mathcal{Y})$ is a **strong left M-triple**.

For a strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$ we put

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(e, t) \in \mathcal{Y} \times T : e \leq t \cdot \varepsilon\} \subseteq \mathcal{X} * T.$$

Then $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is **proper left restriction** with $(e, s)^+ = (e, 1)$.

Theorem A left restriction semigroup S is proper if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ (Branco, Gomes, G).

Important point In the above result, we can take

$$T = S/\sigma_E \text{ and } \mathcal{Y} = E.$$

Proper restriction semigroups: structure

- If S is proper restriction, then as S is proper left restriction,
$$S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$$
where $T = S/\sigma_E$ and $\mathcal{Y} = E$, and as S is proper right restriction,
$$S \cong \mathcal{M}'(\mathcal{Y}, \mathcal{X}', T),$$
where $\mathcal{M}'(\mathcal{Y}, \mathcal{X}', T)$ is constructed from T acting on the right of a semilattice $\mathcal{X}'$.
- Clearly the left and right actions of T must be connected in some way.
- Let $(T, \mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{X}', T)$ be strong left (right) $\mathcal{M}$-triples such that
$$e \leq t \cdot \varepsilon \Rightarrow t \cdot (e \circ t) = e \text{ and } e \leq \varepsilon' \circ t \Rightarrow (t \cdot e) \circ t = e$$
then
$$\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y}) = \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) \cong \mathcal{M}'(\mathcal{Y}, \mathcal{X}', T)$$
is proper restriction.

Structure of proper restriction semigroups: the bad news

Let S be a restriction semigroup.

S **satisfies Condition (EP)** if it satisfies $(\mathrm{EP})^r$ and its dual $(\mathrm{EP})^l$.
$(\mathrm{EP})^r$: for all $s, t, u \in S$, if $s\,\sigma_E\,tu$ then there exists $v \in S$ with $t^+s = tv$ and $u\,\sigma_E\,v$.

Theorem (Cornock, G) Let S be a proper restriction semigroup. Then S is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ if and only if S satisfies (EP).

Free restriction semigroups, and proper inverse semigroups have (EP).

Not all proper restriction semigroups have (EP).

Definition Let T be a monoid, acting partially on the left and right of a semilattice $\mathcal{Y}$, via $\cdot$ and $\circ$ respectively. Suppose that both actions preserve the partial order and the domains of each $t \in T$ are order ideals. Suppose in addition that for $e \in \mathcal{Y}$ and $t \in T$, the following and their duals hold:
(a) if $\exists e \circ t$, then $\exists t \cdot (e \circ t)$ and $t \cdot (e \circ t) = e$;
(b) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $\exists e \circ t$.

Then $(T, \mathcal{Y})$ is a **strong M-pair**.

We put

$$\mathcal{M}(T, \mathcal{Y}) = \{(e, s) \in \mathcal{Y} \times T : \exists e \circ s\}$$

and define operations by

$$(e, s)(f, t) = (s \cdot (e \circ s \wedge f), st), \ (e, s)^{+} = (e, 1) \text{ and } (e, s)^{*} = (e \circ s, 1).$$

A structure theorem for proper restriction semigroups: the result

Theorem (Cornock, G) If $(T, \mathcal{Y})$ is a strong M-pair, then

$$\mathcal{M}(T, \mathcal{Y}) \cong \mathcal{M}'(\mathcal{Y}, T),$$

where $\mathcal{M}'(\mathcal{Y}, T)$ is constructed dually to $\mathcal{M}(T, \mathcal{Y})$.

Theorem (Cornock, G) A semigroup is proper restriction if and only if it is isomorphic to some $\mathcal{M}(T, \mathcal{Y})$.

Corollary (Petrich and Reilly) A semigroup is proper inverse if and only if it is isomorphic to $\mathcal{M}(G, \mathcal{Y})$ for a group G.

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