ABSTRACT. Let $\Delta \subseteq V$ be a proper subset of the vertices $V$ of the defining graph of an irreducible and aperiodic shift of finite type $(\Sigma_A^+, T)$. Let $\Sigma_\Delta$ be the subshift of allowable paths in the graph of $\Sigma_A^+$ which only passes through the vertices of $\Delta$. For a random point $x$ chosen with respect to an equilibrium state $\mu$ of a Hölder potential $\varphi$ on $\Sigma_A^+$, let $\tau_n$ be the point process defined as the sum of Dirac point masses at the times $k > 0$, suitably rescaled, for which the first $n$-symbols of $T^k x$ belong to $\Delta$. We prove that this point process converges in law to a marked Poisson point process of constant parameter measure. The scale is related to the pressure of the restriction of $\varphi$ to $\Sigma_\Delta$ and the parameters of the limit law are explicitly computed.

INTRODUCTION

The study of limit laws for the (rescaled) random times of occurrence of asymptotically rare events has motivated the consideration of dynamically defined hitting time point processes (see definition in Section 1 and the expository notes [4]). Special attention has been given to the case where one considers, for an ergodic dynamical system on a compact metric space, the first hitting time of shrinking neighbourhoods of a generic point. In fact, given any aperiodic ergodic dynamical system, one can get any limit law within a large class of laws [18] by using a suitable family of shrinking neighbourhoods. In contrast with this abstract result, when one considers cylinder sets about a generic point of a system mixing “sufficiently well” its partition, one expects and gets an exponential limit law for the first hitting time, and a Poisson law for the hitting time process; see for instance the papers [1, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 22]. Here, as in [5], we consider another case where the intersection of the shrinking neighbourhoods contains a non-trivial invariant set and we show that a marked Poisson point process appears as the asymptotic limit law.

We should mention that a marked Poisson point process appeared earlier in [22] when Pitskel gave an example of asymptotic return times to shrinking neighbourhoods of a fixed point for Markov Chains. Also in [15] Hirata shows that a
marked Poisson point process appears when considering asymptotic return times to shrinking neighbourhoods of a periodic point in Axiom A diffeomorphisms.

This paper is motivated by the main result in [5], but phrased in the context of symbolic dynamical systems. The problem reads as follows: given two points \( x \) and \( y \) in an aperiodic shift of finite type (randomly chosen according to the equilibrium state of some Hölder potential) consider the times when their orbits under the shift get \( \varepsilon \)-close with respect to the usual distance (take the distance between two points as \( \sum_i |x_i - y_i| \rho^i \) for some fixed \( 0 < \rho < 1 \), say). This defines a point process depending on \( \varepsilon \) and the problem is to prove whether this point process rescaled by \( \varepsilon^{-1} \) converges in law when \( \varepsilon \) tends to zero. Tackling this problem with the technique developed in [5] requires some additional ideas from spectral properties of a perturbed transfer operator studied in [8], and we realized that the above problem could be obtained as an application of a much more general result after the study of random times of asymptotic approach to a subsystem of finite type, and this is the main subject of this paper.

Let \( \Sigma_\Delta \) be a proper subshift of finite type of a one-sided irreducible and aperiodic shift of finite type \( \Sigma_\Delta^+ \) and consider a generic point \( x \in \Sigma_\Delta^+ \) with respect to the equilibrium state \( \mu \) of some Hölder potential \( \varphi \) defined on \( \Sigma_\Delta^+ \). Define a point process on \( [0, \infty) \) by summing Dirac masses at the times \( t > 0 \) for which the orbit of \( x \) under the shift on \( \Sigma_\Delta^+ \) is \( \varepsilon \)-close to \( \Sigma_\Delta \). Using a higher-block representation of \( \Sigma_\Delta^+ \) (see for instance [19]), we may assume the subshift \( \Sigma_\Delta \) is constructed by choosing a proper subset of vertices \( \Delta \subset V \) of the defining graph of \( \Sigma_\Delta^+ \), and defining \( \Sigma_\Delta \) as the subshift of allowable paths in the graph of \( \Sigma_\Delta^+ \) which only passes through vertices of \( \Delta \). Let \( \tau_n \) be the above point process of asymptotic approach to \( \Sigma_\Delta \), redefined as the sum of Dirac masses at the times \( k \geq 1 \), for which the first \( n \)-symbols of \( T^k x \) belong to \( \Delta \) (so we have \( \varepsilon \sim \rho^n \)). Let \( P_\varphi \) denote the pressure of \( \varphi \), and \( P_\Delta \) the pressure of the restriction of \( \varphi \) to the subsystem \( \Sigma_\Delta \) (hence necessarily \( P_\varphi = P_\Delta - P_\varphi < 0 \)).

We prove in this paper that if \( \Sigma_\Delta \) is an irreducible and aperiodic subshift of finite type in its alphabet \( \Delta \), then \( \tau_n \) when rescaled by \( e^{-nP_\varphi} \) converges in law when \( n \to \infty \). The limit law is a marked Poisson point process of constant parameter measure \( \lambda \pi \), where the parameters are given by

\[
\lambda = (1 - e^{P_\varphi}) \int h_\Delta \, d\mu \quad \text{and} \quad \pi_j = (1 - e^{P_\varphi}) e^{(j-1)P_\varphi},
\]

for \( j \geq 1 \). Here \( h_\Delta \) is the density of the Pianigiani-Yorke measure associated to the triple \( (\Sigma_\Delta^+, \Sigma_\Delta, \varphi) \), see Sections 2 and 3, and [8]. This result is mentioned in [4] with a sketch of the proofs and refers to the present paper for the complete proofs. The problem of studying the asymptotic approach of two (or more) shift orbits, mentioned at the beginning, can then be obtained as a consequence of the above result, see the application after Theorems 8 and 9 below.

Let \( \Delta_n \) denote the subset of points \( x \) for which the first \( n \)-symbols of \( T^k x \) belong to \( \Delta \). In the present paper, we study the asymptotic behaviour of \( \mu(\Delta_n) \)
and compare it with the pressure of the restriction of $\varphi$ to $\Sigma_\Delta$. We show that in the case $\Sigma_\Delta$ is irreducible and aperiodic then $e^{n(P_\varphi - P_\Delta)} \mu(\Delta_n)$ has a limit (which can be identified) as $n \to \infty$.

Although we do not know at present if the hitting time point process of $\Delta_n$ exists when $\Sigma_\Delta$ is irreducible but periodic, we show in the present paper that in this case, in general, $e^{n(P_\varphi - P_\Delta)} \mu(\Delta_n)$ may not converge as $n \to \infty$, contrarily to the aperiodic case. Indeed, we provide an explicit example where this phenomenon appears.

We explain in Section 1 the main ingredients of proving that the limit of dynamically defined hitting time point process is a marked Poisson point process by describing two hypotheses (H.1-2) which guarantee convergence in law. We prove in later sections that these hypotheses are satisfied in our setting.

1. Laplace Transforms Technique

We describe the details of the use of Laplace Transforms to study convergence in law of dynamically defined hitting times point processes. We give a general exposition trying to extract the essential elements of the technique which we believe could have independent interest. This follows closely the computations done in [5] and we repeat them in full here for the sake of clarity and completeness and in order to make the present paper self-contained.

Let $(\Omega, \mu, T)$ be an ergodic dynamical system on a standard Borel probability space $(\Omega, \mu)$. Let $\Delta_n$ be a sequence of Borel sets of $\mu$-positive measure, such that $\mu(\Delta_n) \to 0$ (i.e. a sequence of so-called asymptotically rare events). We suppose that a sequence of scales $c_n > 0$ with $c_n \to 0$ is chosen, to be used for rescaling the random times of occurrence of $\Delta_n$. Define the point process $\tau_n$ of hitting times in $\Delta_n$ rescaled by $c_n^{-1}$ as the map $\tau_n : \Omega \to M_\sigma[0, \infty)$ given by

$$\tau_n(\omega) = \sum_{k>0} \chi_{\Delta_n}(T^k \omega) \delta_{kc_n},$$

where $\delta_t$ denotes Dirac measure at the point $t > 0$, and $M_\sigma[0, \infty)$ denotes the Borel $\sigma$-finite measures on $[0, \infty)$. The process of entrances to $\Delta_n$ is given by

$$\tau_n^e(\omega) = \sum_{k>0} \chi_{\Delta_n^c}(T^k \omega) \chi_{\Delta_n^c}(T^{k-1} \omega) \delta_{kc_n},$$

where $\Delta_n^c$ denotes the complement of $\Delta_n$. Let $g : [0, \infty) \to \mathbb{C}$ be a continuous function with compact support. Integrating $g$ by the point process $\tau_n$ gives rise to a random variable

$$X_n(g)(\omega) = \sum_{k>0} \chi_{\Delta_n}(T^k \omega) g(k c_n),$$

which is defined on the probability space $(\Omega, \mu)$. From [20] it is known that convergence in law of $X_n(g)$, for every $g$, is equivalent to convergence in law of
the point process \( \tau_n \). Throughout this paper we denote the expectation with respect to \( \mu \) by

\[
\mathbb{E}(X_n(g)) = \int X_n(g)(\omega) \, d\mu(\omega).
\]

In this Section our aim is to show that under suitable hypothesis on \( \Delta_n \) and \( c_n \), the sequence of point processes \( \tau_n \) converges in law to a marked Poisson point process of constant parameter measure \( \lambda \pi \). Recall that such a process is defined by a map \( \tau : (\Omega, \mathbb{P}) \to \mathcal{M}_{\sigma}[0, \infty) \) given by

\[
\tau(\omega) = \sum_{k>0} L_k(\omega) \delta_{X_k(\omega)},
\]

where for all \( k, k' \), \( D_k = X_k - X_{k-1} \) and \( L_{k'} \) are independent random variables, \( D_k \) are exponentially distributed with parameter \( \lambda > 0 \) (here \( X_0 \equiv 0 \)) and \( L_{k'} \) is a positive integer valued random variable with distribution \( \pi = \{ \pi_j \}_{j>0} \), i.e. \( \mathbb{P}(L_{k'} = j) = \pi_j \). (In the special case of \( \pi_1 = 1 \), \( \tau \) is the Poisson point process of parameter \( \lambda > 0 \).) Integrating \( g \) with respect to \( \tau \) we obtain a random variable \( X(g) \) whose Laplace transform is given by

\[
(3) \quad \mathbb{E}(e^{zX(g)}) = \exp \left\{ \lambda \sum_{j=1}^{\infty} \pi_j \int_0^{\infty} (e^{zj g(y)} - 1) \, dy \right\}.
\]

Regarding the random variable \( X_n(g) \) of (2), its Laplace transform admits the moments expansion

\[
\psi_n(z) = \mathbb{E}(e^{z X_n(g)}) = \sum_{k \geq 0} \frac{\mathbb{E}(X_n(g)^k)}{k!} z^k,
\]

for all \( z \in \mathbb{C} \). By Proposition 8.49 of Breiman [3] we know that if

\[
\nu_k = \lim_{n \to \infty} \mathbb{E}(X_n(g)^k)
\]

exists and satisfies

\[
\limsup_{k \to \infty} \frac{|\nu_k|^{1/k}}{k} < \infty,
\]

then there exists a random variable \( X(g) \) with Laplace transform given by \( \psi(z) = \sum_{k \geq 0} \nu_k z^k/k! \) such that \( X_n(g) \) converges in law to \( X(g) \). Hence our strategy is to show that the Laplace transform of \( X(g) \) coincides with (3) in some disc around the origin of \( \mathbb{C} \) and identify the constants \( \lambda \) and \( \pi_j \).

Let \( \ell(n) \) be an arbitrary sequence of positive real numbers such that \( \ell(n) \to \infty \) as \( n \) diverges. Consider the next conditions on \( \Delta_n \) and \( c_n \). **H.1 (Mean Intermediate Intersections Property)** The following limit exists

\[
C_m = \lim_{n \to \infty} c_n^{-1} \sum_{0=q_0<q_1<\ldots<q_m-1 \quad q_s - q_{s-1} \leq \ell(n)/m} \mathbb{E}
\left( \prod_{s=0}^{m-1} X_{\Delta_n \circ T^{q_s}} \right),
\]
for every fixed \( m > 0 \), and \( C_1 > 0 \). Moreover, there exist \( K, \theta > 0 \) such that \( C_m \leq K \theta^m \).

**H.2** (Relativised Decay of Correlations) There exist \( K_m > 0 \) and \( 0 < \gamma < 1 \) such that for every \( 0 = j_0 < j_1 < \cdots < j_m \) satisfying \( j_s - j_{s-1} \leq \ell(n)/m \), we have for sufficiently large \( n \),

\[
\left| \mathbb{E}\left( \prod_{s=0}^{m} \chi_{\Delta_n \circ T^{j_s} \cdot \Delta_n \circ T^{r+j_m}} \right) - \mathbb{E}\left( \prod_{s=0}^{m} \chi_{\Delta_n \circ T^{j_s}} \right) \mu(B) \right| \leq K_m \gamma^{r+j_m} \mu(B),
\]

for every \( r > 0 \), and for every Borel set \( B \subseteq \Omega \).

Note that if (H.1) is satisfied then \( C_1 = \lim_{n \to \infty} c_n^{-1} \mu(\Delta_n) \) is assumed to exist. Note also that the limit in (H.1) does not depend on the choice of \( \ell(\cdot) \). Hypothesis (H.2) is a type of \( \varphi \)-mixing property, combined with the fact that \( \Delta_n \) has small measure, this explains the introduction of the power \( j_m \) on the right-hand side.

These properties give the following general result.

**Theorem 1.** If \( (\Delta_n, c_n) \) satisfy (H.1-2) then, for any continuous non-negative function \( g \) with compact support on \([0, \infty)\), the limit

\[
\nu_k = \lim_{n \to \infty} \mathbb{E}(X_n(g)^k)
\]

exists for every \( k > 0 \), and it is the \( k \)-th derivative at the origin of the complex function

\[
F_g(z) = \exp \left\{ \sum_{m=1}^{\infty} C_m \int_{0}^{\infty} (e^{zg(t)} - 1)^m dt \right\},
\]

which is well-defined and analytic on a disc around the origin.

For the proof of the above result see Appendix A. In order to identify the limit law of \( \tau_n \) as a marked Poisson point process, one needs to formally solve the equation

\[
\sum_{m=1}^{\infty} C_m \int_{0}^{\infty} (e^{zg(t)} - 1)^m dt = \lambda \sum_{j=1}^{\infty} \pi_j \int_{0}^{\infty} (e^{zjg(y)} - 1) dy
\]

in the constants \( \lambda \) and \( \pi_j \) (where \( \sum \pi_j = 1 \)), and ensure that the series on the right-hand side is absolutely convergent for \( z \) in a neighbourhood of the origin.

In Appendix B we do this computation explicitly for the special case of \( C_m = c \theta^{m-1} \), for some \( c > 0 \) (this is the only case needed in this paper and it is referred to in Section 3), and we obtain in this case

\[
\lambda = \frac{c}{1 + \theta} \quad \text{and} \quad \pi_j = \frac{\theta^{j-1}}{(1 + \theta)^2}.
\]
As already pointed out in [5] and [4], using some facts about convergence of point processes (cf. [9]), Theorem 1 together with the existence of the constants \( \lambda \) and \( \pi_j \) solving (4) prove the next two results.

**Theorem 2.** Suppose \((\Delta_n, c_n)\) satisfy (H.1-2). Then, there exists \( \lambda > 0 \) and a probability measure on the positive integers \( \pi \) such that the process of hitting times \( \tau_n \) in \( \Delta_n \) rescaled by \( c_n^{-1} \) converges in law to a marked Poisson point process with constant parameter measure \( \lambda \pi \), where \( (\lambda, \pi) \) is the solution of (4).

**Theorem 3.** Under the hypotheses of Theorem 2, the process of successive entrances \( \tau_{en} \) to \( \Delta_n \) rescaled by \( c_n^{-1} \) converges in law to a Poisson point process with parameter \( \lambda \).

### 2. Symbolic Dynamics and Pianigiani-Yorke measure

Let \( V = \{1, \ldots, \ell\} \) be a finite set of symbols which we will refer to as the base alphabet. Throughout \( A \) denotes an irreducible and aperiodic \( 0-1 \) \( \ell \times \ell \) matrix which defines the allowable transitions in a directed graph \( G \) of labelled vertices \( V \). Define the space of one-sided allowable paths in the graph \( G \) by

\[
\Sigma_A^+ = \{ x = (x_n) \in V^\mathbb{N} : A(x_{i-1}, x_i) = 1, \forall i \geq 1 \}.
\]

The space \( \Sigma_A^+ \) is compact and metrisable when endowed with the Tychonov product topology (generated by the discrete topology on \( V \)). The shift \( T \) (of finite type) is the map \( T: \Sigma_A^+ \rightarrow \Sigma_A^+ \) defined by \( T(x)_n = x_{n+1} \) for all \( n \geq 0 \). This map is easily seen to be continuous and surjective. The cylinders, denoted by

\[
C[i_0, \ldots, i_m]_k = \{ x \in \Sigma_A^+ : x_{j+k} = i_j, \forall j = 0, \ldots, m \},
\]
form a base of open (and closed) sets in \( \Sigma_A^+ \). Let \( C(\Sigma_A^+) \) denote the space of complex valued continuous functions on \( \Sigma_A^+ \). For \( \psi \in C(\Sigma_A^+) \), consider

\[
\text{var}_n(\psi) = \sup \{ |\psi(x) - \psi(y)| : x_i = y_i, i \leq n \}.
\]

Given \( 0 < \theta < 1 \), define

\[
|\psi|_\theta = \sup \{ \text{var}_n(\psi)/\theta^n \}.
\]

The space \( \mathcal{F}^{+}_\theta = \{ \psi \in C(\Sigma_A^+) : |\psi|_\theta < \infty \} \) is a Banach space when endowed with the norm \( ||\psi||_\theta = ||\psi||_\infty + |\psi|_\theta \), where \( || \cdot ||_\infty \) denotes the supremum norm. The union \( \mathcal{F} = \bigcup_\theta \mathcal{F}_\theta \) is referred to as the space of Hölder continuous functions on \( \Sigma_A^+ \).

Given a potential \( \varphi \in \mathcal{F}_\theta \), let \( \mathcal{L}_\varphi \) be the transfer operator on \( \mathcal{F}_\theta \). It is defined as

\[
(\mathcal{L}_\varphi \psi)(x) = \sum_{T y = x} e^{\varphi(y)} \psi(y).
\]

The operator \( \mathcal{L}_\varphi \) has a maximum positive eigenvalue \( e^{P(\varphi)} \), which is simple and isolated. Moreover, the rest of the spectrum is contained in a disc of radius
strictly less than $e^{P(\varphi)}$ (cf. [2, 23]). The number $P = P(\varphi)$ is called the pressure of $\varphi$. There is a unique $T$-invariant probability measure $\mu = \mu_\varphi$ such that 

$$P(\varphi) = h(\mu) + \int \varphi \, d\mu,$$

where $h(\mu)$ denotes the measure-theoretic entropy of $(T, \mu)$. The pressure $P(\varphi)$ can also be characterised as the maximum of $h(\mu) + \int \varphi \, d\mu$ over all $T$-invariant probabilities $\mu$. The measure $\mu$ is called the equilibrium state of $\varphi$. An eigenfunction $w$ of $L_\varphi$ corresponding to $e^{P(\varphi)}$ may be taken to be strictly positive, in fact one may take $w$ to be the function 

$$w = \lim_{n \to \infty} e^{-nP(\varphi)} L^n_\varphi(1),$$

where $1$ denotes the constant function equal to 1. Replacing $\varphi$ by $\varphi' = \varphi - P(\varphi) + \log(w) - \log(w \circ T)$, we see that $L_\varphi 1 = 1$ and $P(\varphi') = 0$. In this case we say that $\varphi'$ is normalised (cf. [21]). It is easy to see that $\varphi$ and $\varphi'$ have the same equilibrium state $\mu$. In what follows we will assume that $\varphi$ is normalised. Note that in this case the transfer operator $L_\varphi$ satisfies 

$$\int \psi \, d\mu = \int L_\varphi(\psi) \, d\mu,$$

for all $\psi, \psi_1, \psi_2 \in C(\Sigma^+_A)$.

A general subsystem of finite type in $\Sigma^+_A$ is obtained by prescribing a finite number of finite length allowable paths (of possibly different lengths) in $\mathcal{G}$ and defining the subsystem by the infinite allowable paths built out of finite pieces which respect these choices of prescribed paths. However, using a higher-block representation of $\Sigma^+_A$ and choosing a sub-alphabet in the higher-block alphabet, by forcing the transitions to respect the chosen sub-alphabet we obtain the so-called “0-step” subsystem of finite type as above. Hence there is no loss of generality to consider $\Sigma_\Delta$ as in (7).

In this paper we will consider only the case when $\Sigma_\Delta$ is an irreducible and aperiodic subshift of finite type in its alphabet $\Delta$. This means that the restriction of the matrix $A$ to the symbols of $\Delta$ defines a matrix $A_\Delta$ which is irreducible and aperiodic. In particular, the restriction of the shift transformation $T$ to $\Sigma_\Delta$ is topologically mixing in the induced topology from $\Sigma^+_A$.

Let $\varphi_\Delta$ denote the restriction of $\varphi$ to the subsystem $\Sigma_\Delta$. Let $P_\Delta$ be the pressure of $\varphi_\Delta$ with respect to the subsystem $(\Sigma_\Delta, T)$. (Note that since $\varphi$ is assumed to be normalised we have $P_\varphi = 0$, therefore $P_\Delta < 0$.) Let $\mu_\Delta$ denote the equilibrium state
state of $\varphi_{\Delta}$ with respect to the subsystem $(\Sigma_{\Delta}, T)$. Let $w_{\Delta}$ be the strictly positive Hölder continuous function defined on $\Sigma_{\Delta}$ by
\begin{equation}
 w_{\Delta} = \lim_{n \to \infty} e^{-nP_{\Delta}} \mathcal{L}_n(\varphi_{\Delta}).
\end{equation}

Now define the restricted transfer operator $\mathcal{L}_{\Delta}$ acting on the space of Hölder continuous functions $\mathcal{F}_\theta^+$ by
\begin{equation}
 \mathcal{L}_{\Delta}\psi = \mathcal{L}_{\varphi}(\psi \cdot \chi_{\Delta}),
\end{equation}
and consider the subset of $\Sigma_{+}^A$ given by
\begin{equation}
 Z_{\Delta} = \{ x \in \Sigma_{+}^+ : \exists b \in \Delta, A(b, x_0) = 1 \}.
\end{equation}

Note that since $A$ is irreducible and aperiodic in the full alphabet $\mathbf{V}$, $Z_{\Delta}$ is a non-empty finite union of cylinder sets of $\Sigma_{+}^A$. In particular, since $\mu$ is fully supported on $\Sigma_{+}^A$ we have $\mu(Z_{\Delta}) > 0$.

An improvement to the main result of [8] gives the following result. (See Appendix C for a review of the main differences in the proof.)

**Proposition 4.** There exists a unique Hölder continuous function $h_{\Delta}$ defined on the whole space $\Sigma_{+}^A$ such that
\begin{equation}
 \mathcal{L}_{\Delta}(h_{\Delta}) = e^{P_{\Delta}} h_{\Delta},
\end{equation}
and $h_{\Delta}|_{\Sigma_{\Delta}} \equiv w_{\Delta}$, where $w_{\Delta}$ is given by (8). The function $h_{\Delta}$ is strictly positive on $Z_{\Delta}$ and it is zero on the complement $Z_{\Delta}^c$. Moreover,
\begin{equation}
 \left\| e^{-nP_{\Delta}} \mathcal{L}_n(\psi) - h_{\Delta} \int_{\Sigma_{\Delta}} \psi \, d\mu_{\Delta} \right\|_{\infty} \to 0,
\end{equation}
for all $\psi \in C(\Sigma_{+}^A)$.

The Borel measure $\mu_{PY}$ defined by
\begin{equation}
 \mu_{PY}(B) = \int_B h_{\Delta} \, d\mu,
\end{equation}
for every Borel set $B \subseteq \Sigma_{+}^A$, is called the Pianigiani-Yorke measure of the subsystem $(\Sigma_{\Delta}, T)$. This measure is fully supported on $Z_{\Delta}$. In Appendix C we show how the main result of [8] implies the remaining statements of this section.

The measure $\mu_{PY}$ is a quasi-stationary measure satisfying
\begin{equation}
 \mu_{PY}(B) = e^{-P_{\Delta}} \mu_{PY}(T^{-1}B \cap \Delta),
\end{equation}
where we have identified the set of vertices $\Delta$ with the subset of $\Sigma_{+}^+$ given by $\{ x \in \Sigma_{+}^+ : x_0 \in \Delta \}$. For each $n > 0$ consider the set
\begin{equation}
 \Delta_n = \{ x \in \Sigma_{+}^+ : x_i \in \Delta, \text{ for } i = 0, \ldots, n-1 \}.
\end{equation}

We note that $\Delta_n \subseteq \Delta_{n-1}$ for each $n > 0$ and $\cap_{n>0} \Delta_n = \Sigma_{\Delta}$. The Pianigiani-Yorke measure also satisfies
\begin{equation}
 \frac{\mu_{PY}(B)}{\mu_{PY}(\Sigma_{+}^A)} = \lim_{k \to \infty} \mu(T^{-k}B | \Delta_k),
\end{equation}
for all Borel set $B \subseteq \Sigma_{+}^A$. 

\begin{proof}

\end{proof}
for all Borel sets $B$ of $\Sigma^+_A$, and then it carries a statistical information of how subsystems of finite type are embedded into larger systems.

### 3. Limit laws for symbolic systems

Let $\Delta_n$ be defined by (11) and consider the hitting time point process $\tau_n$ of $\Delta_n$ rescaled by $c_n^{-1}$ of (1), where $c_n$ is to be suitably chosen below. The aim is to show that Hypotheses (H.1-2) are satisfied and then to conclude convergence in law of $\tau_n$ to a marked Poisson point process, identifying the parameters $\lambda$ and $\pi_j$.

Usually $c_n = \mu(\Delta_n)$ is the natural choice to rescale dynamically defined hitting time processes (for instance by conditioning the process to starting at $\Delta_n$, one studies asymptotic return times and this scale is very natural in view of Kač’s Lemma, see more comments in [4]). Hence we begin with an asymptotic estimate on $\mu(\Delta_n)$ in terms of the relative pressure $P_\Delta$. Since $P_\Delta$ is an intrinsic constant associated to the triple $(\Sigma^+_A, \Sigma, \varphi)$, we will then take $c_n = e^{nP_\Delta}$ as our scale choice.

**Lemma 5.** Let $h_\Delta$ be the function in Proposition 4. For every $s \geq 0$ we have

$$\lim_{n \to \infty} e^{-nP_\Delta} \mu(\Delta_{n+s}) = e^{sP_\Delta} \int h_\Delta \, d\mu = e^{sP_\Delta} \mu_{PY}(\Sigma^+_A) = \mu_{PY}(\Delta_s),$$

where we have defined $\Delta_0 = \Sigma^+_A$.

**Proof.** Note that

$$\chi_{\Delta_{n+s}} = \prod_{j=0}^{n+s-1} \chi_{\Delta \circ T_j},$$

for every $n > 0$. Therefore

$$\mu(\Delta_{n+s}) = \int \chi_{\Delta_{n+s}} \, d\mu = \int \chi_\Delta \cdot \chi_{\Delta_{n+s-1} \circ T} \, d\mu.$$

Since $L_\varphi$ satisfies property (6) we obtain

$$\mu(\Delta_{n+s}) = \int L_\Delta^1 \cdot \chi_{\Delta_s} \, d\mu,$$

and by induction we have

$$\mu(\Delta_{n+s}) = \int (L_\Delta^n 1) \cdot \chi_{\Delta_s} \, d\mu = \int L_\Delta^{n+s} 1 \, d\mu.$$

Hence Proposition 4 gives

$$\mu(\Delta_{n+s}) = e^{nP_\Delta} \int h_\Delta \, d\mu + o(e^{nP_\Delta}) = e^{(n+s)P_\Delta} \int h_\Delta \, d\mu + o(e^{nP_\Delta}).$$

The next results are used to prove that Hypotheses (H.1-2) are satisfied.
Lemma 6. There exist $K > 0$ and $0 < \gamma < 1$ such that
\[
\left| \mathbb{E}(\chi_{\Delta_n} \cdot \chi_{B^c}T^{s+r}) - \mu(\Delta_n)\mu(B) \right| \leq K \gamma^r e^{sP\Delta} \mu(B),
\]
for every $s, r > 0$ and for every Borel set $B \subseteq \Sigma^+_A$.

Proof. We note that
\[
\mathbb{E}(\chi_{\Delta_n} \cdot \chi_{B^c}T^{s+r}) = \mathbb{E}(\chi_B \cdot \mathcal{L}_{\varphi^r}(\mathcal{L}^*_\Delta 1)).
\]
From the spectral properties of $\mathcal{L}_{\varphi}$, we know that there exists $0 < \gamma < 1$ and $K > 0$ such that for every $k > 0$,
\[
\|\mathcal{L}_{\varphi}^k w\|_{\infty} \leq K \gamma^k \|w\|_\theta,
\]
whenever $w \in \mathcal{F}_\theta^+$ with $\int w \, d\mu = 0$ (cf. [21]). Since $e^{-sP\Delta}\mathcal{L}^*_\Delta 1$ has uniformly bounded Hölder norm (see [8] and Appendix C), taking $w = w_s = e^{-sP\Delta}(\mathcal{L}^*_\Delta 1 - \mu(\Delta_n))$, there exists $K' > 0$ independent of $r$ and $s$ such that
\[
\left| \mathbb{E}(\chi_B \cdot \mathcal{L}_{\varphi}^r w_s) \right| \leq K' \gamma^r \mu(B),
\]
for every $r, s > 0$. Therefore the lemma follows. \qed

Lemma 7. For all integer $m \geq 1$, the limit
\[
\bar{C}_m = \lim_{n \to \infty} \mu(\Delta_n)^{-1} \sum_{0 = q_0 < q_1 < \cdots < q_{m-1} \leq n/m} \mathbb{E}\left( \prod_{s=0}^{m-1} \chi_{\Delta_n sT^{q_s}} \right).
\]
Moreover,
\[
\bar{C}_m = (e^{-P\Delta} - 1)^{-(m-1)},
\]
for all $m \geq 1$.

Proof. For fixed $n > 0$, since $q_s - q_{s-1} \leq n/m \leq n$ for each $s$, we conclude that
\[
\prod_{s=0}^{m-1} \chi_{\Delta_n sT^{q_s}} = \prod_{j=0}^{q_{m-1}+n-1} \chi_{\Delta_n jT^j} = \chi_{\Delta_n q_{m-1}}.
\]
Write $\beta = e^{P\Delta}$. Using Lemma 5 we see that $\mu(\Delta_n q_{m-1})/\mu(\Delta_n)$ is uniformly bounded in $n$ by $C\gamma^{q_{m-1}}$, for some $C > 0$ and $0 < \eta < 1$, and it converges to $\beta^{q_{m-1}}$ as $n \to \infty$. Hence we obtain
\[
\bar{C}_m = \lim_{n \to \infty} \sum_{0 = q_0 < q_1 < \cdots < q_{m-1} \leq n/m} \frac{\mu(\Delta_n q_{m-1})}{\mu(\Delta_n)} = \sum_{0 = q_0 < q_1 < \cdots < q_{m-1}} \beta^{q_{m-1}},
\]
and the latter summation gives $\bar{C}_1 = 1$, and
\[
\bar{C}_m = \sum_{q=m-1}^{\infty} \frac{(q - 1) \cdots (q - m + 2)}{(m - 2)!} \beta^q = \left( \frac{\beta}{1 - \beta} \right)^{m-1} \text{ for } m \geq 2. \qed
Lemma 5 and Lemma 7 show that the pair \((\Delta_n, c_n)\) with \(c_n = e^{nP_{\Delta}}\) satisfy (H.1) with \(\ell(n) = n\). Note also from Lemma 5 that \(e^{-1} \mu(\Delta_n) \to c = \int h_\lambda d\mu\) and hence

\[
C_m = c \tilde{C}_m = c \theta^{m-1},
\]

where \(\theta = (e^{-P_{\Delta}} - 1)^{-1}\). Furthermore, Lemma 6 shows that (H.2) is also satisfied. Consequently, we may apply Theorems 2 and 3 together with the parameters given in (5) to obtain

**Theorem 8.** If \(\Sigma_\Delta\) is an irreducible and aperiodic subshift of finite type of \((\Sigma^+_A, T)\) then the process of hitting times \(\tau_n\) in \(\Delta_n\) scaled by \(e^{-nP_{\Delta}}\) converges in law to a marked Poisson point process with constant parameter measure \(\lambda \pi\), where the parameters are given by

\[
\lambda = (1 - e^{P_{\Delta}}) \int h_\Delta \, d\mu \quad \text{and} \quad \pi_j = (1 - e^{P_{\Delta}}) e^{(j-1)P_{\Delta}},
\]

for \(j \geq 1\), where \(P_{\Delta}\) is the pressure of the restriction of \(\varphi\) to the subsystem \(\Sigma_\Delta\) and \(h_\Delta\) is the density of the Pianigiani-Yorke measure associated to the triple \((\Sigma^+_A, \Sigma_\Delta, \varphi)\).

**Theorem 9.** Under the hypotheses of Theorem 8, the process of successive entrances \(\tau^n\) to \(\Delta_n\) scaled by \(e^{-nP_{\Delta}}\) converges in law to a Poisson point process with parameter \(\lambda\) as above.

**An application.** The above results give an interesting application as follows. Suppose we are given \(N\) points at random \(\omega^{(i)} \in \Sigma^+_A\), independently of one another, and \(\omega^{(i)}\) distributed according to the equilibrium state \(\mu_i\). Record the times \(k > 0\) of \(n\) matchings in a row of all the sequences \(\omega^{(i)}\), i.e. consider the times \(k > 0\) such that \(\omega^{(i)}_k = \omega^{(i')}_{k+s}\) for \(s = 0, \ldots, n-1\), and all \(i, i' \in \{1, \ldots, N\}\). Then rescaling the corresponding point process by the probability of the event hitting \(n\) matchings in a row over all the sequences gives, in the limit, a marked Poisson point process of constant parameter measure. To identify the parameters one only needs to consider the product shift of finite type \(\Sigma^+_A \times \cdots \times \Sigma^+_A\) and notice that \(\mu_1 \times \cdots \times \mu_N\) is the equilibrium state of the potential

\[
\varphi(x^{(1)}, \ldots, x^{(N)}) = \varphi_1(x^{(1)}) + \cdots + \varphi_N(x^{(N)}),
\]

where \(\varphi_i\) is the potential defining \(\mu_i\). Consider the subshift \(\Sigma_\Delta\) as the diagonal subshift obtained by setting \(x^{(i)} = x^{(i')}\) for all \(i, i'\). Let \(\Delta_n\) be the subset of \(\Sigma^+_A \times \cdots \times \Sigma^+_A\) consisting of the points \((x^{(1)}, \ldots, x^{(N)})\) such that \(x^{(i)} = x^{(i')}\) for \(s = 0, \ldots, n-1\), and all \(i, i' \in \{1, \ldots, N\}\). We apply the above results to this situation and conclude that the parameters \(\lambda\) and \(\pi\) of the limiting point process is given by (8), where \(P_{\Delta}\) is replaced by \(P_s = P_{\varphi_s} - P_{\varphi}\), \(P_{\varphi}\) denotes the pressure of \(\varphi\) on \(\Sigma^+_A \times \cdots \times \Sigma^+_A\) and \(P_{\varphi_s}\) is the pressure of the restriction \(\varphi_s\) of the potential.
\(\varphi\) to the subshift \(\Sigma_\Delta\). In the special case of two orbits we also have an analogue of our main result in [5] as follows. Define a distance on \(\Sigma_A^+\) by
\[
d(x, y) = \sum_{k \geq 0} |x_k - y_k| \rho^k,
\]
for \(\rho = e^{P^*}\). We see that given \(\varepsilon > 0\) the point process \(\tau_\varepsilon\) obtained by summing Dirac point masses at the times \(k \geq 1\) such that \(d(T^k x, T^k y) \leq \varepsilon\) converges in law, when rescaled by \(\varepsilon^{-1} \sim e^{-nP^*}\), to a marked Poisson point process of constant parameter measure \(\lambda\pi\) as above.

4. The Periodic Case

Let us consider the case when \(\Sigma_\Delta\) is irreducible but periodic with period \(m > 1\). In this case there exists a decomposition of \(\Delta = \Delta_0 \cup \cdots \cup \Delta_{m-1}\) with the property that if \(i \in \Delta_s\), \(j \in \Delta_{s'}\) are given such that \(A(i, j) = 1\) then necessarily \(s' = s + 1 (\mod m)\). This induces a disjoint partition of \(\Sigma_\Delta = \Omega_0 \cup \cdots \cup \Omega_{m-1}\) such that \(T(\Omega_s) = \Omega_{s+1 (\mod m)}\), which is the so-called cyclically moving partition of \(\Sigma_\Delta\). From the classical Ruelle-Perron-Frobenius theory (cf. [23], [21]), we know that there exist non-negative Hölder continuous functions \(w_0, \ldots, w_{m-1}\) defined on \(\Sigma_\Delta\), and mutually singular probability measures \(\nu_0, \ldots, \nu_{m-1}\) with \(\nu_j\) supported on \(\Omega_j\) satisfying
\[
L_{\varphi_\Delta}(w_j) = e^{P^*} w_{j+1 (\mod m)} ,
\]
and \(\text{supp}(w_j) = \Omega_j\) for \(j = 0, \ldots, m - 1\). Moreover, \(w_\Delta = \sum_{j=0}^{m-1} w_j\) is strictly positive on \(\Sigma_\Delta\) and
\[
\left\| e^{-nP^*} L_{\varphi_\Delta}^n(\psi) - \sum_{j=0}^{m-1} w_{j+n (\mod m)} \int_{\Omega_j} \psi \, d\nu_j \right\|_{\Sigma_\Delta} \longrightarrow 0 ,
\]
for all \(\psi \in C(\Sigma_\Delta)\), where \(\| \cdot \|_{\Sigma_\Delta}\) denotes supremum norm on \(C(\Sigma_\Delta)\). Putting \(\psi = 1\) and replacing \(n\) by \(nm\) in the above expression, we note that \(w_\Delta\) could have been defined uniquely by
\[
w_\Delta = \lim_{n \to \infty} e^{-nmP^*} L_{\varphi_\Delta}^{nm}(1) ,
\]
and then \(L_{\varphi_\Delta}(w_\Delta) = e^{P^*} w_\Delta\). Now we transfer these results from \(\Sigma_\Delta\) to the whole space \(\Sigma_A^+\). Let \(\mathcal{Z}_\Delta\) be defined by (9). Since again \(\mathcal{Z}_\Delta\) is a non-empty finite union of cylinders of \(\Sigma_A^+\), we have \(\mu(\mathcal{Z}_\Delta) > 0\).

Define the constants \(d_j\) by
\[
d_j = \int_{\Omega_{j+1 (\mod m)}} L_{\Delta}(1) \, d\nu_{j+1 (\mod m)} ,
\]
\[\text{In fact, } \nu_j \text{ is the equilibrium state of the potential } S_m(\varphi) = \sum_{i=0}^{m-1} \varphi \circ T^i \text{ restricted to } (\Omega_j, T^m) \text{ and } \nu_{j+1 (\mod m)} = \nu_j \circ T^{-1}.\]
for \( j = 0, \ldots, m - 1 \), we see that \( d_j > 0 \) for all \( j \). Define also the constants \( \alpha_j(k) \) by \( \alpha_j(0) = 1 \) and for \( 1 \leq k \leq m - 1 \),

\[
\alpha_j(k) = e^{-kP_\Delta} \prod_{s=0}^{k-1} d_{j+s \text{ (mod } m)} ,
\]

for \( j = 0, \ldots, m - 1 \). The next is our main result.

**Theorem 10.** There exist a unique choice of non-negative Hölder continuous functions \( h_0, \ldots, h_{m-1} \) defined on the whole space \( \Sigma_+^* \) satisfying

\[
\mathcal{L}_\Delta(h_j) = d_j h_{j+1 \text{ (mod } m)} ,
\]

and \( h_j |_{\Omega_j} \equiv w_j \) for \( j = 0, \ldots, m - 1 \). The function \( h_\Delta = \sum_{j=0}^{m-1} h_j \) is strictly positive on \( Z_\Delta \) and it is zero on the complement \( Z_\Delta^c \). Moreover,

\[
(13) \quad \left\| e^{-nP_\Delta} \mathcal{L}_\Delta(\psi) - \sum_{j=0}^{m-1} \alpha_j(n \text{ (mod } m)) h_{j+n \text{ (mod } m)} \int_{\Omega_j} \psi \, d\nu_j \right\|_{\Sigma_+^*} \to 0 ,
\]

for all \( \psi \in C(\Sigma_+^*) \).

In particular taking \( \psi = 1 \) and integrating the above expression with respect to \( \mu \) we obtain

**Corollary 11.** Let \( \Sigma_\Delta \) be an irreducible and periodic subsystem of finite type with period \( m \). The sets \( \Delta_n \) have the following asymptotic behaviour:

\[
\left| e^{-nP_\Delta} \mu(\Delta_n) - \sum_{j=0}^{m-1} \alpha_j(n \text{ (mod } m)) \int h_{j+n \text{ (mod } m)} \, d\mu \right| \to 0 .
\]

In particular, for each \( k = 0, 1, \ldots, m - 1 \) we have

\[
\lim_{n \to \infty} e^{-(k+nm)P_\Delta} \mu(\Delta_{k+nm}) = \sum_{j=0}^{m-1} \alpha_j(k) \int h_{j+k \text{ (mod } m)} \, d\mu .
\]

In Section 6 we give an explicit example where the above numbers differ for different choices of \( k \), which shows that \( e^{-nP_\Delta} \mu(\Delta_n) \) does not converge in general as \( n \to \infty \) when \( \Sigma_\Delta \) is periodic.

### 5. Proof of Theorem 10

We recall some results from [8]. Let \( \mathcal{C}_p^+(\Sigma_+^*) \) be the set of strictly positive \( p \)-cylindrical functions (i.e. a function depending only on the first \( p \) coordinates of the point). Let \( 0 < \theta < 1 \) be the Hölder exponent of the potential \( \varphi \). Let \( Z_\Delta \) be defined as in (9). Let \( C(Z_\Delta) \) denote the set of continuous functions defined on \( Z_\Delta \). The proof of the following Lemma can be obtained from Appendix C.

**Lemma 12.** For any \( f \in \cup_{p \geq 1} \mathcal{C}_p^+(\Sigma_+^*) \), we have
\( h_\Delta = \lim_{n\to\infty} e^{-nP_\Delta} L^n_\Delta f \) does not depend on the function \( f \in \bigcup_{p\geq 1} C^p_+ (\Sigma_A^+ \bigcap A_{\Delta}^+) \) and it satisfies
\[
L_\Delta(h_\Delta) = e^{P_\Delta} h_\Delta.
\]

Although not explicitly mentioned in [8], the function \( h_\Delta \) is a Hölder continuous function with the same Hölder exponent as the potential \( \varphi \). We also note that, since \( \mu \) is fixed by the dual operator of \( L \) we have
\[
\int_{\Delta_n} f \cdot g \circ T^n \, d\mu = \int L^n_\varphi(\chi_{\Delta_n} \cdot f \cdot g \circ T^n) \, d\mu = \int g \cdot L^n_\Delta(f) \, d\mu.
\]

Now consider the case when \( \Sigma_\Delta \) is irreducible but periodic with period \( m > 1 \). Consider the decomposition of \( \Delta = \Delta_0 \cup \cdots \cup \Delta_{m-1} \) with the property that if \( i \in \Delta_s, j \in \Delta_{s'} \) are given such that \( A(i,j) = 1 \) then necessarily \( s' = s + 1 \) (mod \( m \)). Consider also the corresponding cyclically moving partition \( \Sigma_\Delta = \Omega_0 \cup \cdots \cup \Omega_{m-1} \) such that \( T(\Omega_s) = \Omega_{s+1 \text{ (mod } m)} \), i.e. defining
\[
\Omega_j = \{ x \in \Sigma_\Delta : x_0 = \Delta_j \},
\]
for \( j = 0, \ldots, m - 1 \). Let \( V^{(m)} \) be the sub-alphabet of \( V^m \) defined by
\[
V^{(m)} = \{ (i_0, \ldots, i_{m-1}) \in V^m : i_0 \rightarrow \cdots \rightarrow i_{m-1} \text{ in } G \},
\]
where \( G \) is the defining graph of \( \Sigma_A^+ \). Consider the transition matrix \( A^{(m)} \) indexed by \( V^{(m)} \times V^{(m)} \) given by
\[
A^{(m)}((i_0, \ldots, i_{m-1}), (j_0, \ldots, j_{m-1})) = 1 \text{ if } i_{m-1} \rightarrow j_0 \text{ in } G.
\]

Using the identification
\[
( (x_0, \ldots, x_{m-1}), (x_m, \ldots, x_{2m-1}), \ldots ) \leftrightarrow (x_0, x_1, x_2, \ldots ),
\]
the shift transformation \( T_m \) on \( \Sigma_A^{(m)} \) is naturally topologically conjugate to \( T^m \) on \( \Sigma_A^+ \). In what follows we will abuse the notation and freely identify these transformations and spaces.

The normalised potential \( \varphi \) on \( \Sigma_A^+ \) naturally defines a potential \( \varphi^{(m)} \) on \( \Sigma_A^{(m)} \) by \( \varphi^{(m)} = S_m(\varphi) = \sum_{j=0}^{m-1} \varphi \circ T^j \). Note that \( \varphi^{(m)} \) is a normalised potential for \( T_m \), i.e.
\[
L_{\varphi^{(m)}}(1)(x) = \sum_{y \in T_m(x)} e^{\varphi^{(m)}(y)} = 1,
\]
for all \( x \in \Sigma_A^{(m)} \). Now the sub-alphabet \( \Delta \) of \( V \) defines a sub-alphabet \( \Delta^{(m)} \) of \( V^{(m)} \) by
\[
\Delta^{(m)} = \{ (i_0, \ldots, i_{m-1}) \in V^{(m)} : i_s \in \Delta, \text{ for } s = 0, \ldots, m - 1 \},
\]
and this sub-alphabet can be further decomposed into
\[
\Delta_j^{(m)} = \{ (i_0, \ldots, i_{m-1}) \in \Delta^{(m)} : i_s \in \Delta_{s+j \text{ (mod } m)} \}, \text{ for } s = 0, \ldots, m - 1 \}.
for $j = 0, \ldots, m - 1$. The important fact is that for fixed $j$, $\Sigma_j^{(m)}$ the subsystem of $\Sigma^{(m)}$ obtained by taking transitions through $\Delta_j^{(m)}$ is irreducible and aperiodic in its alphabet $\Delta_j^{(m)}$. Hence the main result of [8] applies and we define a Hölder continuous function $h_j$ by

$$h_j = \lim_{n \to \infty} e^{-nP(\Delta_j^{(m)})} L_{\Delta_j^{(m)}}^n(1),$$

where $P(\Delta_j^{(m)})$ is the pressure of the restriction of $\varphi^{(m)}$ to the subsystem $\Sigma_j^{(m)}$ (hence $P(\Delta_j^{(m)}) = m P_\Delta$ for all $j$), and $L_{\Delta_j^{(m)}}(\psi) = L_{\varphi^{(m)}}(\psi \cdot \chi_{\Delta_j^{(m)}})$ with respect to the shift $T_m$. From Lemma C.3 (ii) extended to continuous functions we obtain

$$\lim_{n \to \infty} e^{-nP\Delta} L_{\Delta_j^{(m)}}^n(\psi) = h_j \int_{\Sigma_j^{(m)}} \psi \, d\nu_j,$$

for every $\psi \in C(\Sigma^{(m)})$ (with the limit being uniform), where $\nu_j$ is the unique equilibrium state of $\varphi^{(m)}$ restricted to the subsystem $\Sigma_j^{(m)}$ with respect to the shift $T_m$.

In view of the identification (15) the function $h_j$ defines a function on $\Sigma_j^+$ in a natural way and $\nu_j$ becomes a probability measure on $\Sigma_j^+$ fully supported on $\Omega_j$. Note that $h_j$ is then strictly positive on

$$\mathcal{Z}_{\Delta_j} = \{ x \in \Sigma_j^+ : \exists b \in \Delta_{j-1} \mod m , A(b, x_0) = 1 \},$$

and it is zero on the complement $\mathcal{Z}_{\Delta_j}^c$. Note also that for each $j$, $\mathcal{Z}_{\Delta_j}$ is a non-empty finite union of cylinders of $\Sigma_j^+$, therefore in particular, $\mu(\mathcal{Z}_{\Delta_j}) > 0$. Applying $L_\Delta$ as

$$L_\Delta(\psi)((x_0, \ldots, x_{m-1}), (x_m, \ldots, x_{2m-1}), \ldots) = \sum_{\{i \in \Delta : A(i, x_0) = 1\}} e^{\varphi^{(m)}((i, x_0, \ldots, x_{m-2}), (x_{m-1}, \ldots, x_{2m-2}), \ldots)} \times \psi((i, x_0, \ldots, x_{m-2}), (x_{m-1}, \ldots, x_{2m-2}), \ldots),$$

we conclude that $L_\Delta \circ L_{\Delta_j}^n = L_{\Delta_j+1 \mod m}^n \circ L_\Delta$. This implies that for all $k \geq 1$ we have $L_\Delta^k \circ L_{\Delta_j}^n = L_{\Delta_j+k \mod m}^n \circ L_\Delta^k$. Putting $\psi = 1$ in (18) we obtain, for fixed $k \geq 1$ and fixed $0 \leq j < m$,

$$L_\Delta^k(h_j) = \lim_{n \to \infty} e^{-nP\Delta} L_\Delta^k(L_{\Delta_j^{(m)}}^n(1)) = \lim_{n \to \infty} e^{-nP\Delta} L_{\Delta_j+k \mod m}^n \circ L_\Delta^k(1)$$

$$= h_{j+k \mod m} \int_{\mathcal{Z}_{\Delta_j+k \mod m}} L_\Delta^k(1) \, d\nu_{j+k \mod m}.$$
Therefore defining the constants $d_j$ by

$$d_j = \int_{\Omega_j+1 (\text{mod } m)} L_\Delta(1) \, d\nu_{j+1 (\text{mod } m)} ,$$

for $j = 0, \ldots, m - 1$, we see that $d_j > 0$ for all $j$ and by (19) we have

$$L_\Delta(h_j) = d_j h_{j+1 (\text{mod } m)} ,$$

for all $j$. Since $L^m_\Delta(h_j) = e^{mP_\Delta} h_j$ for each $j$, by (19) we also see that

$$\prod_{j=0}^{m-1} d_j = e^{mP_\Delta} .$$

At the end of this appendix we give an example where in general one has $d_j$ not necessarily equal to $e^{P_\Delta}$.

The function $h_\Delta = \sum_{j=0}^{m-1} h_j$ is strictly positive on $\mathcal{Z}_\Delta$ and it is zero on the complement $\mathcal{Z}_\Delta^c$. (Note that $\mathcal{Z}_\Delta = \cup_{j=0}^{m-1} \mathcal{Z}_{\Delta_j}$ and this union is not in general a disjoint union, see example below.) The function $h_\Delta$ also satisfies

$$L^m_\Delta(h_\Delta) = e^{mP_\Delta} h_\Delta .$$

Now, from the fact that

$$L^{(m)}_\Delta(\psi) = \sum_{j=0}^{m-1} L^{(m)}\Delta_j(\psi) ,$$

and $L^{(m)}_\Delta \circ L^{(m)}_{\Delta_j} = 0$ if $j \neq j'$, we see that

$$L^n_{\Delta^{(m)}}(\psi) = \sum_{j=0}^{m-1} L^n_{\Delta^{(m)}_{\Delta_j}}(\psi) .$$

Using (19) we have, for fixed $1 \leq k < m$,

$$e^{-(nm+k)P_\Delta} L^{nm+k}_\Delta(\psi) = e^{-(nm+k)P_\Delta} L^k_\Delta(L^{(m)}_\Delta(\psi))$$

$$= \sum_{j=0}^{m-1} e^{-kP_\Delta} L^k_\Delta(e^{-nmP_\Delta} L^{n}_{\Delta_j}(\psi))$$

$$= \sum_{j=0}^{m-1} e^{-kP_\Delta} L^k_\Delta(h_j) \int_{\Omega_j} \psi \, d\nu_j + o(1)$$

$$= \sum_{j=0}^{m-1} \left( e^{-kP_\Delta} \prod_{s=0}^{k-1} d_{j+s (\text{mod } m)} \right) h_{j+k (\text{mod } m)} \int_{\Omega_j} \psi \, d\nu_j + o(1) ,$$
where we have used Lemma C.3 (ii) for continuous functions and \( o(1) \) is with respect to \( n \). Define the constants \( \alpha_j(k) \) by \( \alpha_j(0) = 1 \) and for \( 1 \leq k \leq m - 1 \),

\[
\alpha_j(k) = e^{-k P} \prod_{s=0}^{k-1} d_{j+s \pmod m},
\]

for \( j = 0, \ldots, m - 1 \). Therefore from (20) we finally obtain

\[
\left\| e^{-nP} \mathcal{L}^n_{\Delta}(\psi) - \sum_{j=0}^{m-1} \alpha_j(n \pmod m) h_{j+n \pmod m} \int_{\Omega_j} \psi \, d\nu_j \right\| \xrightarrow{n \to \infty} 0,
\]

for all \( \psi \in \mathcal{C}(\Sigma^+_A) \), which concludes the proof of Theorem 10.

For the proof of Corollary 11, take \( \psi \equiv 1 \) in the above expression, integrate with respect to \( \mu \) and use the fact that \( \int \mathcal{L}^n_A(1) \, d\mu = \mu(\Delta_n) \), to conclude that the sets \( \Delta_n \) have the following asymptotic behaviour:

\[
\left| e^{-nP} \mu(\Delta_n) - \sum_{j=0}^{m-1} \alpha_j(n \pmod m) \int h_{j+n \pmod m} \, d\mu \right| \xrightarrow{n \to \infty} 0.
\]

In particular, for each \( k = 0, 1, \ldots, m - 1 \) we have

\[
\lim_{n \to \infty} e^{-(k+nm)} P \mu(\Delta_{k+nm}) = \sum_{j=0}^{m-1} \alpha_j(k) \int h_{j+k \pmod m} \, d\mu.
\]

6. Illustrative Example

In this section we give an example to illustrate the computations made in the previous section.

**Example.** Let \( V = \{1, 2, 3\} \) and consider the matrix \( A \) given by

\[
A = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]

Let \( \varphi \) be any normalised Hölder continuous potential on \( \Sigma^+_A \), i.e. assume that

\[
\mathcal{L}_\varphi(1)(x) = \sum_{\{i \in V: A(i, x_0) = 1\}} e^{\varphi(ix)} = 1,
\]

for all \( x \in \Sigma^+_A \). Take \( \Delta = \{1, 2\} \). Then \( \Sigma_\Delta \) is the set of periodic orbits \( \{(1, 2, 1, 2, \ldots), (2, 1, 2, 1, \ldots)\} \). Put \( \varphi(1, 2, 1, 2, \ldots) = p \) and \( \varphi(2, 1, 2, 1, \ldots) = q \), and assume \( p \neq q \). There is only one invariant measure for the restriction of the shift \( T \) on \( \Sigma_\Delta \), namely

\[
\mu_\Delta = \frac{1}{2} (\delta_1 + \delta_2),
\]
where $\delta_1$ is Dirac measure at the point $(2, 1, 1, 2, \ldots)$ and $\delta_2$ is Dirac measure at the point $(2, 1, 2, 1, \ldots)$. Since the shift entropy of $\mu_\Delta$ is zero, the restricted pressure $P_\Delta$ is then given by

$$P_\Delta = \int \varphi \, d\mu_\Delta = \frac{1}{2}(p + q).$$

Notice now that $\varphi^{(2)} = \varphi + \varphi_0 T$ when restricted to $\Sigma_\Delta$ is constant with value $p + q$. The pressure of the restriction of $\varphi^{(2)}$ to $\Sigma_\Delta$ with respect to $T^2$ is then given by $p + q = 2P_\Delta$. The set $\Delta$ is further decomposed into $\Delta_{i-1} = \{i\},$ for $i = 1, 2,$ giving the cyclically moving partition $\Sigma_\Delta = \Omega_0 \cup \Omega_1,$ where $\Omega_0 = \{(1, 2, 1, 2, \ldots)\}$ and $\Omega_1 = \{(2, 1, 2, 1, \ldots)\}$. Note then that $T^2$ restricted to $\Sigma_\Delta$ consists in two fixed points. This implies that $\nu_0 = \delta_1$ and $\nu_1 = \delta_2,$ where $\nu_i$ is the equilibrium state of $\varphi^{(2)}$ restricted to $\Omega_i$ with respect to $T^2$. Applying [8] in the case of an aperiodic subsystem consisting of a fixed point for $T^2$ we have from (17), where $\Delta_j^{(m)}$ is defined by (16) and $m = 2$,

$$h_j = \lim_{n\to\infty} e^{-2nF_\Delta} \mathcal{L}_{\Delta_j^{(2)}}(1),$$

for $j = 0, 1.$ Interpreting this we conclude that for $x \in \mathcal{Z}_{\Delta_0} = C[1]_0 \cup C[3]_0$ we have

$$h_0(x) = \lim_{n\to\infty} \exp\left\{S_{2n}(\varphi)\left(1, 2, 1, 2, \ldots, 1, 2, x_0, x_1, \ldots\right) - n(p + q)\right\}$$

$$= \lim_{n\to\infty} \exp\left\{S_{2n}(\varphi)\left(1, 2, 1, 2, \ldots, 2, 1, x_0, x_1, \ldots\right) - S_{2n}(\varphi)\left(1, 2, 1, 2, \ldots\right)\right\},$$

where $S_k(\varphi)$ denotes $\varphi + \varphi_0 T + \cdots + \varphi_0 T^{k-1},$ and $h_0$ is zero on the complement $\mathcal{Z}_{\Delta_0}^c = C[2]_0.$ Also if $x \in \mathcal{Z}_{\Delta_1} = C[2]_0 \cup C[3]_0$ then

$$h_1(x) = \lim_{n\to\infty} \exp\left\{S_{2n}(\varphi)\left(2, 1, 2, 1, \ldots, 2, 1, x_0, x_1, \ldots\right) - n(p + q)\right\}$$

$$= \lim_{n\to\infty} \exp\left\{S_{2n}(\varphi)\left(2, 1, 2, 1, \ldots, 2, 1, x_0, x_1, \ldots\right) - S_{2n}(\varphi)\left(2, 1, 2, 1, \ldots\right)\right\},$$

and $h_1$ is zero on the complement $\mathcal{Z}_{\Delta_1}^c = C[1]_0.$ (Note that $h_0$ and $h_1$ are both strictly positive on the cylinder $C[3]_0.$) Now we compute the constants $d_j,$ for $j = 0, 1.$ We have

$$d_0 = \int_{\Omega_0} \mathcal{L}_\Delta(1) \, d\nu_1 = e^{e^{(1,2,1,\ldots)}} = e^p,$$

and

$$d_1 = \int_{\Omega_0} \mathcal{L}_\Delta(1) \, d\nu_0 = e^{e^{(2,1,2,\ldots)}} = e^q.$$

This provides an example where $d_j \neq e^{p \Delta} = e^{\frac{1}{2}(p + q)},$ since we are assuming $p \neq q.$ One can see directly that $h_0$ and $h_1$ satisfy

$$\mathcal{L}_\Delta(h_0) = d_0 h_1 = e^p h_1$$

and

$$\mathcal{L}_\Delta(h_1) = d_1 h_0 = e^q h_0.$$
The function $h_\Delta = h_0 + h_1$ satisfies $L_\Delta^2(h_\Delta) = e^{2P_\Delta}h_\Delta$, and in the case of this example it is fully supported on $\Sigma_\lambda^+$. Now we compute the constants $\alpha_j(k)$ for $j, k = 0, 1$. We have $\alpha_j(0) = 1$ for $j = 0, 1$,

$$
\alpha_0(1) = e^{-P_\Delta}d_0 = e^{-\frac{1}{2}(p+q)}e^p = e^{\frac{1}{2}(p-q)}, \quad \text{and}
$$

$$
\alpha_1(1) = e^{-P_\Delta}d_1 = e^{-\frac{1}{2}(p+q)}e^q = e^{\frac{1}{2}(q-p)}.
$$

From (13) we conclude that

$$
\left\| e^{-nP_\Delta}\mathcal{L}_\Delta^n(\psi) - \left(\alpha_0(n \mod m) h_n(\mod m) \psi(1, 2, 1, 2, \ldots) + \alpha_1(n \mod m) h_{n+1}(\mod m) \psi(2, 1, 2, 1, \ldots)\right)\right\|_{\Sigma_\lambda^+} \to 0,
$$

for all $\psi \in C(\Sigma_\lambda^+)$. An interesting fact is that, putting $f = g = 1$ in (14) and putting $\psi = 1$ in the above expression we have

$$
\mu(\Delta_n) = \int_{\Delta_n} d\mu = \int L_\Delta^n(1) d\mu = e^{nP_\Delta} \left(\alpha_0(n \mod m) \int h_n(\mod m) d\mu + \alpha_1(n \mod m) \int h_{n+1}(\mod m) d\mu\right) + o(e^{nP_\Delta}).
$$

Therefore

$$
\lim_{n \to \infty} e^{-2nP_\Delta} \mu(\Delta_{2n}) = \alpha_0(0) \int h_0 \, d\mu + \alpha_1(0) \int h_1 \, d\mu = \int (h_0 + h_1) \, d\mu = \int h_\Delta \, d\mu,
$$

but

$$
\lim_{n \to \infty} e^{-(2n+1)P_\Delta} \mu(\Delta_{2n+1}) = \alpha_0(1) \int h_1 \, d\mu + \alpha_1(1) \int h_0 \, d\mu = e^{\frac{1}{2}(q-p)} \int h_0 \, d\mu + e^{\frac{1}{2}(p-q)} \int h_1 \, d\mu.
$$

The latter is not in general equal to $\int h_\Delta \, d\mu$ if $p \neq q$ (see explicit example below). Therefore $\lim_{n \to \infty} e^{-nP_\Delta} \mu(\Delta_n)$ may not exist in general. However, if $p = q$ then $\alpha_j(k) = 1$ for all $j, k$ and then the limit is given by

$$
\lim_{n \to \infty} e^{-nP_\Delta} \mu(\Delta_n) = \int h_\Delta \, d\mu.
$$

Note also that even when $p \neq q$ there are choices of normalised potential $\varphi$ such that

$$
\lim_{n \to \infty} e^{-nP_\Delta} \mu(\Delta_n) = e^{\frac{1}{2}(q-p)} \int h_0 \, d\mu + e^{\frac{1}{2}(p-q)} \int h_1 \, d\mu = \int h_\Delta \, d\mu.
$$
For explicit examples of the above remarks, take for instance \( \varphi \) defined by 
\( \varphi_{|C[1]|} \equiv p \) and \( \varphi_{|C[2]|} \equiv q \). Then necessarily \( h_0 \) is equal to 1 on the cylinders \( C[1] \) and \( C[3] \), and it is equal to 0 on \( C[2] \). Similarly, \( h_1 \) is equal to 1 on \( C[2] \) and \( C[3] \), and it is equal to 0 on \( C[1] \). Therefore

\[
\int h_\Delta \, d\mu = (\mu(C[1]) + \mu(C[3])) + (\mu(C[2]) + \mu(C[3])) = \mu(C[1]) + (1 - \mu(C[1]))
\]

Now the condition of \( \varphi \) being normalised implies that the values of \( \varphi \) on the cylinder \( C[3] \) is uniquely determined. In fact on this cylinder \( \varphi \) is the 2-step cylindrical function given by 
\( \varphi_{|C[3]|} \equiv \log(1 - e^q) \), \( \varphi_{|C[2]|} \equiv \log(1 - e^p) \), and \( \varphi_{|C[3]|} \equiv \log(1 - e^p - e^q) \).

Hence \( \mu \) is the Markov measure defined by the stochastic matrix 
\[
P = \begin{pmatrix}
0 & e^q & 1 - e^q \\
e^p & 0 & 1 - e^p \\
e^q & e^p & 1 - e^p - e^q
\end{pmatrix}.
\]

This matrix has the stationary strictly positive left eigenvector \((p_1, p_2, p_3)\) given by 
\[
(p_1, p_2, p_3) = \left( \frac{e^p}{1 + e^p}, \frac{e^q}{1 + e^q}, 1 - \frac{e^p}{1 + e^p} - \frac{e^q}{1 + e^q} \right).
\]

Therefore, \( \mu(C[i]) = p_i \) for \( i = 1, 2, 3 \), which implies that 
\[
\int h_\Delta \, d\mu = \left( 1 - \frac{e^p}{1 + e^p} \right) + \left( 1 - \frac{e^q}{1 + e^q} \right) = \frac{2 + e^p + e^q}{(1 + e^p)(1 + e^q)}.
\]

Now we compare the above expression with 
\[
e^{\frac{1}{2}(q-p)} \int h_0 \, d\mu + e^{\frac{1}{2}(p-q)} \int h_1 \, d\mu = e^{\frac{1}{2}(q-p)} (1 - p_2) + e^{\frac{1}{2}(p-q)} (1 - p_1)
\]
\[
= \frac{e^{\frac{1}{2}(q-p)}}{1 + e^q} + \frac{e^{\frac{1}{2}(p-q)}}{1 + e^p} = \frac{e^{\frac{1}{2}(q-p)} (1 + e^p) + e^{\frac{1}{2}(p-q)} (1 + e^q)}{(1 + e^p)(1 + e^q)}.
\]

The two expressions coincide if and only if 
\[2 + e^p + e^q = e^{\frac{1}{2}(q-p)} (1 + e^p) + e^{\frac{1}{2}(p-q)} (1 + e^q) .\]

Introducing \( a = e^{\frac{1}{4}(q-p)} \) we see that 
\[2 + e^p + a^2 e^p = a (1 + e^p) + a^{-1} (1 + a^2 e^p),\]

which is equivalent to 
\[a (a^2 - 2a + 1) e^p = a^2 - 2a + 1 .\]
Since $ae^{p} = e^{\frac{1}{2}(p+q)} = e^{P_{\Delta}} < 1$, the above equality holds if and only if $a = 1$ (i.e. if $p = q$). Therefore, whenever $p \neq q$, we have

$$\int h_{\Delta} d\mu \neq e^{\frac{1}{2}(q-p)} \int h_{0} d\mu + e^{\frac{1}{2}(p-q)} \int h_{1} d\mu .$$

Hence, $e^{-nP_{\Delta}} \mu(\Delta_{n})$ does not converge as $n \to \infty$ when $p \neq q$.

**Appendix A. Convergence of Moments**

Here we reproduce the computations of [5] adapted to our general setting.

**Proof of Theorem 1.** We would like to show that $\nu_{k} = \lim_{n \to \infty} E(X_{n}(g)^{k})$ exists and that $\psi(z) = \sum_{k \geq 0} \nu_{k} z^{k} / k!$ takes the form

$$\psi(z) = F_{g}(z) = \exp \left\{ \sum_{m=1}^{\infty} C_{m} \int_{0}^{\infty} \left( e^{zg(t)} - 1 \right)^{m} dt \right\} ,$$

for $z$ in a small disc around the origin, where $C_{m}$ is defined in (H.1). Since the $k$-th derivative of $F_{g}$ at the origin is given by

$$\psi^{(k)}(0) = \sum_{p=1}^{k} \sum_{0 \leq t_{1}, \ldots, 0 < t_{p} \leq k} \frac{k!}{t_{1}! \cdots t_{p}!} \sum_{b=1}^{p} \sum_{0 < n_{1}, \ldots, 0 < n_{b} = p} \prod_{i=1}^{b} C_{n_{i}} \times$$

$$\int_{0}^{\infty} dy_{b} g(y_{b}) \sum_{s=0}^{n_{b}-1} t_{s+n_{1}+\cdots+n_{b-1}+1} \cdots \int_{0}^{y_{2}} dy_{1} g(y_{1}) \sum_{s=0}^{n_{1}-1} t_{s+1} ,$$

we want to prove that $\nu_{k}$ can be expressed by (22) for every $k > 0$.

For $k = 1$ we have

$$E(X_{n}(g)) = E(\chi_{\Delta_{n}}) \sum_{s=1}^{\infty} g(s c_{n}) = c_{n}^{-1} \mu(\Delta_{n}) c_{n} \sum_{s=1}^{\infty} g(s c_{n}) .$$

By (H.1) we know that $c_{n}^{-1} \mu(\Delta_{n})$ converges to $C_{1} > 0$. Since $g$ is continuous with compact support we obtain

$$\lim_{n \to \infty} E(X_{n}(g)) = C_{1} \int_{0}^{\infty} g(y) dy .$$

Now suppose $k > 1$. We know that

$$E(X_{n}(g)^{k}) = \sum_{0 \leq j_{1}, \ldots, 0 \leq j_{k}} E \left( \prod_{s=1}^{k} g(j_{s} c_{n}) \chi_{\Delta_{n} \circ T^{j_{s}}} \right) ,$$

and rearranging this sum we obtain

$$E(X_{n}(g)^{k}) = \sum_{0 \leq j_{1}, \ldots, 0 \leq j_{k}} E \left( \prod_{s=1}^{k} g(j_{s} c_{n}) \chi_{\Delta_{n} \circ T^{j_{s}}} \right) ,$$

and rearranging this sum we obtain
For a fixed set of positive integers \( t_1, \ldots, t_p \) define the summation

\[
S(\{t_i\}, n) = \sum_{b=1}^{p} \sum_{0 < j_1 < j_2 < \ldots < j_p, n_1 + \ldots + n_b = p} \mathbb{E}\left( \prod_{s=1}^{p} g(j_s c_n)^{t_s} \chi_{\Delta_n} \circ T^{j_s} \right),
\]

Decomposing this sum into clusters of consecutive indices differing by at most \( \ell(n)/k \) we obtain

\[
S(\{t_i\}, n) = \sum_{b=1}^{p} \sum_{0 < j_1 < j_2 < \ldots < j_p, n_1 + \ldots + n_b = p} \mathbb{E}\left( \prod_{s=1}^{p} g(j_s c_n)^{t_s} \chi_{\Delta_n} \circ T^{j_s} \right),
\]

where we have defined

\[
Q(n_1, \ldots, n_b) = \\
\{ (j_1, \ldots, j_{n_1} + \ldots + n_b) : j_1 < j_2 < \ldots < j_{n_1} + \ldots + n_b, j_q + 1 - j_q \leq \ell(n)/k \}
\]

if \( q \notin \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_b-1\} \) and else \( j_q+1 - j_q > \ell(n)/k \).}

Now we use the ‘relativised’ decay of correlations (condition H.2) between the different clusters. Fixing the positive integers \( n_1, \ldots, n_b \) and fixing \( (j_1, \ldots, j_{n_1} + \ldots + n_b) \in Q(n_1, \ldots, n_b) \),

we have

\[
\mathbb{E}\left( \prod_{s=1}^{n_1 + \ldots + n_b} \chi_{\Delta_n} \circ T^{j_s} \right) =
\]

\[
\mathbb{E}\left( \prod_{s=1}^{n_1} \chi_{\Delta_n} \circ T^{j_s} \cdot \prod_{s=n_1+1}^{n_1+n_2} \chi_{\Delta_n} \circ T^{j_s} \cdot \ldots \cdot \prod_{s=n_1+\ldots+n_{b-2}+1}^{n_1+\ldots+n_b} \chi_{\Delta_n} \circ T^{j_s} \right).
\]

Therefore we can write

\[
\mathbb{E}\left( \prod_{s=1}^{n_1 + \ldots + n_b} \chi_{\Delta_n} \circ T^{j_s} \right) = \mathbb{E}\left( \prod_{s=1}^{n_1} \chi_{\Delta_n} \circ T^{j_s-j_1} \cdot \chi_B \circ T^{j_{n_1}+1-j_1} \right),
\]

where \( B \) is a finite union of pre-images under \( T \) of \( \Delta_n \). Note that \( j_{n_1+1} - j_1 > j_{n_1} \), therefore we can apply (H.2) to get

\[
\mathbb{E}\left( \prod_{s=1}^{n_1 + \ldots + n_b} \chi_{\Delta_n} \circ T^{j_s} \right) = \left( \mathbb{E}\left( \prod_{s=1}^{n_1} \chi_{\Delta_n} \circ T^{j_s-j_1} \right) + R(j_1, \ldots, j_{n_1+1}) \right) \mathbb{E}(\chi_B),
\]

where the remainder \( R \) satisfies

\[
|R(j_1, \ldots, j_{n_1+1})| \leq K_{n_1} \gamma^{j_{n_1+1}-j_1}.
\]
Now using induction on the remaining clusters we obtain for \( b > 1 \),

\[
\mathbb{E}\left( \prod_{s=1}^{n_1 + \ldots + n_h} X_{\Delta_b} \circ T^{j_s} \right) =
\]

\[
\left( \mathbb{E}\left( \prod_{s=1}^{n_1} X_{\Delta_b} \circ T^{j_{s-j_1}} \right) + R(j_1, \ldots, j_{n_1+1}) \right) \times \cdots
\]

\[
\cdots \times \left( \mathbb{E}\left( \prod_{s=n_1+\ldots+n_{b-2}+1}^{n_1+\ldots+n_{b-1}} X_{\Delta_b} \circ T^{j_s} \right) + R(j_{n_1+\ldots+n_{b-2}+1}, \ldots, j_{n_1+\ldots+n_{b-1}+1}) \right)
\]

\[
\times \mathbb{E}\left( \prod_{s=n_1+\ldots+n_{b-1}+1}^{n_1+\ldots+n_b} X_{\Delta_b} \circ T^{j_s} \right) .
\]

The above expression implies that

\[
\mathbb{E}(X_n(g)^k) = \sum_{p=1}^{k} \sum_{0 < t_1, \ldots, t_p < q} \frac{k!}{t_1 \cdots t_p} \sum_{b=1}^{p} \sum_{0 < n_1, \ldots, 0 < n_b \text{ and } n_1 + \cdots + n_b = p} \sum_{(j_1, \ldots, j_p) \in \mathcal{Q}(n_1, \ldots, n_b)} \prod_{m=1}^{b} \mathbb{E}\left( \prod_{s=n_1+\ldots+n_{m-1}+1}^{n_1+\ldots+n_m} X_{\Delta_b} \circ T^{j_s} \right) \prod_{s=n_1+\ldots+n_{m-1}+1}^{n_1+\ldots+n_m} g^{j_s}(j_s, c_n) + \mathcal{R}(n, k),
\]

where we have set \( n_0 = 0 \), and \( \mathcal{R}(n, k) \) is a remainder term. For fixed indices \( p, t_1, \ldots, t_p, b, n_1, \ldots, n_b \) and for \( (j_1, \ldots, j_p) \in \mathcal{Q}(n_1, \ldots, n_b) \), define a double sequence of integers \( (q_{m,s}) \) with \( 1 \leq m \leq b \) and \( 0 \leq s \leq n_{m-1} - 1 \) by

\[
q_{m,s} = j_{s+n_1+\ldots+n_{m-1}+1} - j_{n_1+\ldots+n_{m-1}+1}.
\]

We then obtain

\[
\sum_{(j_1, \ldots, j_p) \in \mathcal{Q}(n_1, \ldots, n_b)} \prod_{m=1}^{b} \mathbb{E}\left( \prod_{s=n_1+\ldots+n_{m-1}+1}^{n_1+\ldots+n_m} X_{\Delta_b} \circ T^{j_s} \right) \prod_{s=n_1+\ldots+n_{m-1}+1}^{n_1+\ldots+n_m} g^{j_s}(j_s, c_n)
\]

\[
\times \sum_{0 < q_{i,1} < \cdots < q_{i,n_i-1} \text{ for } i=1, \ldots, b} \sum_{q_{i,s}+q_{i,s} \leq \ell(n)/k} c_{n,b} \prod_{i=1}^{b} \mathbb{E}\left( \prod_{s=0}^{n_i-1} X_{\Delta_b} \circ T^{q_{i,s}} \right) \times
\]

\[
\sum_{j_1 < j_{n_1+1} < \cdots < j_{n_1+\ldots+n_{b-1}+1} \text{ for } j_{n_1+\ldots+n_{b-1}+1} > q_{r,n_r-1}+\ell(n)/k} c_{n,b} \prod_{i=1}^{b} \prod_{s=0}^{n_i-1} g^{j_{s+n_1+\ldots+n_i-1}+1}(j_{n_1+\ldots+n_i-1}+1+q_{i,s}, c_n) .
\]
From (H.1) and the elementary properties of the Riemann integral we see that the expression (25) converges to
\[ \prod_{i=1}^{b} C_{n_i} \int_{0}^{\infty} dy_{b} g(y_{b}) \sum_{i=0}^{n_{b}-1} t_{b+n_{1}+\cdots+n_{b-1}+1} \cdots \int_{0}^{y_{2}} dy_{1} g(y_{1}) \sum_{i=0}^{n_{1}-1} t_{1+1}. \]
Hence the proof of Theorem 1 is finished, provided we show that \( R(n, k) = o(n) \), for each fixed \( k \).

We illustrate the estimation of the remainder in the case \( b = 2 \) for fixed \( n_1 \) and \( n_2 \). (The general case can be obtained in a similar manner as indicated at the end of this proof.) In this case, the remainder is composed by a finite sum of expressions of the form
\[ \sum_{(j_1, \ldots, j_{n_1}) \in \mathcal{Q}(n_1, n_2)} R(j_1, \ldots, j_{n_1}) E \left( \prod_{s=n_1+1}^{n_1+n_2} \chi_{\Delta_{\circ} T^{j_s}} \right) \prod_{s=1}^{n_1+n_2} g^{t_s}(j_s c_n). \]
Using (23) this term is bounded by
\[ (\|g\|_{\infty})^{n_1} K_{n_1} \times \sum_{(j_1, \ldots, j_{n_1}) \in \mathcal{Q}(n_1, n_2)} \gamma^{j_{n_1+1} - j_{n_1}} E \left( \prod_{s=n_1+1}^{n_1+n_2} \chi_{\Delta_{\circ} T^{j_s}} \right) \prod_{s=n_1+1}^{n_1+n_2} g^{t_s}(j_s c_n). \]
Since \( j_{q+1} - j_q \leq \ell(n)/k \) for all \( q < n_1 \), when we perform the sum over the indices \( j_1, \ldots, j_{n_1} \), we obtain a factor of \( (n_1\ell(n)/k)^{n_1} \). Using the fact that \( j_{n_1+1} - j_{n_1} > \ell(n)/k \) and introducing the variables \( q_{m,s} \) with \( m = 2 \) as before, we see that (26) is bounded by
\[ (\|g\|_{\infty})^{k} K_{n_1} \times \left( \frac{n_1\ell(n)}{k} \right)^{n_1} \sum_{s > \ell(n)/k} \gamma^s \times \]
\[ \sum_{0 < q_1 < \cdots < q_{n_2-1}} \sum_{q_{2,+1} - q_{2,s} \leq \ell(n)/k} E \left( \prod_{s=0}^{n_2-1} \chi_{\Delta_{\circ} T^{q_{2,s}}} \right) \prod_{j_{n_1+1}}^{n_2-1} g^{t_{j_{n_1+1}+q_{2,s}}}(j_{n_1+1} + q_{2,s} c_n). \]
When \( n \) diverges, the second part of (27) is bounded by an integral (multiplying and dividing by \( c_n \)), whereas the first part of (27) is of the order
\[ \left( \frac{n_1\ell(n)}{k} \right)^{n_1} \gamma^{\ell(n)/k}, \]
which clearly tends to zero as \( n \) diverges because \( n_1 \) is bounded.

Now, for the general estimate of the remainder, we note that equation (24) shows that \( R(n, k) \) is a sum of products of \( b \) terms of the form \( E \left( \prod \chi_{\Delta_{\circ} T^{j_s}} \right) \) or \( R(\ldots) \) and there is at least one of the latter type. Introducing the indices \( m, s \) and \( q_{m,s} \) as before, the summation over the indices can be performed similarly. Hence one obtains an expression very similar to equation (25) except that one
POISSON PROCESSES FOR SUBSHIFTS OF FINITE TYPE

multiplies and divides by a power of \(c_n\) which equals the number of factors of the form \(E(\prod \chi_{\Delta_i} T)\). Using the analogous estimates as in (23) for the terms \(R(\ldots)\) one readily sees that the remainder is \(o(n)\).

□

Appendix B. Identification of the parameters

Here we assume \(C_m = c \theta^{m-1}\) for some \(c, \theta > 0\), and we find an explicit solution (in the constants \(\lambda\) and \(\pi_j\)), where \(\sum \pi_j = 1\) of

\[
\sum_{m=1}^{\infty} C_m \int (e^{\zeta g(t)} - 1)^m dt = \lambda \sum_{j=1}^{\infty} \pi_j \int_0^{\infty} (e^{\zeta j g(y)} - 1) dy,
\]

for \(z\) in some disc around the origin in \(\mathbb{C}\). Consider the analytic function \(\Phi(u) = \sum_{m=1}^{\infty} C_m u^m\) defined in a neighbourhood of the origin. Setting \(u = (1 + u) - 1\) and using analytic continuation we obtain

\[
\Phi(u) = \frac{c}{\theta} \sum_{m=1}^{\infty} (\theta u)^m = \frac{c u}{1 - \theta u} = \frac{c u}{(1 + \theta) - \theta (1 + u)}
\]

\[
= c (1 - \theta') \left[(1 + u) - 1\right] \sum_{j=0}^{\infty} \left(\theta' (1 + u)\right)^j,
\]

where we have introduced \(\theta' = \theta/(1 + \theta)\). Since finding a formal solution for \(\lambda\) and \(\pi_j\) such that

\[
\Phi(u) = \sum_{m=1}^{\infty} C_m u^m = \lambda \left(\sum_{j=1}^{\infty} \pi_j (1 + u)^j - 1\right)
\]

gives a formal solution of (28), we use (29) to compare the coefficients of \((1 + u)^j\) to conclude that

\[
\lambda = c (1 - \theta') = \frac{c}{1 + \theta} \quad \text{and} \quad \pi_j = (1 - \theta') (\theta')^{j-1} = \frac{\theta^{j-1}}{(1 + \theta)^j}.
\]

Hence the right-hand side of (28) is an analytic function in a neighbourhood of the origin.

Appendix C. Eigenfunctions for the restricted transfer operator

This is a review of the paper [8] with comments on some improvements of their main result, which are used in the present paper.

The main difference between our setting and the one used in [8] is the fact that in [8] there is a fixed initial finite alphabet \(S\) (our set of vertices \(V\)) and the whole space is a subshift of finite type \(X_{L'}\) of \(X = S^\mathbb{Z}\) defined by an irreducible and aperiodic transition matrix \(L'\) in the alphabet \(S\) (our subshift \(\Sigma^\alpha\)). Then the authors consider a subsystem of \(X_{L'}\) given by a transition matrix \(L\) in the alphabet \(S\), where \(L\) imposes more restrictions than \(L'\) (i.e. if \(L = [\ell_{ij}]\) and \(L' = [\ell'_{ij}]\), then \(\ell'_{ij} = 0\) implies \(\ell_{ij} = 0\)). The important thing is that \(L\) is assumed
to be irreducible and aperiodic in the full alphabet $S$, therefore the allowable paths of the corresponding subshifts $X_L$ and $X_{L'}$ go through all the symbols of the initial alphabet $S$. In our setting, we choose a strictly smaller alphabet $\Delta \subset V$ and consider the allowable paths of $\Sigma^+_{\Delta}$ which go through vertices of $\Delta$ defining then a subshift $\Sigma_{\Delta}$ with alphabet $\Delta$. If we assume now that $\Sigma_{\Delta}$ is irreducible and aperiodic in its alphabet $\Delta$, then there may not exist a strictly positive eigenfunction of the restricted transfer operator $L_{\Delta}$ associated to the eigenvalue $e^{P_{\Delta}}$, in contrast with [8], where the restricted transfer operator $\mathcal{L}$ of $X_L$ is shown to have a strictly positive eigenfunction associated to the corresponding eigenvalue $\alpha_L$. The following provides an example. Take the set of vertices $V = \{1, 2, 3, 4\}$ and the matrix $A$ given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. $$

Let $\Delta = \{1, 2\}$ and then $\Sigma_{\Delta}$ is the full two shift on the symbols $\{1, 2\}$. Any function $\psi$ defined on $\Sigma^+_{\Delta}$ satisfies $L_{\Delta}(\psi)(x) = 0$ whenever $x_0 = 4$, therefore $L_{\Delta}$ does not have a strictly positive eigenfunction. In fact, in general if $Z_{\Delta}$ is the subset of $\Sigma^+_{\Delta}$ given by

(30) \[ Z_{\Delta} = \{ x \in \Sigma^+_{\Delta}: \exists b \in \Delta, \ A(b, x_0) = 1 \}, \]

then for any function $\psi$, $L_{\Delta}(\psi)(x) = 0$ whenever $x \notin Z_{\Delta}$. Assuming $\Sigma_{\Delta}$ is irreducible and aperiodic in its alphabet $\Delta$, the next comments show that $L_{\Delta}$ has an eigenfunction associated to $e^{P_{\Delta}}$, which is strictly positive on $Z_{\Delta}$ and it is zero on the complement $Z_{\Delta}^c$.

Since we would not want to rewrite the paper [8], we will only mention the main differences. For $x \in \Sigma^+_{\Delta} \setminus \Sigma_{\Delta}$, let $N(x) = \inf\{ n \geq 0: x_n \notin \Delta \}$. Fix some point $z \in \Sigma_{\Delta}$. Using the fact that $A$ is irreducible and aperiodic there exists $q > 0$ such that $A^q$ is a strictly positive matrix. This means that for any symbol $s \in V$ there exists an allowable path of length $q$ in the graph of $\Sigma^+_{\Delta}$ which starts at $s$ and ends at $z_0$. Let $s \to \psi_1(s) \to \cdots \to \psi_{q-1}(s) \to z_0$ be such a path, where $\psi_i(s) \in V$, for $i = 1, \ldots, q - 1$. Define $\pi: \Sigma^+_{\Delta} \to \Sigma_{\Delta}$ by $\pi(x) = x$ if $x \in \Sigma_{\Delta}$, and for $x \in \Sigma^+_{\Delta} \setminus \Sigma_{\Delta}$ define

$$\pi(x) = (x_0, \ldots, x_{N(x)}, \psi_1(x_{N(x)}), \ldots, \psi_{q-1}(x_{N(x)}), z_0, z_1, \ldots).$$

Let $C^+_{p}(\Sigma^+_{\Delta})$ be the set of strictly positive $p$-cylindrical functions (i.e. a function depending only on the first $p$ coordinates of the point). Let $0 < \theta < 1$ be the Hölder exponent of the potential $\varphi$. Let $Z_{\Delta}$ be defined as in (30).
Lemma C.1. There exists $c > 0$ such that for any $p > 0$, for any $k \geq p$, and for any $f \in C^+_p(\Sigma^+_A)$, we have
\[ e^{-c\theta^N(x)} \leq \frac{\mathcal{L}_\Delta^k f(x)}{\mathcal{L}_\Delta f(\pi(x))} \leq e^{c\theta^N(x)}, \]
for all $x \in Z_\Delta$; and $\mathcal{L}_\Delta f(x) = 0$ if $x \notin Z_\Delta$.

The proof of the above Lemma is exactly the same as the proof of Lemma 1 of [8]. For the next result, we note that if $f \in C^+_p(\Sigma^+\Delta)$ then $f$ can be extended in a natural way to a function defined on $\Delta_p = \{x \in \Sigma^+_A: x_i \in \Delta, i = 0, \ldots, p-1\}$.

Lemma C.2. There exists $0 < r < 1$ and $c(f) > 0$ such that for any $n > 2p$, and for any $f \in C^+_p(\Sigma^+\Delta)$, we have
\[ e^{-c(f)r^n} \leq e^{-nP_\Delta L^n\Delta f(x)} \leq e^{c(f)r^n}, \]
for all $x \in Z_\Delta \subseteq \Sigma^+_A$.

Again the proof of the above Lemma is exactly the same as the proof of Lemma 2 of [8]. Let $C(Z_\Delta)$ denote the set of continuous functions defined on $Z_\Delta$.

Lemma C.3. For any $f \in \bigcup_{p \geq 1} C^+_p(\Sigma^+_A)$, we have
\begin{enumerate}
\item[(i)] $\{e^{-nP_\Delta L^n\Delta f}\}_{n \geq 0}$ is a Cauchy sequence in $C(\Sigma^+_A)$;
\item[(ii)] $h_\Delta = \lim_{n \to \infty} e^{-nP_\Delta L^n\Delta f}$ does not depend on the function $f \in \bigcup_{p \geq 1} C^+_p(\Sigma^+_A)$ and it satisfies
\[ \mathcal{L}_\Delta(h_\Delta) = e^{P_\Delta h_\Delta}. \]
\end{enumerate}

The above Lemma is the same as Lemma 3 of [8]. The proof of Lemma 3 of [8] implies that $\{e^{-nP_\Delta L^n\Delta f}\}_{n \geq 0}$ is a Cauchy sequence in $C(Z_\Delta)$. Since on the complement $Z_\Delta^c$ the sequence is identically zero, we conclude that (i) holds. The proof of (ii) is the same as the proof of Lemma 3 (ii) in [8]. We note that this proof implies that $h_\Delta$ is strictly positive on $Z_\Delta$, and it is zero on the complement $Z_\Delta^c$. Since the transfer operator $\mathcal{L}$ on the subsystem $\Sigma_\Delta$ coincides with $\mathcal{L}_\Delta$ for points in $\Sigma_\Delta$, we conclude from (ii) that $h_\Delta|_{\Sigma_\Delta} \equiv w_\Delta$.

Although not explicitly mentioned in [8], the function $h_\Delta$ is a Hölder continuous function with the Hölder exponent of the potential $\varphi$. This is because from (i) and (ii) we have $h_\Delta = \lim_{n \to \infty} e^{-nP_\Delta L^n\Delta(1)}$ in the supremum norm $\| \cdot \|_\infty$ on $\Sigma^+_A$. Hence $\|e^{-nP_\Delta L^n\Delta(1)}\|_\infty$ is a bounded sequence. Now, if $x, y \in \Sigma^+_A$ are such that $x_i = y_i$, for $i = 0, \ldots, k-1$ and $k \geq 1$, then either $x, y \in Z_\Delta$ or $x, y \in Z_\Delta^c$. 
Therefore we have
\[ e^{-nP_\Delta} |L^0_\Delta 1(x) - L^0_\Delta 1(y)| \]
\[ \leq \sum_{(i_0, \ldots, i_{n-1}) \in \Delta} e^{-nP_\Delta} \left| e^{\varphi(i_0, i_{n-1}, y)} - e^{\varphi(i_0, i_{n-1}, x)} - 1 \right| \]
\[ \leq \|e^{-nP_\Delta} L^0_\Delta(1)\|_\infty \|e^{\theta n + k} - 1\| \leq C \theta^k , \]
where \( C \) is independent of \( n, k \) and \( x, y \). Hence \( \text{var}_k(h_\Delta) \leq C \theta^k \) and \( h_\Delta \) is \( \theta \)-Hölder.

From (ii) one can extend the convergence from \( C^+_p(\Sigma^+_A) \) to \( C(\Sigma^+_A) \), which is the same proof as in [8]. This proves Proposition 4 as stated in the present paper. For the remaining comments in Section 2, we mention the corresponding changes in the expressions (9), (10) and (11) of the main result of [8] in our setting. Consider the Pianigiani-Yorke measure \( \mu_{PY} \) defined by
\[ \mu_{PY}(B) = \int_B h_\Delta \, d\mu , \]
for every Borel subset \( B \subseteq \Sigma^+_A \). First we note that for \( f, g \in L^1(\mu) \) we have for every \( n \geq 1 \), \( L^n_\Delta(f \cdot \chi_{\Delta_n}) = L^n_\Delta(f) \) and
\[ L^n_\Delta(f \cdot g \circ T^n) = L^n_\Delta(\chi_{\Delta_n} \cdot f \cdot g \circ T^n) = g L^n_\Delta(f \cdot \chi_{\Delta_n}) = g L^n_\Delta(f) . \]
On the other hand, for \( n \geq 1 \) we also have \( L^n_\Delta(f \cdot g \circ T^n) = L^n(\chi_{\Delta_n} \cdot f \cdot g \circ T^n) \).

Since \( \mu \) is fixed by the dual operator of \( \mathcal{L} \) we have
\[ (31) \quad \int_{\Delta_n} f \cdot g \circ T^n \, d\mu = \int L^n(\chi_{\Delta_n} \cdot f \cdot g \circ T^n) \, d\mu = \int g \cdot L^n_\Delta(f) \, d\mu . \]

Let \( B \subseteq \Sigma^+_A \) be a Borel subset. Putting \( g = \chi_B \cdot f = h_\Delta \) in the above expression and noting that \( L^n_\Delta(h_\Delta) = e^{nP_\Delta} h_\Delta \) we obtain
\[ \mu_{PY}(T^{-n}B \cap \Delta_n) = \int_{T^{-n}B \cap \Delta_n} h_\Delta \, d\mu = \int_{\Delta_n} h_\Delta \cdot \chi_B \circ T^n \, d\mu = \int \chi_B \cdot L^n_\Delta(h_\Delta) \, d\mu = e^{nP_\Delta} \int \chi_B \cdot h_\Delta \, d\mu = e^{nP_\Delta} \mu_{PY}(B) . \]
This proves (10), since for \( n = 1 \) we obtain
\[ \mu_{PY}(T^{-1}B \cap \Delta) = e^{P_\Delta} \mu_{PY}(B) , \]
where we identified \( \Delta \) with the set \( \Delta_1 \). Putting \( g = \chi_B \) and \( f = 1 \) in (14) gives
\[ \frac{\mu(T^{-n}B \cap \Delta_n)}{\mu(\Delta_n)} = \frac{\int_{\Delta_n} \chi_B \circ T^n \, d\mu}{\int_{\Delta_n} d\mu} = \frac{\int \chi_B \cdot L^n_\Delta(1) \, d\mu}{\int L^n_\Delta(1) \, d\mu} = \frac{\int_B e^{-nP_\Delta} L^n_\Delta(1) \, d\mu}{\int e^{-nP_\Delta} L^n_\Delta(1) \, d\mu} . \]
Taking the limit when $n \to \infty$ proves (12), since
\[
\lim_{n \to \infty} \mu(T^{-n}B | \Delta_n) = \frac{\int B h_{\Delta} \, d\mu}{\int h_{\Delta} \, d\mu} = \frac{\mu_{\text{PY}}(B)}{\mu_{\text{PY}}(\Sigma^+_A)}.
\]
Although $\Sigma_{\Delta}$ (which is the support of $\mu_{\Delta}$) has $\mu$-measure zero, an interesting fact is that
\[
(32) \quad \mu_{\Delta}(B) = \lim_{n \to \infty} \mu(B | \Delta_n),
\]
for every closed and open subset $B \subseteq \Sigma^+_A$. (Since $\mu$ and $\mu_{\Delta}$ are ergodic measures for $T$, they are mutually singular, therefore the above is untrue in general for all Borel sets $B$.) Now, assume $B$ is a closed and open subset of $\Sigma^+_A$ and then $g = \chi_B$ is a continuous function on $\Sigma^+_A$. Note that
\[
\mu(B | \Delta_n) = \frac{\mu(B \cap \Delta_n)}{\mu(\Delta_n)} = \frac{\int_{\Delta_n} \chi_B \, d\mu}{\int_{\Delta_n} d\mu} = \frac{\int e^{-nP_{\Delta} L^n(\chi_B)} \, d\mu}{\int e^{-nP_{\Delta} L^n(1)} \, d\mu}.
\]
Taking the limit when $n \to \infty$ and using an extension of Lemma C.3 (ii) to continuous functions, we obtain (32).

**References**


Centre de Physique Théorique, CNRS UMR 7644, Ecole Polytechnique, F-91128 Palaiseau Cedex, FRANCE
E-mail address: jean-rene.chazottes@cpht.polytechnique.fr

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
E-mail address: zc3@york.ac.uk

Centre de Physique Théorique, CNRS UMR 7644, Ecole Polytechnique, F-91128 Palaiseau Cedex, FRANCE
E-mail address: pierre.collet@cpht.polytechnique.fr