Magnetic mirrors and pressure-driven instabilities

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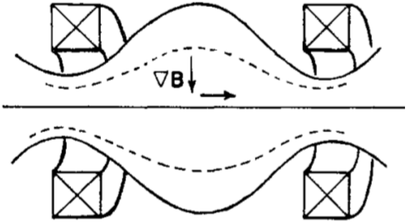
Plasma stability

- The early history of Magnetic Confinement Fusion was the search for configurations which are stable on a large scale
- Last lecture we looked at instabilities driven by plasma currents (kinks)
- One of the important parameters for achieving economical fusion power is plasma beta $\beta = \frac{2\mu_0 p}{B^2}$, the ratio of plasma to magnetic pressure
- Pressure-driven instabilities often set the limits on performance, so understanding and avoiding or mitigating them is important
- First we'll look in more detail at magnetic mirrors...

Magnetic mirrors

- Magnetic mirrors confine particles by exploiting the conservation of magnetic moment $\mu = mv_\perp^2 / (2B)$.
- If the particle's kinetic energy is also conserved then $v_{||}^2 + v_\perp^2 =$const
- This leads to a condition that particles are reflected if

$$\frac{v_\perp}{v} > \sqrt{\frac{B_{\min}}{B_{\max}}}$$



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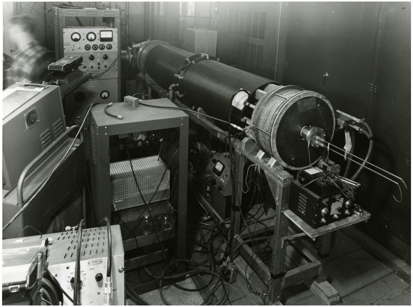
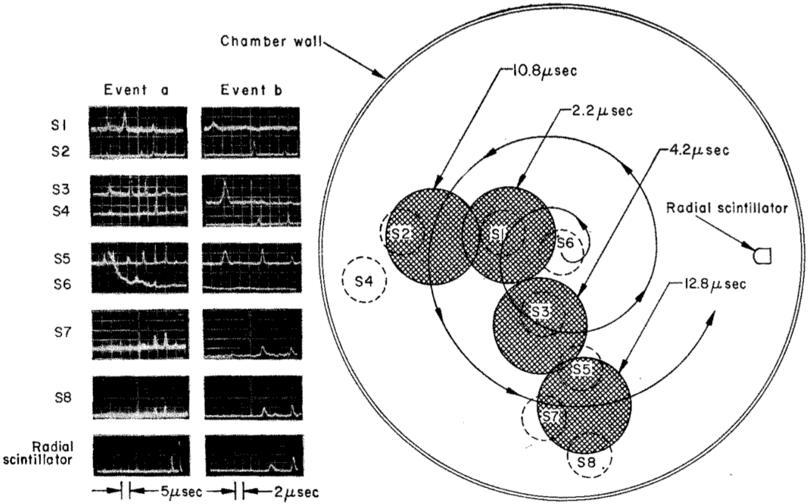


Cucumber (or "Q-cumber"), built around 1954. UCRL / LLNL.

Magnetic mirror instability



[W. A. Perkins and R. F. Post, Physics of Fluids 6 (1963) 1537]

- In a curved magnetic field particles drift. Since $\Omega = qB/m$, the sign is different for electrons and ions.
- If there is a perturbation to the plasma, then this leads to charge separation and an electric field
- Depending on the direction of the curvature, the resulting $\mathbf{E} \times \mathbf{B}$ drift can either reduce or enhance the original perturbation

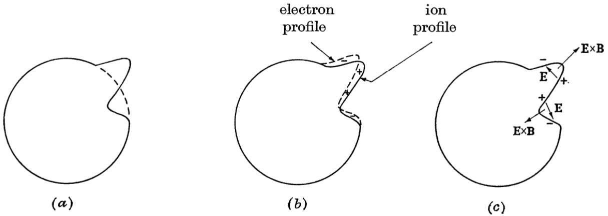


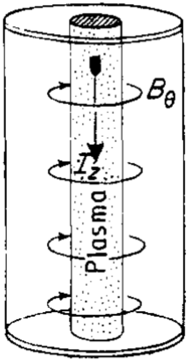
FIGURE 2. Development of flute instability. (a) Initial disturbance. (b) Effect of ion and electron azimuthal drifts. (c) Resulting $E \times B$ drifts increase amplitude.

[J.B.Taylor "Plasma Containment and Stability Theory" Proc. Royal Soc. A Vol 304, No. 1478 (1968) pp 335-360]

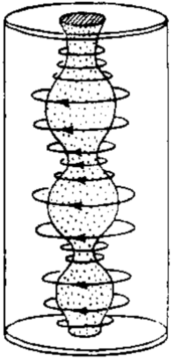
Interchange instability in a Z-pinch

The same process occurs in Z-pinches

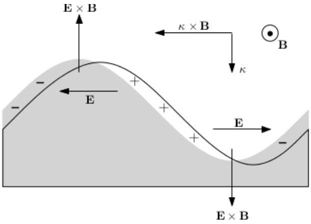
- Most unstable mode has $m = 0$
- Known as a **sausage** instability



Equilibrium pinch



Sausage instability



Interchange instability

- Configurations where the magnetic field curves towards the plasma are unstable to interchange modes. This is known as **bad curvature**.
- If we can reverse the sign of the curvature, then the drift reverses:

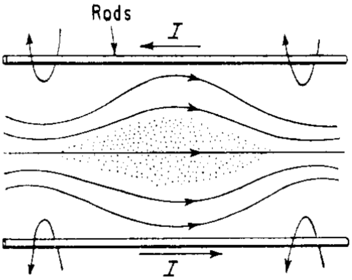
$$\underline{v}_R = \frac{v_{||}^2}{\Omega}\frac{\underline{R}_C \times \underline{B}}{R_C^2 B} = -\frac{v_{||}^2}{\Omega}\underline{\kappa} \times \underline{b}$$

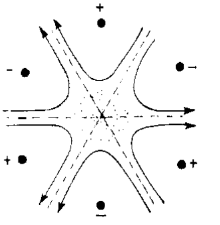
- Configurations where magnetic fields curve away from the plasma have **good curvature**, and are stable to interchange modes.
- Configurations with good curvature also have the property that the magnetic field increases in all directions from the plasma: a **minimum B** or **magnetic well**.

[J.D.Jukes "Plasma stability in magnetic traps I" Rep. Prog. Phys. 30 (1967) 333]

Configurations which have good curvature everywhere are possible, but only with open magnetic field-lines.

- Coils carrying current in opposite directions produce a null region where the fields cancel, with $|B|$ increasing in all directions
- By combining with a mirror field, these "Ioffe bars"[1] can create a trap with a minimum in $|B|$





Field lines from six bars

[1] Proposed by M.S.Ioffe, Kurchatov Institute, 1962

Stabilised magnetic mirrors

- One disadvantage of these coils is that the magnetic field is not axisymmetric, which leads to poorer particle confinement
- Better performance can be achieved by using minimum B "anchors" at the end of axisymmetric straight sections
- Configurations known as Baseball and Yin-Yang coils

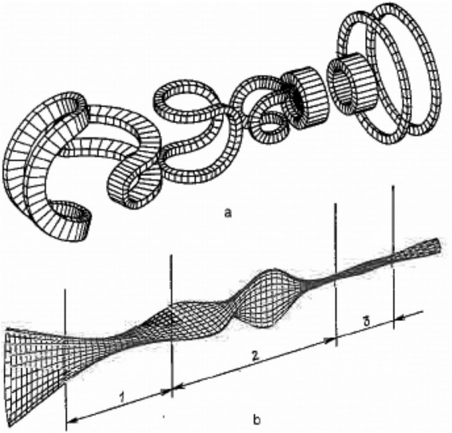


FIG. 2. The quadrupole mirror system of the MFTF-B facility: (a) the magnet system; (b) one of the flux surfaces. Zone 1 is the MHD-stable "anchor," and zone 3 is an almost axisymmetric ambipolar "plug," whereas zone 2 is a transition region; the central solenoid begins at the right upper corner and is terminated at the opposite end by the same complex system reversed left to right and rotated by 90° around the magnetic axis.

[D.D. Ryutov *et al* Phys. Plasmas 18, 092301 (2011)]

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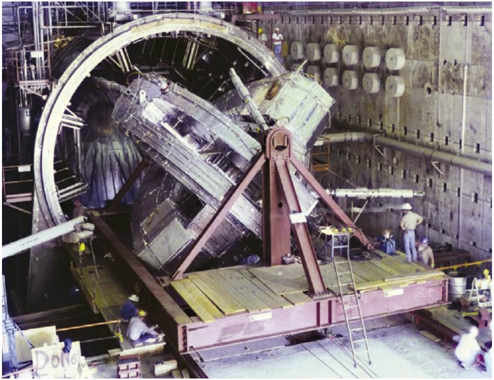
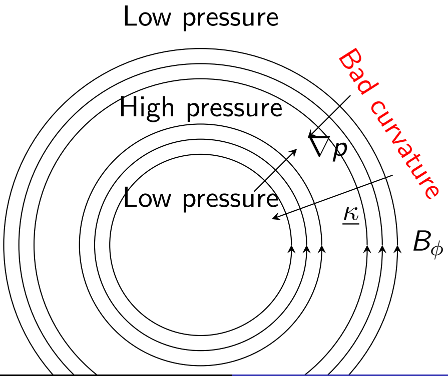


MFTF-B under construction, ca. 1985

[D.D. Ryutov *et al* Phys. Plasmas 18, 092301 (2011)]

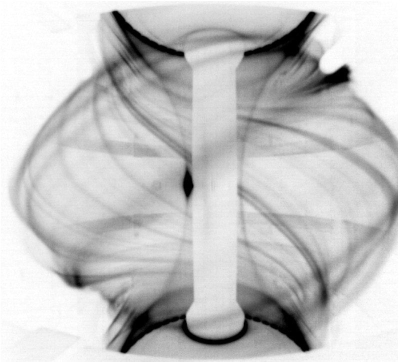
Tokamak stability

- In a tokamak, the toroidal field is curved and the pressure is highest in the core
  $\Rightarrow$ on the **outboard side** (large R) the curvature is bad, whilst on the **inboard side** (small R) the curvature is good
- Many tokamak instabilities have maximum amplitudes on the outboard side, called **ballooning** type modes



Edge Localised Modes

- In a tokamak the field is on average minimum B, so flute interchange is stable if $q > 1$ (Mercier criterion).
- Unfortunately this doesn't mean that tokamaks are immune from pressure-driven instabilities
- During high-performance mode (H-mode), steep gradients form close to the plasma edge.



- These collapse quasi-periodically in eruptions called **Edge Localised Modes** (ELMs).
- Leading theory is peeling-ballooning modes (Connor, Hastie, Wilson)
- Pressure-driven (ballooning) and edge current (peeling)

Energy and plasma stability

- To calculate whether a plasma is unstable we need to consider both sources of instability such as pressure gradients, but also stabilising effects
- Last lecture we saw that bending field-lines produced a force $\underline{F}$ which opposed the motion i.e $\underline{F} \cdot \underline{v} < 0$
- This means that the instability is having to do work to bend the field-lines
- To be unstable, the energy available has to be greater than the energy needed to overcome this force

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Analogous to a ball on a hill

Energy and plasma stability

The difference between stable and unstable situations is the change in potential energy δW due to a small perturbation

- $\delta W < 0$ **Unstable**: potential energy converted to kinetic energy
- $\delta W > 0$ **Stable**: kinetic to potential, then oscillates
- $\delta W = 0$ **Marginal**: Like a ball on a flat surface

A plasma is stable if $\delta W > 0$ for **all** possible perturbations, and unstable if **any** perturbation results in $\delta W < 0$

To calculate δW we'll use the ideal MHD equations...

To calculate the change in energy from a small perturbation we need to first linearise the equations:

$$n = n_0 + \epsilon n_1 \qquad \underline{v} = \underline{v}_0 + \epsilon \underline{v}_1 \qquad \ldots$$

which after substituting into the ideal MHD equations, and assuming a stationary equilibrium $\underline{v}_0 = 0$ gives:

$$\begin{aligned}
\frac{\partial}{\partial t} n_1 &= -n_0 \nabla \cdot \underline{v}_1 - \underline{v}_1 \cdot \nabla n_0 \\
\frac{\partial}{\partial t} \underline{v}_1 &= \frac{1}{m_i n_0}\left[-\nabla p_1 + \frac{1}{\mu_0}\left(\nabla \times \underline{B}_1\right) \times \underline{B}_0 + \frac{1}{\mu_0}\left(\nabla \times \underline{B}_0\right) \times \underline{B}_1\right] \\
\frac{\partial}{\partial t} p_1 &= -\gamma p_0 \nabla \cdot \underline{v}_1 - \underline{v}_1 \cdot \nabla p_0 \\
\frac{\partial}{\partial t} \underline{B}_1 &= \nabla \times \left(\underline{v}_1 \times \underline{B}_0\right)
\end{aligned}$$

Ideal MHD dispacement

In ideal MHD all perturbed quantities n_1, $\underline{v}_1$, p_1, and $\underline{B}_1$ can be written in terms of a single **displacement** $\underline{\xi}(\underline{x})$ which is the distance the fluid has moved from equilibrium.

- The velocity is $\underline{v}_1 = \frac{\partial \underline{\xi}}{\partial t}$
- Substitute this into the other equations:

$$\frac{\partial}{\partial t} n_1 = -n_0 \nabla \cdot \frac{\partial \underline{\xi}}{\partial t} - \frac{\partial \underline{\xi}}{\partial t} \cdot \nabla n_0$$

  Since the equilibrium quantities do not depend on time, this can be integrated trivially:

$$\Rightarrow n_1 = -n_0 \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla n_0$$

- Similarly:

$$\begin{aligned} p_1 &= -p_0 \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla p_0 \\ \underline{B}_1 &= \nabla \times \left(\underline{\xi}_1 \times \underline{B}_0 \right) \end{aligned}$$

Note: This only works for ideal MHD: resistivity breaks this

Ideal MHD normal modes

Substituting into the equation for velocity gives:

$$m_i n_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \nabla \underbrace{\left(\underline{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{\xi}\right)}_{-p_1} + \frac{1}{\mu_0}\left(\nabla \times \underline{B}_1\right) \times \underline{B}_0 + \frac{1}{\mu_0}\left(\nabla \times \underline{B}_0\right) \times \underline{B}$$

This is a linear operator

$$m_i n_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = F\left(\underline{\xi}\right) \qquad \text{Fancy form of } m\underline{a} = \underline{F}$$

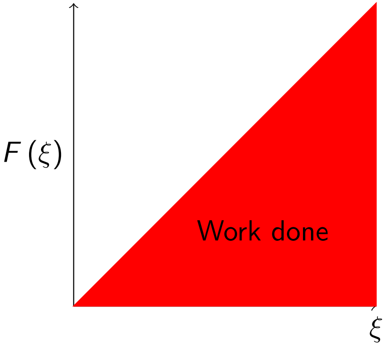
Solutions are linear $\underline{\xi}\left(\underline{x}, t\right) = \underline{\xi}\left(\underline{x}\right) e^{-i\omega t}$ and include the shear Alfvén and magnetosonic waves.
Hence we can write an eigenvalue equation with eigenfunction $\underline{\xi}$

$$-m_i n_0 \omega^2 \underline{\xi} = F\left(\underline{\xi}\right)$$

Ideal MHD energy principle

To calculate the work done on the plasma, we need force times distance

- When $\xi = 0$, $F(\xi) = 0$ (an equilibrium)
- $F(\xi)$ is linear in ξ by construction
- A more formal proof can be made. See handout and textbook e.g. Freidberg "Ideal MHD".



$$\boxed{\delta W = -\frac{1}{2}\int d^3x F\left(\underline{\xi}\right)\cdot\underline{\xi}}$$

Ideal MHD energy equation (intuitive form)

The plasma contribution δW_p can be rearranged:

$$
\begin{aligned}
\delta W_p = & \\
\frac{1}{2}\int d^3x\Bigg[\quad & \frac{\left|\underline{B}_1\right|^2}{\mu_0} \quad \text{Field-line bending} \geq 0 \\
& + \frac{B^2}{\mu_0}\left|\nabla\cdot\underline{\xi}_\perp + 2\underline{\xi}_\perp\cdot\kappa\right|^2 \quad \text{Magnetic compression} \geq 0 \\
& + \gamma p_0\left|\nabla\cdot\underline{\xi}\right|^2 \quad \text{Plasma compression} \geq 0 \\
& - 2\left(\underline{\xi}_\perp\cdot\nabla p\right)\left(\underline{\kappa}\cdot\underline{\xi}_\perp^*\right) \quad \text{Pressure/curvature drive, + or} - \\
& - \underline{B}_1\cdot\left(\underline{\xi}_\perp\times\underline{b}\right)j_{||}\Bigg] \quad \text{Parallel current drive, + or} -
\end{aligned}
$$

This is a very useful form of the energy equation because it makes clear the balance between destabilising and stabilising effects

Instability drives

The first three terms in this equation are always ≥ 0 and so are stabilising, but the last two can be positive (stabilising) or negative (destabilising):

- $-2\left(\underline{\xi}_{\perp}\cdot\nabla p\right)\left(\underline{\kappa}\cdot\underline{\xi}_{\perp}^{*}\right)$ depends on ∇p and κ: if $\nabla p\cdot\kappa > 0$ then this is destabilising. Instabilities driven by this term are often called pressure-driven
  $\rightarrow$ This is the interchange instability drive we saw earlier.
- $-\underline{B}_1\cdot\left(\underline{\xi}_{\perp}\times\underline{b}\right)j_{||}$ depends on the parallel current $j_{||}$ and leads to parallel current-driven kink modes

Compression

The term $\gamma p_0 \left|\nabla \cdot \underline{\xi}\right|^2$ represents compression of plasma

- The only place in this equation where the parallel displacement $\xi_{||} \equiv \underline{b} \cdot \underline{\xi}$ enters explicitly is this term
- Therefore, we can choose $\epsilon_{||}$ to minimise $\nabla \cdot \underline{\xi}$
- For a fluid or plasma motion parallel to $\underline{B}$,

$$\left|\nabla \cdot \underline{v}\right| \sim M_S^2 \frac{v}{L}$$

  close to marginal stability where L is a typical length and M_S is the Mach number
- Perpendicular to the field, a similar expression applies, but with the Alfvénic Mach number:

$$\left|\nabla \cdot \underline{v}\right| \sim M_A^2 \frac{v}{L}$$

$\Rightarrow$ Close to marginal stability, plasma instabilities tend to be incompressible

$\frac{\left|\underline{B}_1\right|^2}{\mu_0}$ is the energy which goes into bending field-lines, and is always stabilising. The perturbed magnetic field $\underline{B}_1$ is given by

$$\underline{B}_1 = \nabla\times\left(\underline{\xi}_1 \times \underline{B}_0\right) = \underline{\xi}\underbrace{\left(\nabla\cdot\underline{B}_0\right)}_{=0} - \underline{B}_0\left(\nabla\cdot\underline{\xi}\right) + \left(\underline{B}_0\cdot\nabla\right)\underline{\xi} - \left(\underline{\xi}\cdot\nabla\right)\underline{B}_0$$

Assuming we're already minimising the compression, neglect the $\nabla\cdot\underline{\xi}$ term.

$$\Rightarrow \text{look for modes which minimise } \left(\underline{B}_0\cdot\nabla\right)\underline{\xi} - \left(\underline{\xi}\cdot\nabla\right)\underline{B}_0$$

Field-line bending

Trying to minimise $\left(\underline{B}_0 \cdot \nabla\right)\underline{\xi} - \left(\underline{\xi} \cdot \nabla\right)\underline{B}_0$
Consider a perturbation of the form

$$\underline{\xi}\left(r, \theta, \phi\right) = \hat{\underline{\xi}}\left(r\right) e^{i\left(m\theta - n\phi\right)}$$

The first of these terms $\left(\underline{B}_0 \cdot \nabla\right)\underline{\xi}$ can be written in a cylinder (large aspect-ratio tokamak) as:

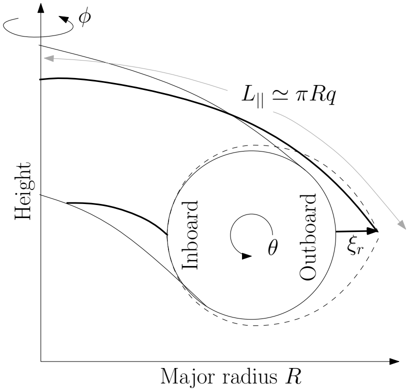
$$\left(\underline{B}_0 \cdot \nabla\right)\underline{\xi} = \left[\frac{B_\theta}{r}\frac{\partial}{\partial\theta} + \frac{B_\phi}{R}\frac{\partial}{\partial\phi}\right]\underline{\xi} = i\left[m\frac{B_\theta}{r} - n\frac{B_\phi}{R}\right]\underline{\xi}$$

rearranging:

$$\left(\underline{B}_0 \cdot \nabla\right)\underline{\xi} = i\frac{B_\phi}{R}\left(\frac{m}{q} - n\right) \quad \text{where} \quad \boxed{q = \frac{rB_\phi}{RB_\theta}}$$

This is minimised when $q \simeq m/n$ so instabilities tend to localise around resonant surfaces

Using the ideal MHD energy equation, we can estimate the pressure limit for a **ballooning mode**:



Field-line bending so mode is maximum on the outboard side, minimum on inboard side. Consider case when parallel bending dominates:

$$\frac{|B_1|^2}{2\mu_0} \simeq \frac{|B_0 \cdot \nabla \xi_r|^2}{2\mu_0}$$

Using the length along field-lines,

$$\frac{|B_1|^2}{2\mu_0} \sim \frac{\left|B_0 \xi_r / L_{||}\right|^2}{2\mu_0} = \frac{B_0^2}{2\mu_0} \frac{\xi_r^2}{\pi^2 q^2 R^2}$$

This gives the energy (density) needed to bend field-lines

For ballooning modes to be stable, the energy available from the pressure gradient has to be less than this field-line bending i.e.

$$\left(\xi\cdot\nabla p\right)\left(\underline{\kappa}\cdot\xi_\perp^*\right) < \frac{B_0^2}{2\mu_0}\frac{\xi_r^2}{\pi^2 q^2 R^2}$$

Taking $\underline{\xi}$ at the outboard midplane (maximum perturbation), ∇p and κ are both in the direction of $\underline{\xi}_r$. Therefore,

$$\left(\xi\cdot\nabla p\right)\left(\underline{\kappa}\cdot\xi_\perp^*\right) \simeq \xi_r^2\kappa\frac{dp}{dr}$$

For stability then:

$$\xi_r^2\kappa\frac{dp}{dr} < \frac{B_0^2}{2\mu_0}\frac{\xi_r^2}{\pi^2 q^2 R^2}$$

Since $\kappa\sim -1/R$, this becomes

$$-\frac{dp}{dr} < \frac{B_0^2}{2\mu_0}\frac{1}{\pi^2 q^2 R}$$

Beta limits: Ballooning modes

We can use this to get a beta limit by setting $\frac{dp}{dr} \sim -p_0/r$ where p_0 is the core pressure and r the minor radius.

$$\frac{\mu_0 p_0}{B_0^2} = \beta < \frac{1}{2\pi^2}\frac{r}{qR}$$

If $q \simeq 2$, $r/R = \epsilon \simeq 1/3$ this gives a limit of $\beta \sim 0.5\%$

- This is in the right ballpark for conventional tokamaks: a couple of percent is quite typical
- More important than a global *beta* limit is the effect ELMs have on divertor power loads
- A "natural" ELM on ITER is predicted to produce 20MW/m^2, and must be reduced by a factor of ~ 20 for acceptable component lifetimes
- Research ongoing into means of controlling these events

Summary

- Pressure driven instabilities can be destabilised when $\kappa \cdot \nabla p > 0$ (bad curvature regions)
- Plasma compression is always stabilising, so tends to be minimised close to marginal stability
- To minimise parallel field bending, modes tend to be localised around resonant surfaces $q = m/n$
- Interchange modes are constant along B to minimise field-line bending, but are usually stable in tokamaks. Exceptions are in the SOL and if $q < 1$.
- In tokamaks, ballooning modes have some variation along B so that they can maximise their amplitude in the bad curvature region