Resistive instabilities

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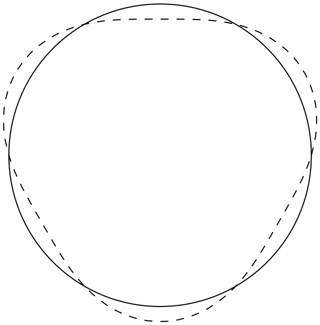
Instability overview

- Instabilities are often classified based on whether the boundary of the plasma moves:
  - **Internal** modes only move a region inside the plasma, and the boundary remains stationary
  - **External** modes involve motion of the plasma boundary, and so usually the whole plasma. These tend to be more dangerous to plasma stability
- There are two commonly used terms for current-driven instablities:
  - **Kink** modes are ideal MHD instabilities which involve a motion or deformation of flux surfaces, and apart from $m = 1$ (internal kink) require motion of the boundary (external kink)
  - **Tearing** instabilites are resistive, and involve the tearing or breaking of magnetic flux surfaces.

Flux surface perturbations

It is possible to use the energy equation to derive kink stability, but the calculations are pretty long. Instead, consider perturbations $\delta\psi$ perpendicular to flux surfaces

$$\delta\underline{B} = \underline{\hat{\phi}} \times \nabla\delta\psi \qquad \mu_0\delta J_\phi = \nabla^2\delta\psi$$



In this case the change in B field is

$$\delta B_r = -\frac{1}{r}\frac{\partial\delta\psi}{\partial\theta} \qquad \delta B_\theta = \frac{\partial\delta\psi}{\partial r}$$

We're going to assume a sinusoidal perturbation, so all quantities of the form:

$$\boxed{\delta\psi = \delta\hat{\psi}e^{i(m\theta - n\phi)}}$$

Flux surface perturbations

We can look at the form this perturbation should take by starting with the equilibrium

$$\boxed{\underline{J} \times \underline{B} = \nabla p}$$

If we're close to marginal stability, then the growth-rate is small and so the force imbalance is also small. Hence this equation still holds and we can linearise it as

$$\delta \underline{J} \times \underline{B}_0 + \underline{J}_0 \times \delta \underline{B} = \nabla \delta p$$

Taking the curl of this equation gets rid of the pressure term and so

$$\nabla \times \left[\delta \underline{J} \times \underline{B}_0 + \underline{J}_0 \times \delta \underline{B}\right] = 0$$

This can be expanded out, and assuming that B and δB change slowly along B (large aspect-ratio) becomes

$$\left(B \cdot \nabla\right) \delta \underline{J} + \left(\delta \underline{B} \cdot \nabla\right) \underline{J}_0 \simeq 0$$

Flux surface perturbations

Taking the toroidal component of this equation gives:

$$\left(B\cdot\nabla\right)\delta J_\phi + \left(\delta \underline{B}\cdot\nabla\right)J_\phi \simeq 0$$

If we assume large aspect-ratio (cylinder), then J_ϕ only depends on minor radius r and so

$$\left(\delta\underline{B}\cdot\nabla\right)\underline{J}_\phi = \delta B_r\cdot\frac{dJ_\phi}{dr} = -\frac{1}{r}\frac{\partial\delta\psi}{\partial\theta}\frac{dJ_\phi}{dr}$$

The first term is

$$\left(B\cdot\nabla\right)\delta J_\phi = \left[B_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + B_\phi\frac{1}{R}\frac{\partial}{\partial\phi}\right]\delta J_\phi$$

Since all perturbations are $\propto \exp\left[i\left(m\theta - n\phi\right)\right]$, $\frac{\partial}{\partial\theta}\rightarrow im$ and $\frac{\partial}{\partial\phi}\rightarrow -in$

$$i\left[\frac{B_\theta m}{r} - \frac{B_\phi n}{R}\right]\frac{1}{\mu_0}\nabla^2\delta\psi - i\frac{m}{r}\delta\psi\frac{dJ_\phi}{dr} = 0$$

Cylindrical Tearing Mode equation

This rearranges using $q = \frac{B_\phi r}{B_\theta R}$ to give the **Cylindrical Tearing Mode equation**

$$\nabla^2 \delta\psi - \frac{\mu_0 \frac{dJ_\phi}{dr}}{B_\theta \left[1 - qn/m\right]} \delta\psi = 0$$

This equation gives the form of current-driven instabilities close to marginal stability and assumes:

- Close to marginal $\Rightarrow$ time derivatives and force imbalances are small. Plasma is incompressible.
- Large aspect ratio tokamak (cylindrical)
- No pressure effects, so low β plasmas

What happens when $q = m/n$? This is the resonance condition which minimises field-line bending

Cylindrical Tearing Mode equation

When $q = m/n$, the second derivative of $\delta\psi$ has goes to infinity. This corresponds to a perturbed current sheet

$$\delta J_\phi = \frac{1}{\mu_0}\nabla^2\delta\psi$$

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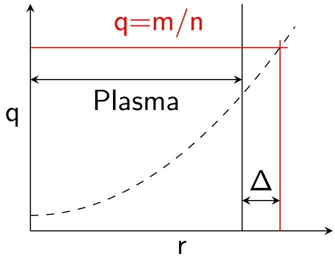
$\Rightarrow$ in ideal MHD, this magnetic surface must be outside the plasma (special exception is the $m = 1$ internal kink)



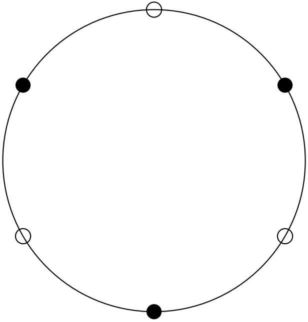
- The resonant q surface must be in the vacuum close to the plasma edge
- Since the plasma edge moves these are called **external kink** modes.
- Sensitive to boundary conditions and proximity of surface to plasma edge Δ
- Example is the **peeling mode**

Cylindrical Tearing Mode equation

In ideal MHD we are limited to either $m = 1$ internal kink, or external kinks with the resonant surface with $q = m/n$ just outside the plasma.

$$\delta J_\phi = \frac{1}{\mu_0}\nabla^2\delta\psi$$

If there is some resistivity, the current is spread over a finite region and δJ_ϕ can be finite



- On the surface where $q = m/n$ We have picked out particular field-lines and driven currents along them. This is called current **filamentation**.

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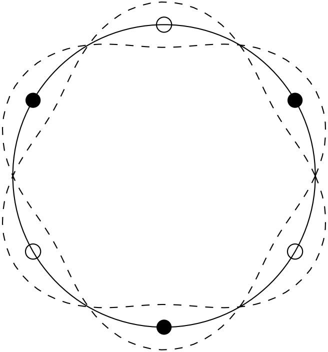
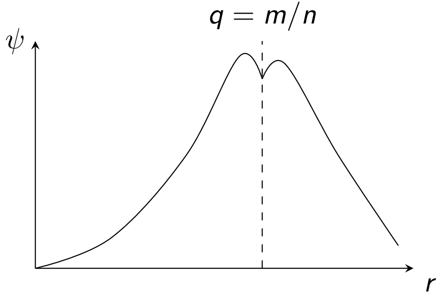


- On the surface where $q = m/n$ We have picked out particular field-lines and driven currents along them. This is called current **filamentation**.
- These perturbed currents produce a chain of magnetic islands
- This is called a **Tearing mode**
- Since this is a change in topology, it is forbidden in ideal MHD

Solving Cylindrical Tearing Mode equation



- The cylindrical Tearing mode breaks down at the resonant surface
- Instead, solve from inner and outer boundaries, matching the value of $\delta\psi$ at the resonant surface

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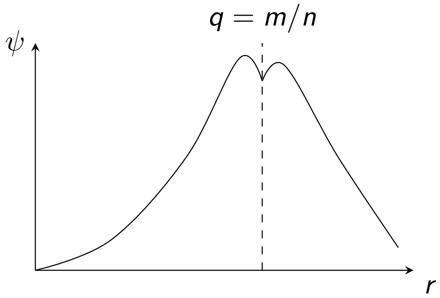


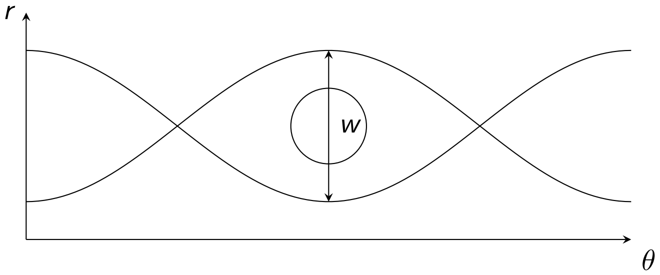
- The cylindrical Tearing mode breaks down at the resonant surface
- Instead, solve from inner and outer boundaries, matching the value of $\delta\psi$ at the resonant surface

- The jump in the gradient of $\delta\psi$ corresponds to currents on the resonant surface
- It is quantified by the **Tearing stability index**:

$$\boxed{\Delta' \equiv \left[\frac{1}{\delta\psi}\frac{\partial\delta\psi}{\partial r}\right]_-^+}$$

  where the limits are either side of the surface
- Δ' is related to the δW of a tearing mode and is $\boxed{\text{unstable if } \Delta' > 0}$

Mode width



- The growth of the island is limited by diffusion of the radial magnetic field which tears the flux surface into the island

$$\frac{\partial \delta B_r}{\partial t} \simeq \frac{\eta}{\mu_0} \nabla^2 \delta B_r$$

- These modes tend to be narrow in r compared with the θ or ϕ directions

$$\Rightarrow \nabla^2 \simeq \frac{\partial^2}{\partial r^2} \qquad \frac{\partial \delta B_r}{\partial t} \simeq \frac{\eta}{\mu_0} \frac{\partial^2 \delta B_r}{\partial r^2}$$

Mode width

Now integrate this over the width of the island w:

$$\int_{-w/2}^{+w/2} \frac{\partial \delta B_r}{\partial t} dr \simeq \frac{\eta}{\mu_0} \int_{-w/2}^{+w/2} \frac{\partial^2 \delta B_r}{\partial r^2} dr$$

Since we are looking at a sinusoidal perturbation $\delta\psi \propto \exp\left[i\left(m\theta - n\phi\right)\right]$:

$$\delta B_r = -\frac{1}{r}\frac{\partial \delta\psi}{\partial \theta} \qquad \Rightarrow \delta B_r = -i\frac{m}{r}\delta\psi$$

Provided the island is quite small, $\delta\psi$ and so δB_r doesn't change much across w, only it's gradient. Therefore,

$$\int_{-w/2}^{+w/2} \frac{\partial \delta B_r}{\partial t} dr \simeq w \frac{\partial \delta B_r}{\partial t}$$

This is called the **constant ψ approximation**.

Rutherford equation

The second term is just the change in the gradient of δB_r from one side of the island to the other:

$$\frac{\eta}{\mu_0}\int_{-w/2}^{w/2}\frac{\partial^2\delta B_r}{\partial r^2}dr = \frac{\eta}{\mu_0}\left[\frac{\partial\delta B_r}{\partial r}\right]_{-w/2}^{+w/2}$$

Therefore,

$$\int_{-w/2}^{+w/2}\frac{\partial\delta B_r}{\partial t}dr \simeq \frac{\eta}{\mu_0}\left[\frac{\partial\delta B_r}{\partial r}\right]_{-w/2}^{+w/2}$$

Using $\delta B_r = -im\delta\psi/r$ and $B_r \propto w^2$

$$\Rightarrow \frac{dw}{dt} \simeq \frac{\eta}{2\mu_0}\left[\frac{1}{\delta\psi}\frac{\partial\delta\psi}{\partial r}\right]_{-w/2}^{+w/2}$$

This is the **Rutherford equation** (1973). This is $\dfrac{dw}{dt} \simeq \dfrac{\eta}{2\mu_0}\Delta'$ so if $\Delta' > 0$ then the mode will grow

Summary of current-driven modes

- $m = 1$ is a special case because it's a shift of the flux surfaces. **Internal kinks** Can occur when $q < 1$ in the plasma
- **External kinks** where the plasma boundary moves are possible when the resonant q surface is just outside the plasma
- In the presence of resistivity, flux surfaces can be torn through resistive diffusion
- This leads to **tearing modes** governed by Δ'
- The stability of all these modes depends on
  - Boundary conditions e.g. the wall of the vessel. This will be important for locked modes and Resistive Wall Modes
  - Plasma profiles: Solution of the CTM depends on J'_ϕ across entire profile, not just local (unlike e.g. Interchange modes).
  - Locations of resonant q surfaces
- Low m, n numbers are most favourable as they minimise field-line bending. Commonly observed are:
  - $q = 1 : \; m, n = 1, 1$ Sawteeth instabilities
  - $q = 2 : \; m, n = 2, 1$ tearing mode
  - $q = 3/2 : \; m, n = 3, 2$ tearing mode

Complications

- A more detailed analysis of tearing modes shows that only 2,1 modes should be unstable except in "pathological" cases.
- *But* 3,2 modes are common in high temperature plasmas. What is causing these?

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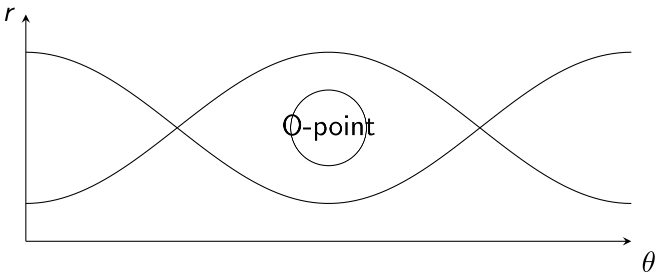
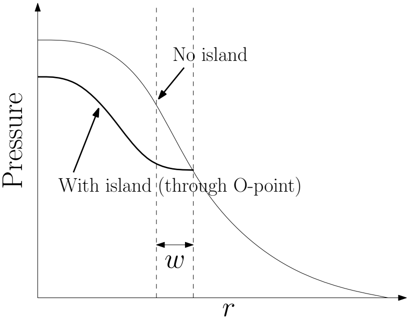
Look at what happens to the plasma with a tearing mode island



Transport of heat and particles is very fast along field-lines, so pressure will be equalised around the island flux surfaces.

Profile modification



- This means that if we plot the pressure through the O-point (widest point) of the island, we'll see a flat region
- This is bad news for confinement, as this part of the plasma is now providing little insulation of the core

Bootstrap current reminder

In lecture 6 we looked at Neoclassical currents. We saw that the bootstrap current is driven by a density or temperature gradient through collisions between trapped and passing particles. We derived the expression

$$\nu J_b \sim \nu/\epsilon J_t \qquad \Rightarrow J_b \sim \sqrt{\epsilon}\frac{T}{B_\theta}\frac{dn}{dr}$$

and in general the expression is a more complicated function of density and temperature. They are all approximately

$$J_b \sim \frac{\sqrt{\epsilon}}{B_\theta}\frac{dp}{dr}$$

Hence as the pressure gradient is modified, so is the bootstrap current. This introduces an interaction between plasma pressure and current-driven instabilities.

Modified Rutherford equation

Ohm's law is modified as the bootstap current needs no electric field to drive it

$$\delta E_{||} = \eta \left(\delta j_{||} - \delta j_{||b}\right)$$

This modifies the Rutherford equation to

$$\frac{dw}{dt} \simeq \frac{\eta}{2\mu_0}\left[\Delta' - \alpha \frac{\sqrt{\epsilon}}{w\left(r_s/a\right)}\frac{d\beta_p}{dr}\right]$$

where $\beta_p = \frac{\mu_0 p}{\left(\epsilon B_\theta\right)^2}$ is the poloidal beta.

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- α is an $O(1)$ constant, > 0 and so always destabilising
- Note that as $w \rightarrow 0$, $\frac{dw}{dt} \rightarrow \infty$
- The bootstrap current should drive all tearing modes unstable. Tokamaks shouldn't work at all!

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- The bootstrap current should drive all tearing modes unstable. Tokamaks shouldn't work at all!
- There must be another effect which prevents this

Incomplete flattening

The most popular theory[1] is that small islands do not lead to flattening of the pressure profile.

- The distance $L_{||}$ along a field-line from one side of an island is bigger than the perpendicular distance across the island $L_{\perp}$

$$L_{||} \sim \frac{r_s^2}{\epsilon w} \gg L_{\perp} = w$$

  where r_s is the minor radius of the magnetic island
- As the island gets smaller, $L_{||}$ gets longer.
- At a critical island size w_c parallel and perpendicular conduction are similar, and the pressure is **incompletely flattened**

$$\chi_{||}\nabla_{||}^2 T \sim \chi_{\perp}\nabla_{\perp}^2 T \qquad \rightarrow \frac{w_c}{r_s} \sim \left(\frac{\chi_{\perp}}{\epsilon^2\chi_{||}}\right)^{1/4}$$

- Therefore, small islands do not affect the bootstrap current and there is a minimum size for islands to grow.

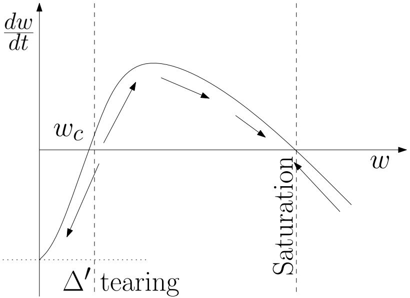
[1] R.Fitzpatrick, 1995

Modified Rutherford equation

Incomplete flattening modifies the Rutherford equation to something of the form:

$$\frac{dw}{dt} \simeq \frac{\eta}{2\mu_0}\left[\Delta' - \frac{\alpha}{r_s}\frac{\sqrt{\epsilon}\left(w/r_s\right)}{\left(w_c/r_s\right)^2 + \left(w/r_s\right)^2}\frac{d\beta_p}{dr}\right]$$

When w is small, this limits the effect of the bootstrap current on tearing modes. Including some additional nonlinear effects gives an equation of this form:



- At small w stability is determined by Δ' (usually stable)
- Above w_c profiles are flattened, mode grows
- At some amplitute the mode saturates

As the pressure (β) is increased, the critical size for an NTM to grow gets smaller. At some point a sufficient "kick" will occur to start the mode

- First observed on TFTR ("supershot" scenarios), now observed in all high-performance tokamaks
- Common triggers are sawteeth, fishbones, and ELMs
- Tend to be what limits β

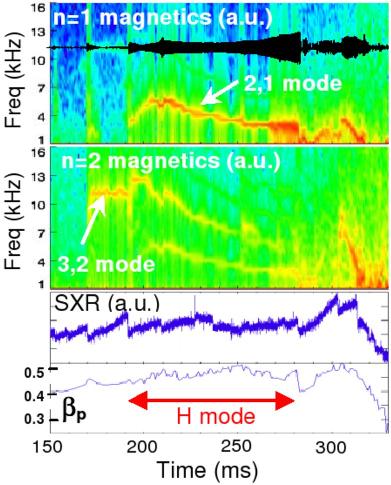


Figure: MAST shot 2952

Beta limits: Normalised beta and the Troyon limit

From ballooning, we have the beta limit $\beta \sim r/\left(q^2 R\right)$. Using $q = \dfrac{rB_\phi}{RB_\theta}$ and $B_\theta \sim I_\phi / r$ where I_ϕ is the toroidal current this becomes:

$$\beta \sim \frac{r}{R}\frac{R^2\left(I_\phi/r\right)^2}{r^2 B_\phi^2} = \left(\frac{I_\phi}{rB_\phi}\right)^2 \frac{R}{r}$$

- Experiments and more careful calculations find a linear dependence on $I_\phi / \left(rB_\phi\right)$
- The quantity $\beta_N \equiv \dfrac{\beta\left(\%\right)}{I_\phi/\left(rB_\phi\right)}$ is called the **Normalised Beta**
- Based on simulations, the limit $\beta_N < 2.8$ is called the **Troyon limit**
- Further optimisation of current profile (l_i)

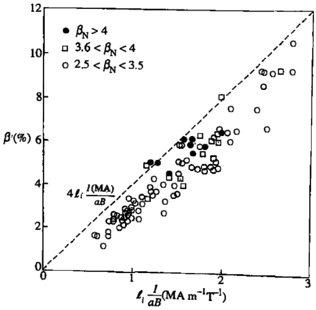


Figure: Wesson 16.6.4

Interaction between NTM and walls

- Like external kinks, NTMs can be stabilised (to a degree) by machine walls
- If an instability is rotating at a frequency Ω then the $\underline{B}$ field at the wall will reverse direction on a timescale $\tau = 2\pi/\Omega$
- If this is much faster than τ_w then the field doesn't have time to diffuse into the wall
- Hence if $\Omega \gg 1/\tau_w$ then the wall appears ideal

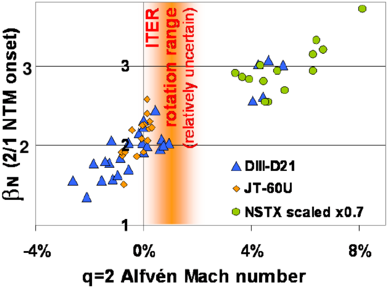


Figure: R.Buttery, IAEA 2008

Mode locking

- If a current is being driven in a resistive wall then there must be some heating $W_\eta = \int \eta j^2 d^3x$
- This energy must come from plasma rotation. Currents in the wall produce a torque on the plasma which brakes the rotation
- Rotational energy $\propto \Omega^2$ so

$$\Omega \frac{d\Omega}{dt} \propto \int \eta j^2 d^3x \propto -\tau_w \left(\Omega B_{r,wall}\right)^2$$

- The magnetic field at the island and wall are approximately: $B^2_{r,island} \propto \left(1 + \Omega^2\tau_w^2\right) B^2_{r,wall}$

Therefore,

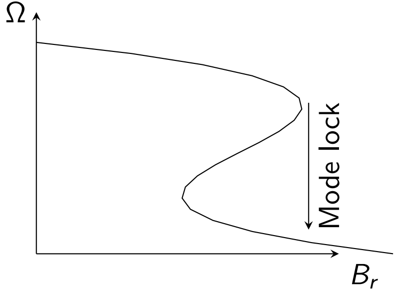
$$\frac{d\Omega}{dt} = -\frac{a\Omega\tau_w b_r^2}{1 + \Omega^2\tau_w^2}$$

For large tokamaks $\Omega\tau_w << 1$ and $\Omega \propto e^{t\sqrt{\tau_w B_r}}$

Mode locking

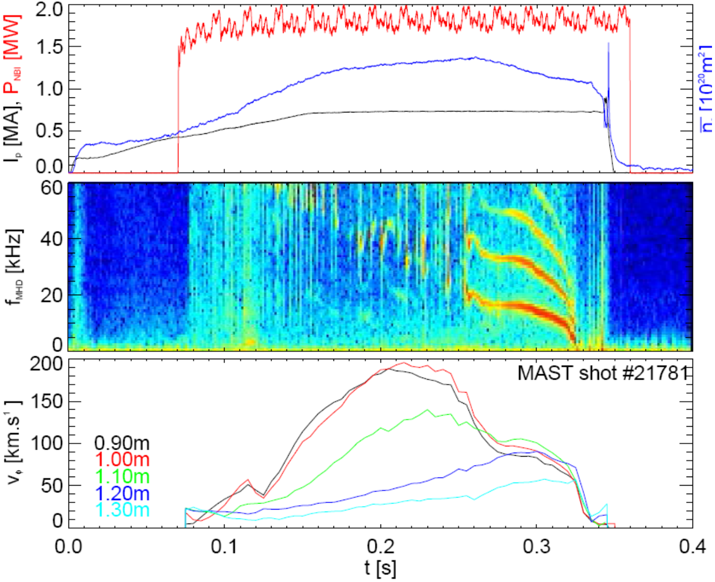
- Interaction between instability and the wall slows the mode
- Rotation slows and wall stabilisation becomes less effective
- A simple model is that an island rotation is driven by drag with the rest of the plasma, and braked by the wall

$$\frac{d\Omega}{dt} \propto \nu\left(\Omega_0 - \Omega\right) - \frac{\Omega\tau_w B_r^2}{1 + \Omega^2\tau_w^2}$$



- Final braking caused by error fields: non-axisymmetric fields caused by finite number of coils or manufacturing imperfections
- Sudden braking of the plasma, often leading to a disruption

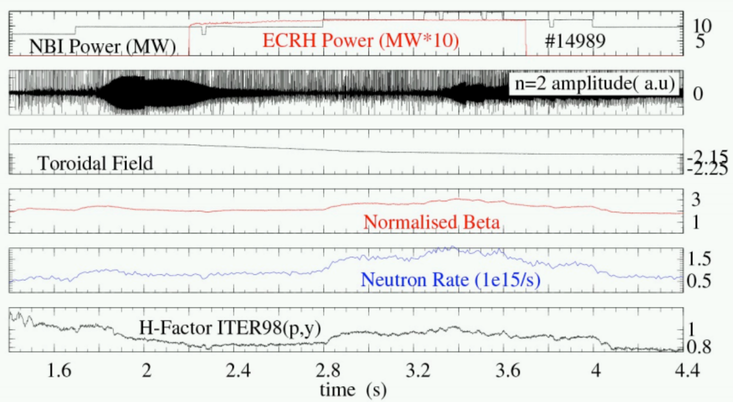
Mode locking



Stabilising NTMs

There are several schemes being investigated to control NTMs.

- The most successful one so far has been to use Electron Cyclotron Current Drive (ECCD).
- Idea is to restore the missing bootstrap current in the island
    - Localised current drive: ECCD, LHCD, ...
    - Localised heating: changes resistivity and so current profile



Stabilising NTMs

- The absorption location of RF waves can be varied by changing the launcher angle, plasma major radius or toroidal field
- Suppression methods typically use magnetics signals to find start of an NTM
- "Search and suppress" methods scan the alignment, stopping when the island shows a response
- Active tracking uses experimental measurements to reconstruct the location of the $q = 3/2$ and $q = 2$ surfaces, then targets these with the launcher. Very complicated and needs fast calculations
- If the q surface location is known, preemptive suppression can be used to prevent NTMs from starting
- Active suppression of 3,2 and 2,1 modes on ITER using upper ECRH launcher under development

Stabilising NTMs: alternatives

In addition to active suppression, other things can be done to reduce NTMs

- The source of seed islands should be reduced. Suppressing sawteeth or making them smaller
- Driving plasma rotation or rotation shear helps suppress NTMs, but not so applicable to ITER
- Kinetic effects (fast particles) have been found to have an effect on mode stability, and being investigated

Summary

- **Neoclassical Tearing Modes** are resistive (tearing) modes driven unstable by the bootstrap current (neoclassical effects).
- NTMs need a finite size "seed" to grow. This can emerge from background noise if the pressure gradient is high enough, but often will be triggered by another instability e.g. Sawtooth.
- Because NTMs are destabilised by the bootstrap current, they appear at high β, and tend to limit tokamak performance
- Experimentally the beta limit is found to vary linearly with $I_\phi/\left(rB_\phi\right)$ (**Troyon limit**) and so the **normalised beta** is defined as

$$\beta_N \equiv \frac{\beta\left(\%\right)}{I_\phi/\left(rB_\phi\right)}$$

- The vessel wall influences the growth-rate of NTMs and external kinks, and rotation makes a resistive wall appear ideal