Neoclassical currents

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Last time

- Ideal MHD gives the relation $\underline{J} \times \underline{B} = \nabla p$ for equilibrium
- This lead to the Grad-Shafranov equation which is used to design and interpret tokamak experiments
- This relation determines the perpendicular current, but says nothing about the parallel current

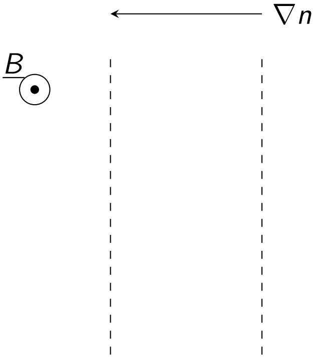
In this lecture we'll look in more detail about currents in tokamak plasmas

Diamagnetic current

You have already seen one current in MHD equilibrium

$$\underline{J} \times \underline{B} = \nabla P \quad \Rightarrow J_{\perp}^{DIA} = \frac{\underline{B} \times \nabla P}{B^2}$$

This is called the **Diamagnetic current** (Recall that this also means that $\underline{B} \cdot \nabla P = 0$ so $P = P(\psi)$; $P =$ poloidal flux)



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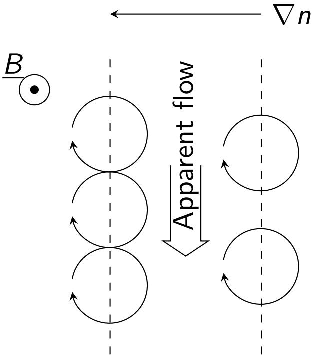

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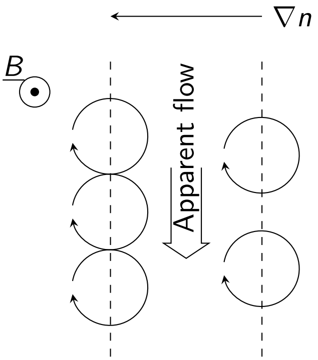

No other current can flow perpendicular to the magnetic field lines (in ideal MHD at least) $\Rightarrow$ All other currents must be along the magnetic field.

Note that J^{DIA} is a classical current - it exists in a cylinder or slab as well as a torus.

Divergence of diamagnetic current

Consider the divergence of the diamagnetic current:

$$\nabla \cdot \underline{J}_\perp^{DIA} = \nabla \cdot \left[\frac{\underline{B} \times \nabla P}{B^2}\right]$$

Noting the vector identity

$$\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\nabla \times \underline{A}) - \underline{A} \cdot (\nabla \times \underline{B})$$

then we have

$$\nabla \cdot \underline{J}_\perp^{DIA} = (\underline{B} \times \nabla P) \cdot \nabla \left(\frac{1}{B^2}\right) + \frac{1}{B^2} \nabla P \cdot \underbrace{\nabla \times \underline{B}}_{\mu_0 \underline{J}} - \underline{B} \cdot \underbrace{(\nabla \times \nabla P)}_{=0 \text{ (identity)}}$$

$$\Rightarrow \nabla \cdot \underline{J}_\perp^{DIA} = (\underline{B} \times \nabla P) \cdot \nabla \left(\frac{1}{B^2}\right) + \frac{\mu_0}{B^2} \underbrace{\nabla P \cdot \underline{J}}_{=0}$$

$$\nabla \cdot \underline{J}_\perp^{DIA} = (\underline{B} \times \nabla P) \cdot \nabla \left(\frac{1}{B^2}\right)$$

Return current

Recall from last lecture that $p = p(\psi)$, so $\nabla p = \frac{dp}{d\psi}\nabla\psi$. Using the expression $\underline{B} = f(\psi)\nabla\phi + \nabla\phi\times\nabla\psi$, we can write

$$\begin{aligned}
\underline{B}\times\nabla p &= \left(f(\psi)\nabla\phi + \nabla\phi\times\nabla\psi\right)\times\frac{dp}{d\psi}\nabla\psi \\
&= \frac{dp}{d\psi}\Bigg[f\left(\nabla\phi\times\nabla\psi\right) + \underbrace{\left(\nabla\phi\cdot\nabla\psi\right)}_{=0}\nabla\psi - \underbrace{\left|\nabla\psi\right|^2}_{R^2B_\theta^2}\nabla\phi] \\
&= \frac{dp}{d\psi}\left[f\left(\underline{B} - f\nabla\phi\right) - R^2B_\theta^2\nabla\phi\right] \\
&= \frac{dp}{d\psi}[f\underline{B} - \underbrace{f^2}_{R^2B_\phi^2}\nabla\phi - R^2B_\theta^2\nabla\phi] \\
&= \frac{dp}{d\psi}\left[f\underline{B} - R^2B^2\nabla\phi\right]
\end{aligned}$$

Return current

Using this,

$$\begin{aligned}
\nabla \cdot \underline{J}_\perp^{DIA} &= \left(\underline{B} \times \nabla P\right) \cdot \nabla \left(\frac{1}{B^2}\right) \\
&= \frac{dp}{d\psi}\left(f\underline{B} - R^2 B^2 \nabla\phi\right) \cdot \nabla \left(\frac{1}{B^2}\right) \\
&= \frac{dp}{d\psi} f \left(\underline{B} \cdot \nabla\right) \left(\frac{1}{B^2}\right)
\end{aligned}$$

- For a cylinder, the magnitude of B doesn't vary along $\underline{B}$ so $\nabla \cdot \underline{J}_\perp^{DIA} = 0$
- In a torus however, field-lines go between high and low B regions, so $\nabla \cdot \underline{J}_\perp^{DIA} \neq 0$

Return current

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$$
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- For a cylinder, the magnitude of B doesn't vary along $\underline{B}$ so $\nabla \cdot \underline{J}_\perp^{DIA} = 0$
- In a torus however, field-lines go between high and low B regions, so $\nabla \cdot \underline{J}_\perp^{DIA} \neq 0$
- Divergence of the total current must be zero $\boxed{\nabla \cdot \underline{J} = 0}$

$\Rightarrow$ there must be another current

Pfirsch-Schlüter current

- We need to add another current to make $\nabla \cdot \underline{J} = 0$
- This can't be perpendicular to $\underline{B}$ as this is fixed by force balance: $\nabla \cdot \underline{J}_\perp^{DIA}$ is the only perpendicular current
- The current must therefore have the form $\underline{J} = \underline{J}_\perp^{DIA} + \underline{J}_{||}$ where $\underline{J}_{||} = J_{||}\underline{B}/B$

$$\nabla \cdot \underline{J} = \underbrace{\nabla \cdot \underline{J}_\perp^{DIA}}_{\frac{dp}{d\psi} f(\underline{B}\cdot\nabla)\cdot(1/B^2)} + \underbrace{\nabla \cdot \left(\frac{J_{||}\underline{B}}{B}\right)}_{\underline{B}\cdot\nabla\left(J_{||}/B\right)} = 0$$

$$\Rightarrow \underline{B} \cdot \nabla \left[\frac{J_{||}}{B} + \frac{f}{B^2}\frac{dp}{d\psi}\right] = 0$$

Pfirsch-Schlüter current

$$\underline{B} \cdot \nabla \left[\frac{J_{||}}{B} + \frac{f}{B^2}\frac{dp}{d\psi}\right] = 0$$

If the parallel gradient $(\underline{B} \cdot \nabla)$ of a quantity is zero, then it must be constant on flux surfaces

$$\frac{J_{||}}{B} + \frac{f}{B^2}\frac{dp}{d\psi} = C\left(\psi\right)$$

The parallel current must therefore satisfy

$$J_{||} = -\frac{f}{B}\frac{dp}{d\psi} + C\left(\psi\right)B$$

Now we need to determine $C\left(\psi\right)$

Constraint on $C(\psi)$

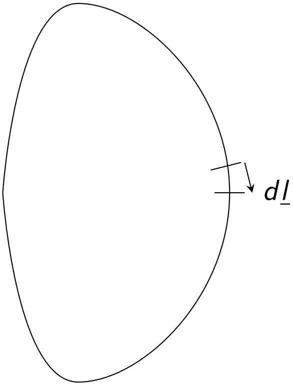
To get a constraint on $C(\psi)$, we assume steady state. In this case, we can write

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} = 0$$

and therefore:

$$\oint \underline{E} \cdot d\underline{l} = \oint E_\theta dl_\theta = 0$$

where $d\underline{l}$ is a line element in the poloidal direction along the flux surface



This poloidal electric field is due to parallel resistivity

$$E_{||} = \eta_{||} J_{||} \qquad \Rightarrow \frac{B_\theta}{B} E_\theta + \frac{B_\phi}{B} E_\phi = \eta_{||} J_{||}$$

Constraint on $C(\psi)$

combining $\frac{B_\theta}{B}E_\theta + \frac{B_\phi}{B}E_\phi = \eta_{||}J_{||}$, $\oint E_\theta dl_\theta = 0$ gives:

$$\oint\left(\eta_{||}J_{||}\frac{B}{B_\theta} - \frac{B_\phi E_\phi}{B_\theta}\right)dl_\theta = 0$$

Assuming that $\eta_{||} \neq 0$ and using $J_{||} = -\frac{f}{B}\frac{dp}{d\psi} + C(\psi)B$

$$\oint\left(\left[-\frac{f}{B}\frac{dp}{d\psi} + C(\psi)B\right]\frac{B}{B_\theta} - \frac{B_\phi E_\phi}{\eta_{||}B_\theta}\right)dl_\theta = 0$$

Because f, p and C are functions of ψ only, they don't depend on l_θ. Assuming that $\eta_{||}$ is also constant on flux surfaces,

$$-f\frac{dp}{d\psi}\oint 1/B_\theta dl_\theta + C(\psi)\oint B^2/B_\theta dl_\theta - \oint\frac{B_\phi E_\phi}{\eta_{||}B_\theta}dl_\theta = 0$$

Constraint on $C(\psi)$

At this point, it's useful to divide through by $\oint dl_\theta$. This is just the distance around the flux-surface, and means that each term can be written as averages around a flux surface:

$$\boxed{\langle x \rangle \equiv \frac{\oint x dl_\theta}{\oint dl_\theta}}$$

We can therefore write:

$$-f\frac{dp}{d\psi}\langle 1/B_\theta\rangle + C(\psi)\left\langle B^2/B_\theta\right\rangle - \frac{\langle B_\phi E_\phi/B_\theta\rangle}{\eta_{||}}$$

and so

$$C(\psi) = f\frac{dp}{d\psi}\frac{\langle 1/B_\theta\rangle}{\langle B^2/B_\theta\rangle} + \frac{\langle B_\phi E_\phi/B_\theta\rangle}{\eta_{||}\langle B^2/B_\theta\rangle}$$

Pfirsch-Schlüter current

This therefore gives the parallel current:

$$J_{||} = \underbrace{-f\frac{dp}{d\psi}\left(\frac{1}{B} - \frac{\langle 1/B_\theta\rangle B}{\langle B^2/B_\theta\rangle}\right)}_{\text{Pfirsch-Schlüter current}} + \underbrace{\frac{\langle B_\phi E_\phi/B_\theta\rangle B}{\eta_{||}\langle B^2/B_\theta\rangle}}_{\text{Induced current}}$$

This parallel current is a combination of the current induced by an applied toroidal field E_ϕ, and the **Pfirsch-Schlüter current**

$$J_{||}^{PS} = -f\frac{dp}{d\psi}\left(\frac{1}{B} - \frac{\langle 1/B_\theta\rangle B}{\langle B^2/B_\theta\rangle}\right)$$

This current doesn't provide force balance since it is parallel to $\underline{B}$, but modifies the equilibrium. It is a neoclassical effect because it doesn't appear in a cylinder.

Other currents

There can be other currents in a tokamak, but:

- They *must* flow parallel to the magnetic field. The diamagnetic current is the only current that can flow across field lines
- The current density must be divergence free. From earlier, $\underline{B}\cdot\nabla\left(J_{||}/B\right) = 0$ and so

$$\underline{J} = C\left(\psi\right)\underline{B}$$

  Taking dot-product with $\underline{B}$, $\underline{J}\cdot\underline{B} = C\left(\psi\right)B^2$. The poloidal average of this

$$\left\langle\underline{J}\cdot\underline{B}\right\rangle = C\left(\psi\right)\left\langle B^2\right\rangle$$

$$\Rightarrow \underline{J} = \frac{\left\langle\underline{J}\cdot\underline{B}\right\rangle}{\left\langle B^2\right\rangle}\underline{B}$$

Banana current

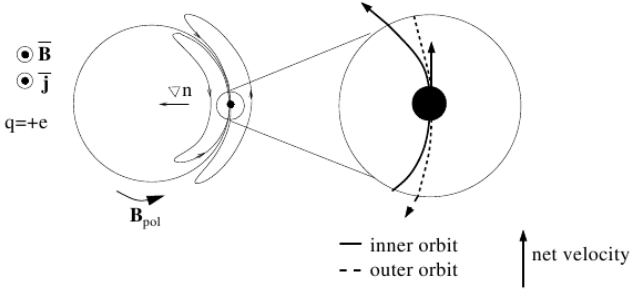
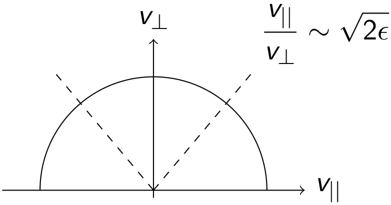


Figure: Banana current: Samuli Saarelma's PhD thesis, 2005

- Consider two banana orbits which are touching (shown above)
- If there is a density gradient, then there are more particles on the inner orbit than the outer
- There is therefore a net flow of trapped particles

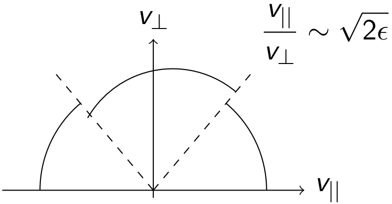
We can visualise this by looking at the distribution function of the particles

- In a uniform plasma particles are distributed in velocity space on the unit circle



We can visualise this by looking at the distribution function of the particles

- In a uniform plasma particles are distributed in velocity space on the unit circle
- If there is a density gradient then at a particular location there are more trapped particles going one way than the other



This gives rise to a current $J_t \simeq \Delta n v_{||} e$. Since the electrons drift in opposite direction to ions, the contribution adds together.

Banana current

- Parallel velocity of trapped particles $v_{||} \simeq \sqrt{\epsilon} v_{th}$
- What is the difference in density between two banana orbits? Depends on the gradient of the number of trapped particles $\sqrt{\epsilon} n$

$$\Delta n \simeq \delta r_b \frac{d}{dr}\left(\sqrt{\epsilon} n\right)$$

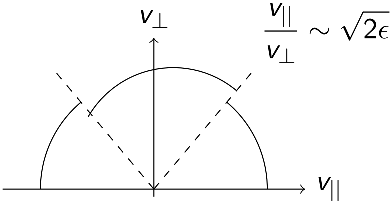
- The current driven along the field-lines is therefore

$$J_t \simeq e \delta r_b \sqrt{\epsilon} \frac{dn}{dr} \sqrt{\epsilon} v_{th} \sim e \frac{\sqrt{\epsilon} v_{th}^2}{\Omega} \frac{dn}{dr} \sim \epsilon^{3/2} \frac{T}{B_\theta} \frac{dn}{dr}$$

- Here we have assumed no temperature gradient, but this will also contribute to the banana current: If particles are moving faster on the inner banana orbit than on the outer orbit, then this will also give rise to a current.

Bootstrap current

- Passing particles also drift across flux surfaces, but unlike trapped particles there is no ambiguity in their radial position. $\Rightarrow$ do not directly contribute to the parallel current
- However if we look again at the distribution function:



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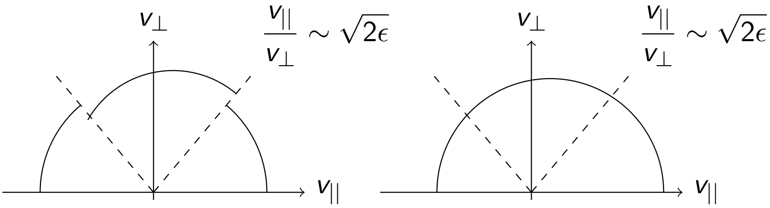

  there is a discontinuity at the trapped-passing boundary
- Collisions are a diffusion in velocity, and transfer momentum between trapped and passing particles
- Collisions smooth out the distribution function, resulting in a transfer of momentum from trapped to passing particles

This results in a current called the **bootstrap current**

Bootstrap current

- We can estimate the bootstrap current J_b by considering the momentum transfer between trapped and passing particles. The rate is given by the effective collision frequency

$$\nu_{eff} \simeq \nu/\epsilon$$

- The rate of transfer of momentum from trapped to passing particles $\sim \frac{\nu}{\epsilon} J_t$
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- Collisions between passing particles provides a friction $\sim \nu J_b$
- Balancing the momentum transfer against the friction gives

$$\nu J_b \sim \nu/\epsilon J_t \qquad \Rightarrow J_b \sim \sqrt{\epsilon}\frac{T}{B_\theta}\frac{dn}{dr}$$

Hence J_b is roughly R/r times larger than the banana current

- Collisions play a crucial role, but do not appear in the final result provided that ν_* is small enough

Bootstrap current

The full calculation results in an expression

$$\left\langle \underline{J}_b \cdot \underline{B} \right\rangle = \sqrt{2\epsilon} f\left(\psi\right) p\left(\psi\right)\left[\frac{a_1}{n}\frac{\partial n}{\partial \psi} + \frac{a_2}{T_e}\frac{\partial T_e}{\partial \psi} + \frac{a_3}{T_i}\frac{\partial T_i}{\partial \psi}\right]$$

where the constants a_1, a_2 and a_3 are complicated functions of geometry and collisionality.

- Note that $\frac{\partial n}{\partial \psi} \simeq \frac{1}{RB_\theta}\frac{\partial n}{\partial r}$, and $p\left(\psi\right)/n = T_e + T_i$. The first term gives a similar expression to our approximation
- Even though collisions are crucial to the bootstrap current, it is quite insensitive to collision frequency provided it's not zero or too high
- At high collisionality, there are no trapped particles and so $\left\langle \underline{J}_b \cdot \underline{B} \right\rangle \rightarrow 0$

Importance of the bootstrap current

The bootstrap current is present in all (hot) tokamak plasmas, and is driven by density and temperature gradients

- This current exists independently of any other current drive, and provides some of the poloidal field
- This ability of the plasma to generate its own poloidal field and so "lift itself up by the bootstraps" gave rise to the name
- In Advanced Tokamak (AT) scenarios, the bootstrap current provides the majority of the current e.g. ITER AT $\simeq$ 70%. JT-60 has been operated with 80% bootstrap fraction[1]
- This current is vital for economic steady-state operation, as it greatly reduces the current which must be driven externally

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- This current is vital for economic steady-state operation, as it greatly reduces the current which must be driven externally
- Bootstrap current also has a bad side: in steep pressure gradient regions at the plasma edge, the bootstrap current can drive instabilities (peeling modes)

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Summary

- The only current which flows perpendicular to $\underline{B}$ is the **diamagnetic current** $\underline{J}_\perp^{DIA}$. This satisfies $\underline{J}_\perp^{DIA} \times \underline{B} = \nabla P$
- This has non-zero divergence, and so there must be a parallel current: the **Pfirsch-Schlüter current**. This can be thought of as the return current for the net vertical current needed to balance the hoop force
- Trapped particle (banana) orbits have a finite radial width. Radial density and temperature gradients therefore distort distribution functions, leading to the **banana current**
- Collisions transfer momentum from trapped to passing particles, leading to the **bootstrap current** which is $\sim R/r$ times larger than the banana current
- This bootstrap current can provide the majority of the toroidal current needed to produce the poloidal magnetic field in a tokamak