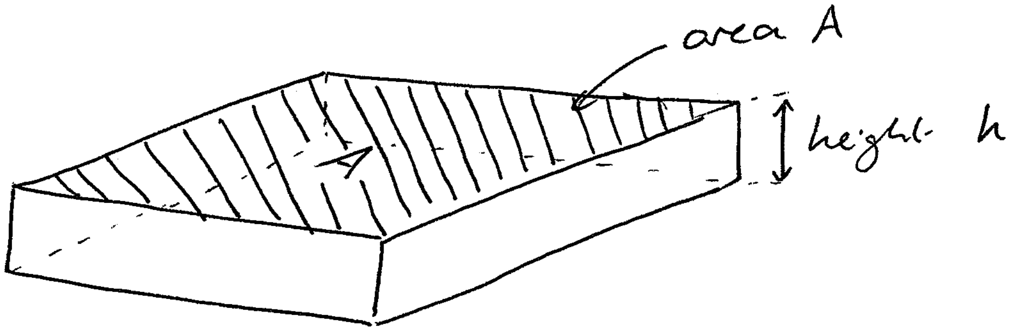
VERTICAL STRUCTURE AND STABILITY

Although the atmosphere varies considerably in the horizontal direction, many features can be understood by considering only variations with height.

Last lecture we saw that the atmosphere has large variations in pressure, temperature, and density. In this lecture we'll look at some of the reasons why.

1. Pressure

The pressure of the air is determined by the weight of the atmosphere above it. We can therefore work out the pressure variation by considering a slab of air, supporting the air above it, and being supported by the air below



Forces on the parcel of air:

- Downwards pressure of air above $\quad F = -P_{above} A$
- Downwards force due to gravity $\quad -g \underbrace{\rho V}_{Mass}$
- Upwards pressure of air below $\quad + P_{below} A$

Assume the air is stationary in force balance

$$\Rightarrow \quad -P_{above}A - \rho g V + P_{below}A = 0$$

$$\underbrace{(P_{above} - P_{below})}_{\Delta P} A = -\rho g V$$

change in pressure going up. Negative as expected so pressure falls with height.

Volume $V = A\Delta h$ $\qquad \Delta P \cancel{A} = -\rho g \cancel{A}\Delta h$

$$\Rightarrow \quad \frac{\Delta P}{\Delta h} = -\rho g$$

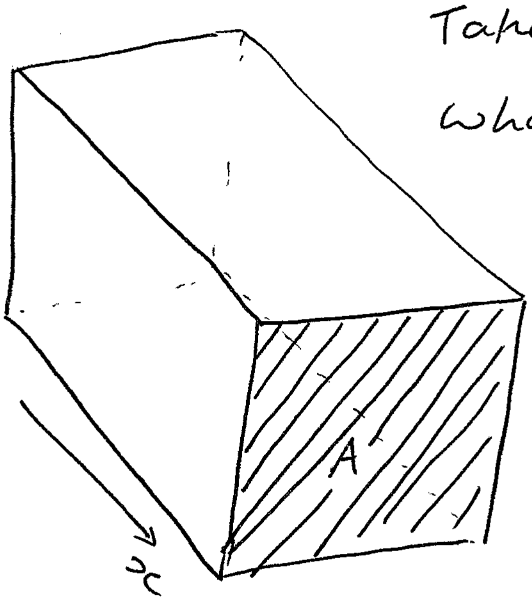
taking $\Delta h \to 0$, $\quad \underline{\underline{\dfrac{dP}{dh} = -\rho g}}$

The density ρ is not very convenient to use, so we would like to relate this to a temperature.

This can be done using an equation of state, in particular the <u>IDEAL GAS LAW</u>.

IDEAL GAS LAW

Consider N molecules in a box of volume V.



Take one of the faces, of area A.
What is the force on this surface?

Force is due to molecules colliding with and bouncing off the surface

Newton's Law $F = ma = m\dfrac{\Delta V}{\Delta t}$

Typical molecule has mass m, velocity V_x in the x direction. When it bounces off the wall, the velocity changes from $+V_x$ to $-V_x$

$$\Rightarrow \Delta V = 2V_x$$

How many collisions are there per second?

In a time Δt, particles with a velocity V_x will hit the wall if they are within a distance $V_x \Delta t$ of the wall

Volume of this region = $A V_x \Delta t$

Number of particles in this volume $= \frac{N}{V} A \Delta t V_x$

Only half the molecules (on average) are going towards the wall, and so the number of molecules which hit the wall per second is:

$$\frac{1}{2} \frac{N}{V} A V_x$$

The total force on the wall is therefore

$$F = \frac{1}{2} \frac{N}{V} A V_x \times 2 m_a V_x$$

Pressure is just Force per unit area
units Pascals, N/m^2

$$\boxed{P = F/A}$$

Pressure on walls of box is therefore

$$P = \frac{N}{V} m_a V_x^2$$

The quantity mV_x^2 is just $2\times$ the kinetic energy in the x direction. Averaging over all molecules, this is proportional to temperature:

$$\frac{1}{2} m_a \langle V_x^2 \rangle \propto T$$

$$= \frac{1}{2} k_B T$$

Boltzmann constant

Putting this together gives

$$\boxed{PV = N k_B T}$$

Putting together the equation for Hydrostatic force balance, and the Ideal Gas Law

$$\frac{dP}{dh} = -\rho g \qquad PV = N k_B T$$

$$P = \frac{N}{V} k_B T$$

Density $\rho = \frac{M}{V} = \frac{N m_a}{V}$ ← mass of one molecule (average)

$$\Rightarrow \frac{N}{V} = \frac{\rho}{m_a} \qquad P = \frac{\rho}{m_a} k_B T$$

or

$$\underline{\rho = \frac{m_a P}{k_B T}}$$

$$\Rightarrow \frac{dP}{dh} = -\frac{m_a P}{k_B T} g \qquad \frac{dP}{P} = -\frac{m_a g}{k_B T} dh$$

Can integrate both sides, assuming T is constant

$$\ln P = -\frac{m_a g}{k_B T} h + c$$

$$\Rightarrow P = P_0 e^{-h \frac{m_a g}{k_B T}} = P_0 e^{-h/H}$$

↑ scale height

The scale Height $H = \frac{k_B T}{m_a g}$ is the height change over which the pressure falls by a factor of $e \simeq 2.717$.

At $T = 210$ °K, $H \simeq 6$ km

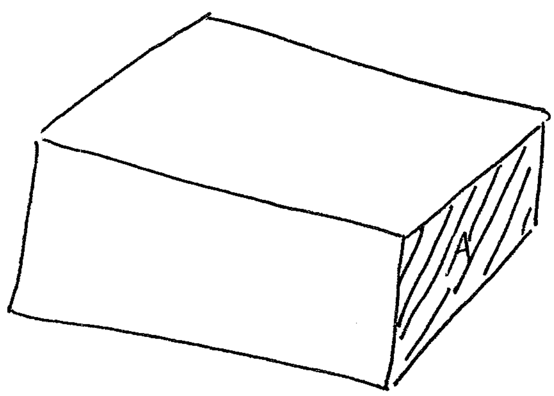
$T = 290$ °K $H \sim 8.5$ km

The pressure therefore falls exponentially with height, but to get the correct scale length we need to know the temperature.

Temperature variation

Consider this simple model for heat transport in the atmosphere: Air is heated close to the ground, it rises and so its pressure falls to match its surroundings. Since air is a poor conductor of heat, it does not exchange any energy by conduction with its surroundings ("adiabatic").

Consider our box of gas from before



N molecules in a volume V

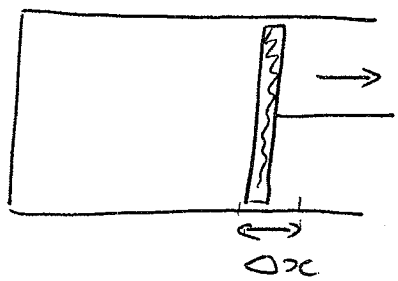
Surface of area A experiences a force due to the gas inside of

$$F = PA = \frac{N}{V} k_B T A$$

Now move this box of gas upwards. The pressure outside the box falls with altitude.

$\Rightarrow$ Inwards force due to gas outside falls.

Imagine you now allow one of the walls of this box to move like a piston



As the pressure outside drops, the wall is pushed outwards a distance Δx

Since the gas is exerting a force on the wall, which has moved through a distance Δx, the gas has done work $W = F \cdot \Delta x$

$$= \underbrace{PA\Delta x}_{\text{change in volume } \Delta V}$$

$$\Rightarrow W = P\Delta V$$

This work must come from the internal energy of the gas: the kinetic and vibrational energy of the molecules. This is just proportional to the temperature and each molecule has on average an energy of $\gamma k_B T$ where γ depends on the number of degrees of freedom

e.g. $\gamma = \frac{3}{2}$ for monatomic (V_x, V_y, V_z), $\gamma = \frac{5}{2}$ for diatomic at low temperatures $(V_x, V_y, V_z + 2 \text{ rotation})$

$\therefore$ Total energy of N molecules is

$$E = N\gamma k_B T$$

We assume that no molecules leave our slab of air as it rises (N in box stays constant)

So

$$\Delta T = \frac{\Delta E}{N\gamma k_B} \qquad \Delta E = -W = -P\Delta V$$

$$\Rightarrow \Delta T = -\frac{P\Delta V}{N\gamma k_B} \qquad \text{so} \qquad \frac{\Delta V}{\Delta T} = -\frac{N\gamma k_B}{P}$$

take limit of small changes

$$\rightarrow \quad \frac{dV}{dT} = -\frac{N\gamma k_B}{P}$$

We have now calculated what the change in temperature is for a given change in volume.

The temperature change will affect the pressure, so we need the equation of state to get the change in volume

$$PV = Nk_BT$$

Differentiate $P(h)$, $V(h)$, $T(h)$ with respect to height h:

$$\frac{dP}{dh}V + P\frac{dV}{dh} = Nk_B\frac{dT}{dh}$$

From the chain rule $\quad \dfrac{dV}{dh} = \dfrac{dV}{dT}\cdot\dfrac{dT}{dh}$

Therefore $\quad \dfrac{dP}{dh}V + P\left(-\dfrac{N\gamma k_B}{P}\right)\dfrac{dT}{dh} = Nk_B\dfrac{dT}{dh}$

$$\frac{dP}{dh}V = Nk_B\frac{dT}{dh}(1+\gamma)$$

Divide through by V
use $N/V = \rho/m_a$

The density ρ is the total mass $N m_a$ divided by the volume V.
$$\rho = \frac{N m_a}{V}$$

$$\Rightarrow \quad \frac{dT}{dh} \frac{\rho}{m_a} k_B (1+\gamma) = \frac{dP}{dh}$$

This relates the change in temperature to the change in pressure. We already know $\frac{dP}{dh}$

$$\frac{dP}{dh} = -\rho g$$

$$\frac{dT}{dh} \frac{\cancel{\rho}}{m_a} k_B (1+\gamma) = -\cancel{\rho} g$$

$$\Rightarrow \quad \frac{dT}{dh} = -\frac{g\, m_a}{k_B (1+\gamma)} = -g/c_p = -\Gamma_d$$

c_p is the specific heat at constant pressure

$$c_p = \frac{k_B (1+\gamma)}{m_a}$$

$\Gamma_d = g/c_p$ is the dry adiabatic lapse rate putting in some values:

• Mean molecular weight of the atmosphere is
$\sim 29 \quad (N_2 = 2 \times 14 = 28)$

$$\Rightarrow m_a \approx 29 \times 1.67 \times 10^{-27}\,\text{kg} = 4.8 \times 10^{-26}\,\text{kg}$$

$k_B \approx 1.38 \times 10^{-23}\,\text{J/K}$

Mostly diatomic molecules $\rightarrow \gamma \approx 5/2$

This gives $\quad C_p = \dfrac{k_B(1 + 5/2)}{m_a} \simeq 997$

Measured value is $\quad C_p \simeq 1005$

$$\frac{dT}{dh} = -g/C_p \simeq -10\,\text{K/km}$$

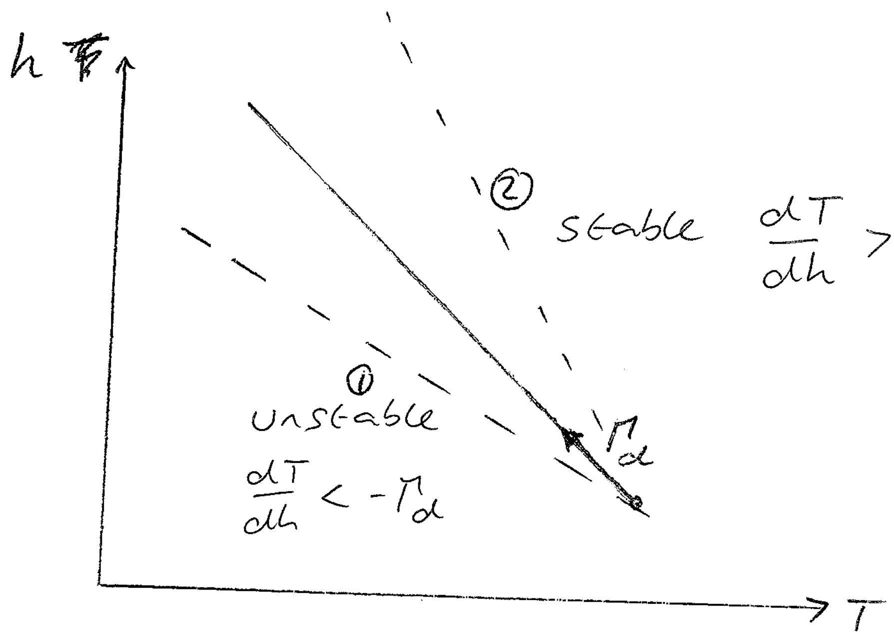
This is known as the <u>ADIABATIC LAPSE RATE</u>, Γ_d of <u>DRY AIR</u>. Haven't considered effect of water yet...

VERTICAL STABILITY

We have derived that a parcel of air which moves upwards and doesn't conduct heat (adiabatic) will cool at a rate of $\Gamma_d \sim -10$ K/km. due to expansion.

We can imagine three scenarios:

1. The temperature in the atmosphere falls faster than Γ_d



In this case a parcel of air which moves upwards will move along the Γ_d line and so will be hotter than its surroundings since hot air is less dense, it will continue to rise

→ UNSTABLE

This is what happens when fluids start to convect.

2. The temperature of the atmosphere falls slower than Γ_d. In this case a rising air packet becomes cooler than its surroundings. It is therefore less dense, and will tend to return to its equilibrium location

$\rightarrow$ STABLE

- In the stratosphere the temperature is increasing with height: $\frac{dT}{dh} > 0$

$\rightarrow$ This is very stable so little convection.

- Since air tends to be pushed back to its equilibrium location, this leads to oscillations known as <u>buoyancy oscillations</u>

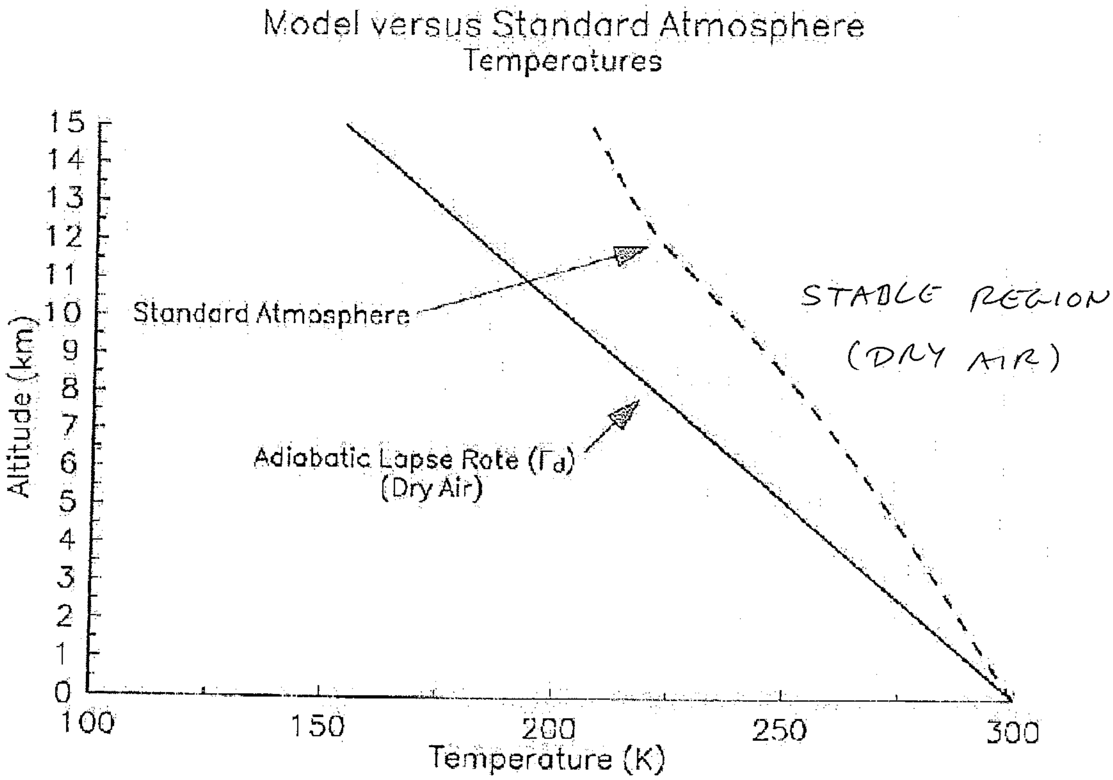
3. If $\frac{dT}{dh} = \Gamma_d$, then a displaced air packet will have the same temperature as its surroundings and so will be "statically neutral".

The result is that an atmosphere heated from below should be unstable, convection will carry energy upwards until the temperature gradient reaches Γ_d at which point convection will stop

$\Rightarrow$ Expect atmosphere (troposphere) to have

$$\frac{dT}{dh} \approx \Gamma_d \approx -10\,K/km$$

How well does this simple model agree with measurements?



The dry adiabatic lapse rate predicts

$$\frac{dT}{dh} = \Gamma_d \simeq -10\,K/km$$

Observed in the troposphere $\frac{dT}{dh} \sim -6\,K/km$

Difference is due to several effects, most importantly water vapour which modifies C_p and changes the lapse rate.

SUMMARY

- Force balance for a slab of air (Hydrostatic equilibrium) gives variation of pressure with height h:

$$\frac{dP}{dh} = -\rho g$$

- The IDEAL GAS LAW equation of state is

$$PV = N k_B T$$

- Using this, the pressure can be written as

$$P = P_0 e^{-h/H} \quad \text{with} \quad H = \frac{k_B T}{m_a g}$$

SCALE HEIGHT

By considering the work done by a rising packet of air as it expands, the temperature change can be related to the pressure change:

$$\frac{dT}{dh} \frac{\rho}{m_a} k_B (1+\gamma) = \frac{dP}{dh}$$

This gives the ADIABATIC LAPSE RATE

$$\frac{dT}{dh} = -g/c_p = -\Gamma_d \qquad c_p \simeq 1005 \ \mathrm{J/kg/K}$$

$$\simeq -10 \ \mathrm{K/km}$$

- If $\frac{dT}{dh} < -\Gamma_d$ (steep change in T) then the atmosphere is unstable, and will convect. If $\frac{dT}{dh} > -\Gamma_d$ then it is stable e.g. in the stratosphere.