Small-scale instabilities

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This lecture

- We've looked at MHD instabilities, which tend to be the fastest and most dangerous for confinement
- In this lecture we'll look at other instabilities which degrade confinement but don't lead to catastrophic results.
- These result from treating electrons and ions separately and so are two-fluid or kinetic effects
- They are essentially electrostatic, and are driven by gradients in temperature and density
- These are thought to be the origin of turbulence in confinement devices, and so anomalous transport

References:

- J. Wesson "Tokamaks", sections 8.2 - 8.5
- J.W.Connor, H.R.Wilson "Survey of theories of anomalous transport" *Plasma Phys. Control. Fusion* **36** 719-795 (1994)
- B.D.Scott "Computation of turbulence in magnetically confined plasmas" *Plasma Phys. Control. Fusion* **48** B277 (2006)

Electron response

- Because electrons move quickly along magnetic fields, they are often assumed to quickly reach equilibrium on the timescale of instabilities.
- This simplifies the analysis as we can concentrate on the ions, and assume that the electrons follow.

The momentum equation for electrons is

$$n_e m_e \left(\frac{\partial \underline{v}_e}{\partial t} + \underline{v}_e \cdot \nabla \underline{v}_e \right) = -\nabla p_e - n_e e \left(\underline{E} + \underline{v}_e \times \underline{B} \right)$$

Parallel to the magnetic field, and setting the left side to zero

$$n_e e E_{||} + \nabla_{||} p_e = 0$$

Electron response

Linearising this equation, neglecting temperature variations because parallel thermal conduction is fast (so $\nabla_{||} T \simeq 0$), and assuming electrostatic perturbations so $E_{||} = -\nabla_{||}\phi$

$$-\left(n_0 + \delta n\right) e\nabla_{||}\left(\phi_0 + \delta\phi\right) + \nabla_{||}\left[T_0\left(n_0 + \delta n\right)\right] = 0$$

$$-n_0 e\nabla_{||}\delta\phi - \delta n e\cancelto{0}{\nabla_{||}\phi_0} + \nabla_{||}\left(\delta n T_0\right) = 0$$

Since there are no parallel gradients of equilibrium ϕ_0 and n_0, this becomes

$$-\nabla_{||}\left(n_0 e\delta\phi\right) + \nabla_{||}\left(\delta n T_0\right) = 0$$

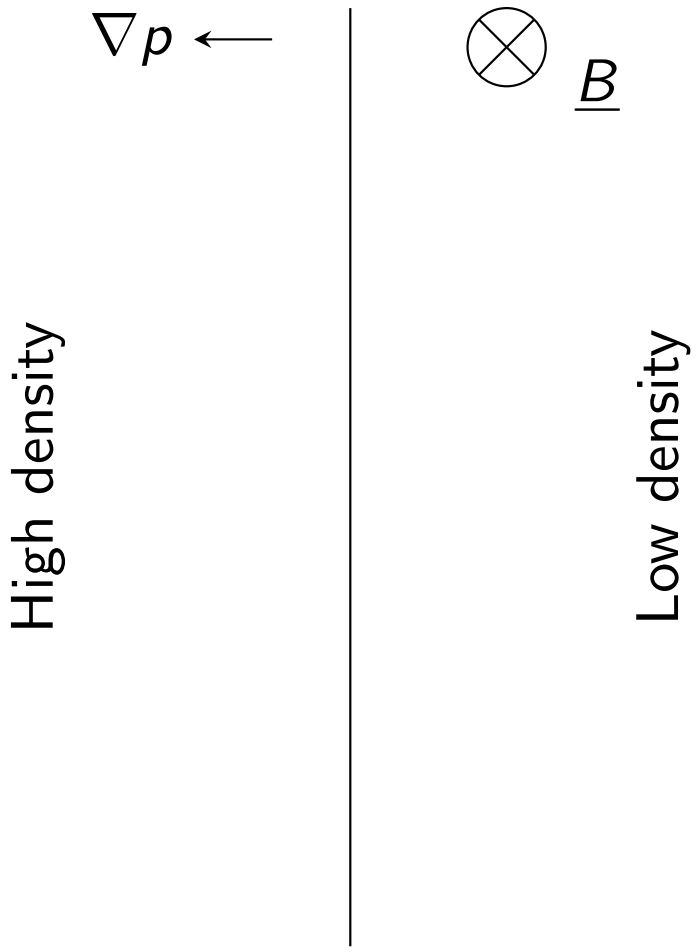
and so:

$$n_0 e\delta\phi = \delta n T_0 \qquad \Rightarrow \frac{\delta n}{n_0} = \frac{e\delta\phi}{T_0}$$

This is called the **adiabatic** or **Boltzmann response**

Electron drift waves

Consider a slab of plasma with the $\underline{B}$ into the page, and density increasing from right to left



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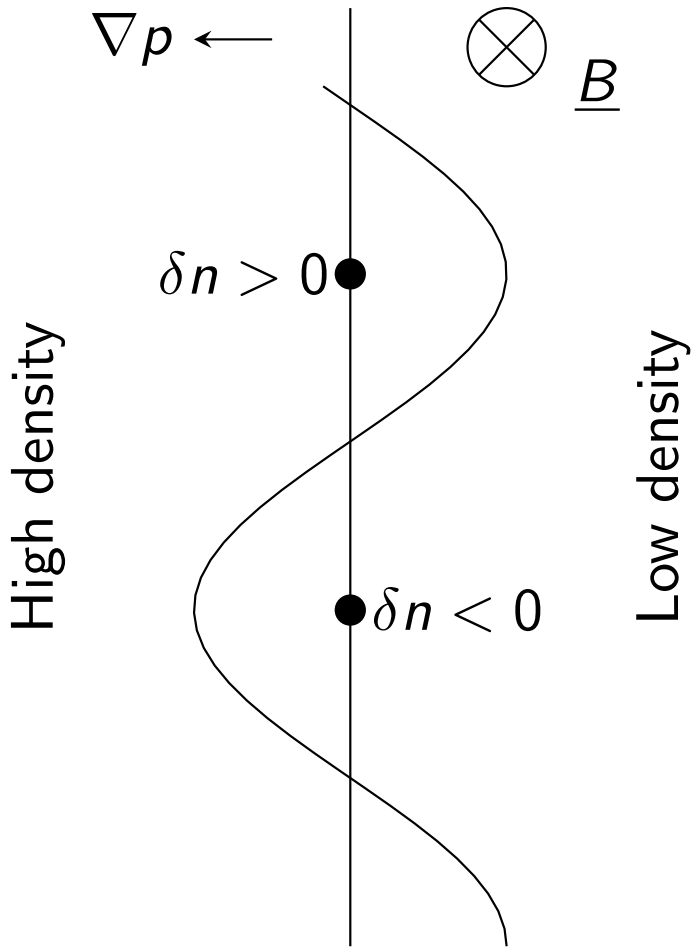

- A small density perturbation

Electron drift waves

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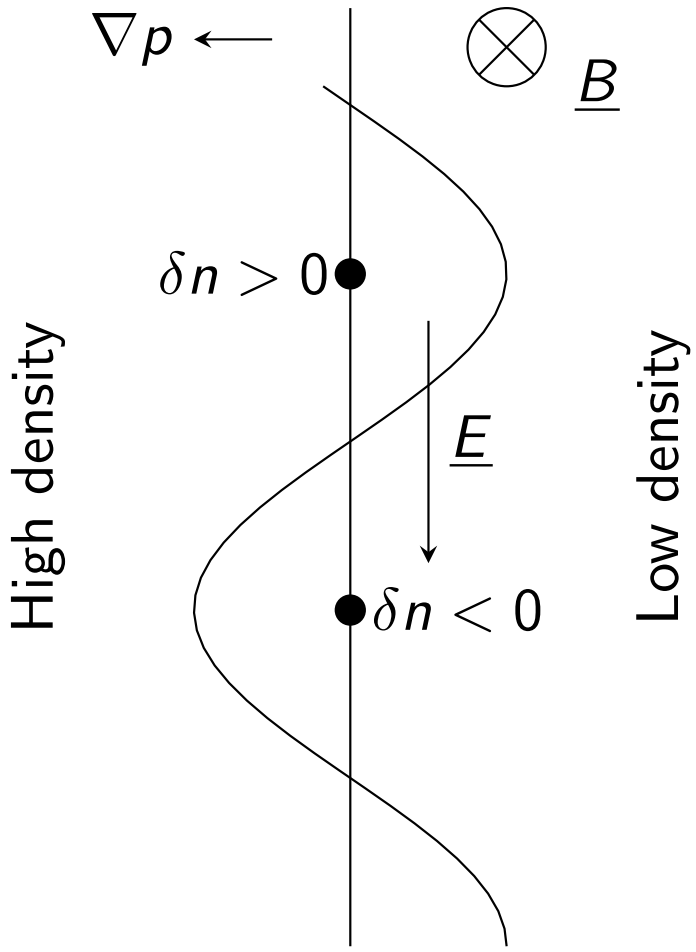

- A small density perturbation
- The electrons move along the field and establish force balance and so $\delta\phi \simeq \frac{T_0}{e}\frac{\delta n}{n_0}$

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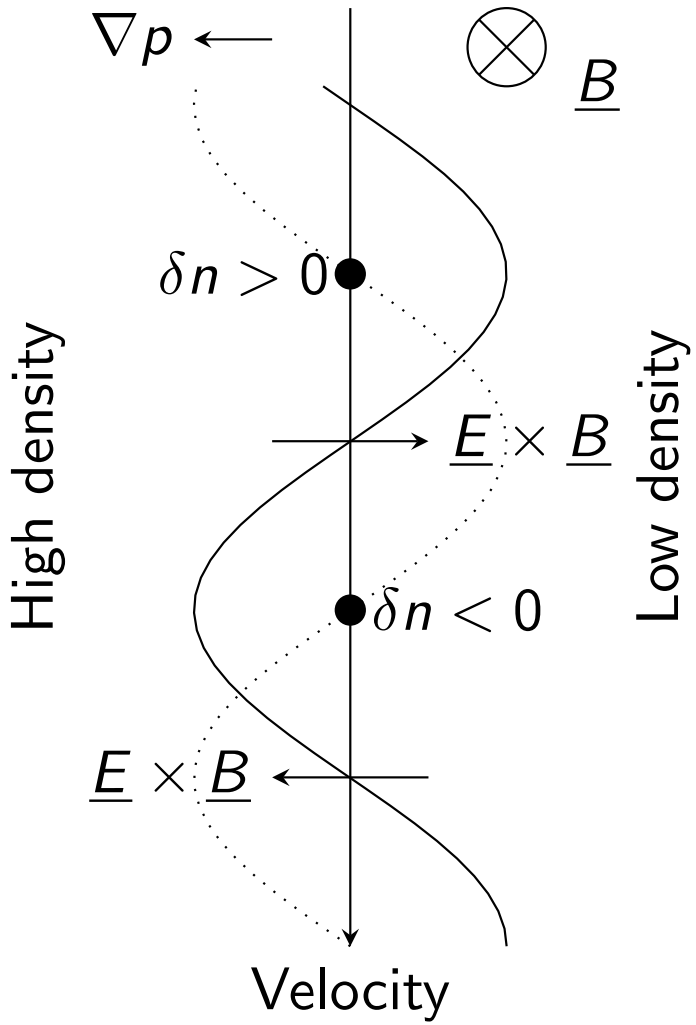

- A small density perturbation
- The electrons move along the field and establish force balance and so $\delta\phi \simeq \frac{T_0}{e}\frac{\delta n}{n_0}$
- This gives an $\underline{E} \times \underline{B}$ velocity which is 90° out of phase with the density
- This is a wave which propagates perpendicular to ∇n and $\underline{B}$
- Because $\delta\underline{v}$ and δn are out of phase, there is no net radial transport.

Electron drift waves

We can derive the dispersion relation of this wave using the ion density (continuity) equation

$$\frac{\partial \delta n_i}{\partial t} = -\nabla \cdot \left[\left(n_0 + \delta n_i \right) \delta \underline{v} \right]$$

As before, put in a solution $\delta n_i \propto \exp\left(-i\omega t\right)$. so that $\frac{\partial \delta n_i}{\partial t} \rightarrow -i\omega$. For the ions, assume that radial $\underline{E} \times \underline{B}$ is the dominant motion (not true for electrons!)

$$\Rightarrow -i\omega \delta n_i = v_{ExB,r} \frac{dn_0}{dr}$$

Electron drift waves

The radial $\underline{E} \times \underline{B}$ velocity is given by the poloidal gradient of the electrostatic potential (since B field is mainly toroidal).
Taking a single wave of the form $\exp(ik_\theta r\theta)$

$$v_{ExB,r} = -\frac{1}{B}\frac{\partial \delta\phi}{\partial r\theta} = -\frac{1}{B} ik_\theta \delta\phi \Rightarrow n_i = \frac{k_\theta \delta\phi}{\omega B}\frac{dn_0}{dr}$$

Using quasi-neutrality, $n_i \simeq n_e$ and so

$$\frac{k_\theta \cancel{\delta\phi}}{\omega B}\frac{dn_0}{dr} = \frac{n_0 e \cancel{\delta\phi}}{T_0}$$

The wave frequency is therefore

$$\omega = \frac{k_\theta T_0}{eBn_0}\frac{dn_0}{dr} = \frac{k_\theta}{eBn_0}\frac{dp_0}{dr} = k_\theta v_* = \omega_*$$

This velocity v_* is the **diamagnetic drift** velocity, and is the reason why this category of waves are known as **drift waves**.

Electron drift wave growth

- If velocity and density perturbations are out of phase then there is no net radial transport and the wave doesn't grow.
- If the electrons can't keep up with the wave then this leads to a phase shift and growth of the mode
- Finite electron mass, or just about any form of dissipation e.g. resistivity or Landau damping will have this effect.
- This is often called the **Universal instability** because all useful plasmas have density gradients and some dissipation.
- In fact magnetic shear (change in q with radius) can stabilise these modes
- In toroidal geometry these waves are destabilised by trapped particles and become unstable above a threshold given by the magnetic shear

Ion Temperature Gradient (ITG or η_i) mode

- Start with a small fluctuation in temperature
- The sum of curvature and grad-B drifts is

$$\underline{v}_{\nabla B} + \underline{v}_R \simeq \frac{\left(v_\perp^2/2 + v_{||}^2\right)}{R\Omega}\underline{e}_z$$

so hotter particles drift faster

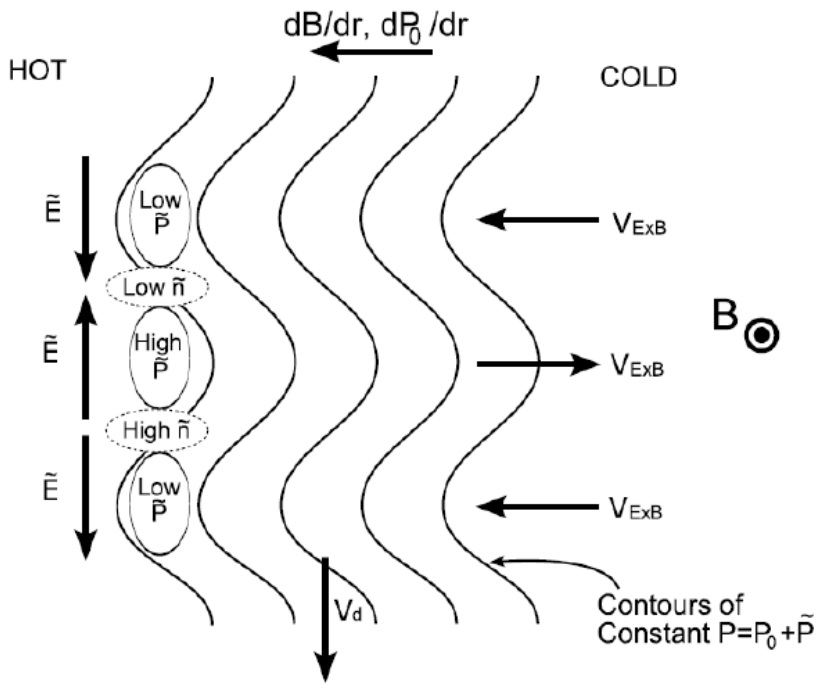

Figure: D.Applegate's thesis

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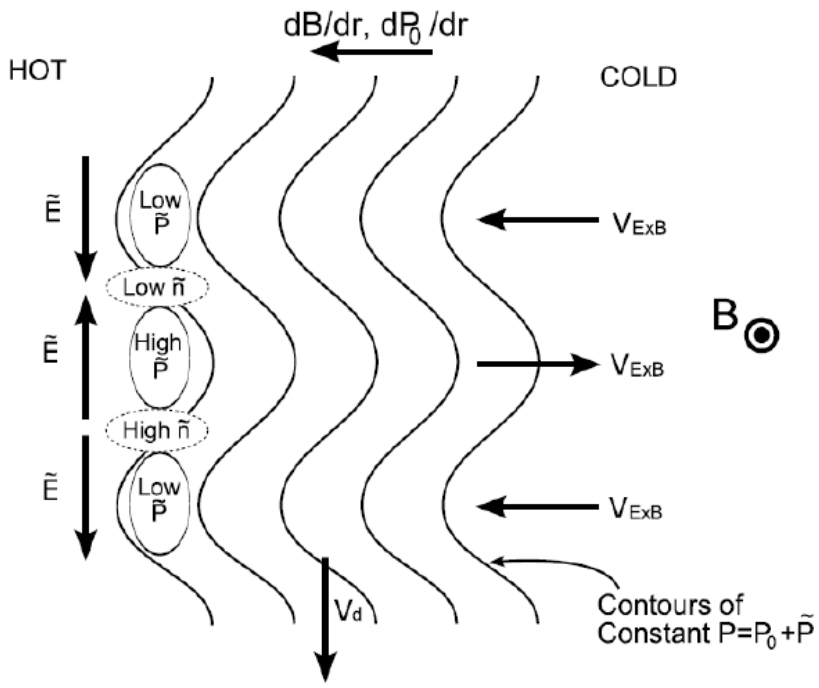
  so hotter particles drift faster
- This leads to regions of higher and lower density
- Using the electron adiabatic response, this gives an $\underline{E} \times \underline{B}$ drift which amplifies the original perturbation



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Ion Temperature Gradient (ITG or η_i) mode

- The ITG mode is stabilised by magnetic shear, but less so than the electron drift wave
- A proper analysis gives a threshold for stability (see figure)
- The mode gets its name because there is a threshold in ion temperature gradient at a given density gradient
- This is thought to be why tokamak profiles are "stiff" - the gradient tend to be fixed.

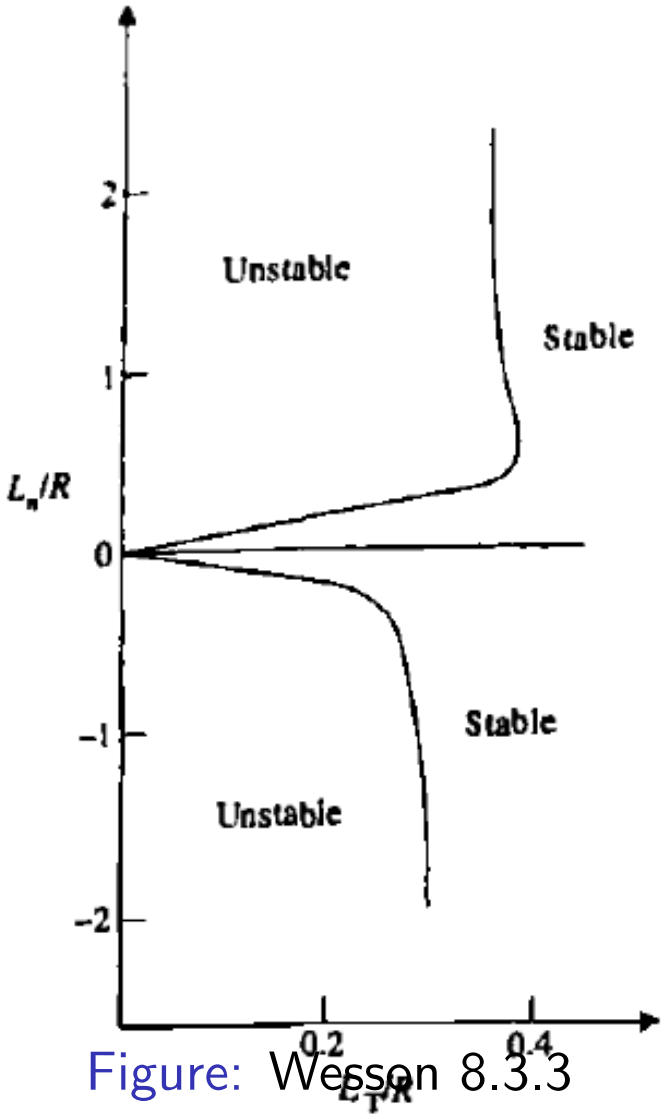


Figure: Wesson 8.3.3

Electron Temperature Gradient (ETG)

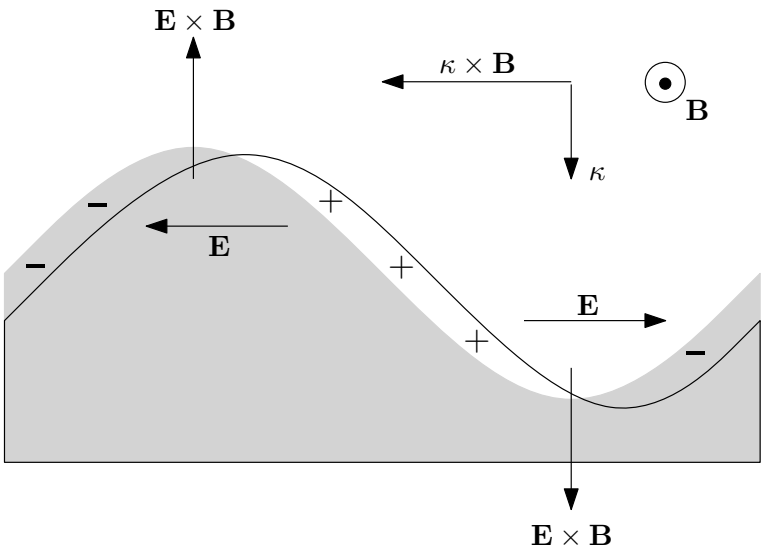
- Because the ITG instability depends on the ion ∇B and curvature drift, it has wavelengths of a few ion Larmor radii.
- There is another instability which has a scale between the electron and ion Larmor radii.
- Because it is smaller than the ion Larmor radius, the instability doesn't "know" that ions are on orbits. Instead, it sees a Boltzmann-like response for the ions, similar to the electron behavior in the ITG mode.
- The ETG mode is therefore very similar to the ITG mode, but with the roles of the electrons and ions reversed
- **Sidenote:** there has been a long and ongoing debate over whether ETG or ITG is more important for tokamak transport

Trapped particle modes

Particle trapping in toroidal plasmas means there are two populations of particles:

- passing particles which have a net parallel velocity and which explore all parts of the torus
- trapped particles which have little net parallel motion, and which only explore part of the torus

This leads to an instability similar to sausage / interchange modes



- For passing particles the good and bad curvature averages out and is stable
- Trapped particles only see the bad curvature side so have a net drift
- The passing particles act like a background with a Boltzmann response

Small-scale instabilities and turbulence

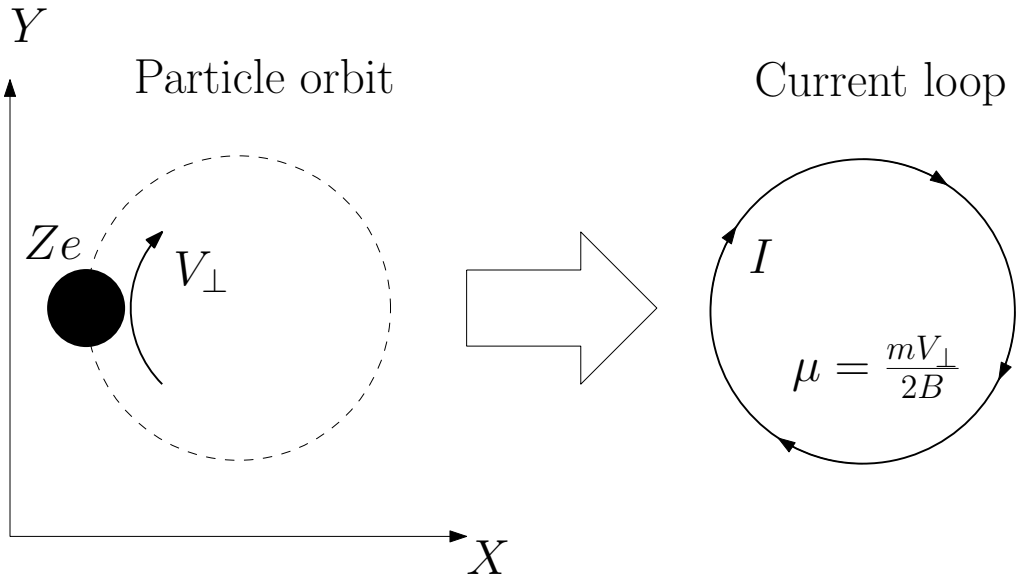
Problem: The full 6-D Vlasov equation is too difficult to solve in most situations of interest, but the plasma core is not collisional enough for a fluid (MHD-like) model to be valid

How is plasma turbulence calculated?

- **Gyrokinetics**: Remove fast timescales and reduce number of dimensions
- **Numerical tricks**: Speed up calculations by many orders of magnitude
- **High Performance Computing**: Algorithms needed to parallelise efficiently across thousands of processors

Gyrokinetics

- Recall that the Vlasov equation describes a collection of particles, each with a position **x** and velocity **v**. Both are 3D, so this is a 6D problem.
- In a strong field, these particles are gyrating quickly ($\sim$ GHz) around the magnetic field, much faster than the turbulence we want to calculate ($\sim$ 100 kHz)
- We can think of these particles as small current loops



Gyrokinetics describes the dynamics of these current loops

(Short) History of Gyrokinetics

One of the major achievements in plasma theory

- Early work on linear theory e.g. J.B. Taylor, R.J. Hastie (1968), Rutherford and Frieman (1968), P.Catto (1978)
- Nonlinear theory: Frieman & Chen (1982), and first simulations: W. W. Lee (1983)
- Hamiltonian formulation: R.G.Littlejohn (1979,1982), Dublin et al. (1983) ensures conservation of energy
- Modern gyrokinetics uses sophisticated mathematics of differential geometry and field theories

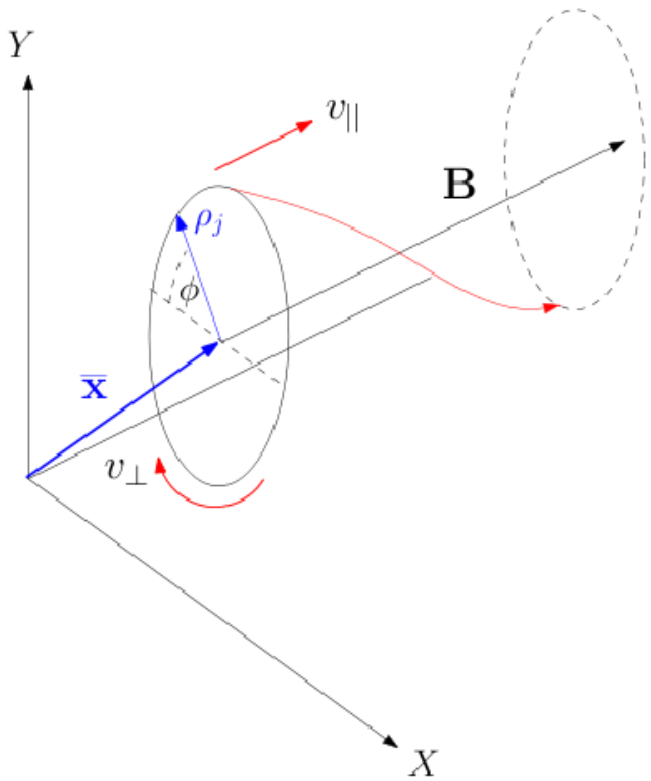
Here I will give only a brief outline of the basic versions

Re-writing Vlasov equation

Consider a particle at position $\mathbf{x}$ with velocity $\mathbf{v}$

- We need 6 numbers to describe the position of this particle in phase space
- We're free to choose what coordinates to use:

$$(\mathbf{x}, \mathbf{v}) \rightarrow \left(\overline{\mathbf{x}}, v_{||}, v_{\perp}, \phi\right)$$



where $\overline{\mathbf{x}}$ is the middle of the orbit, $v_{||}$ is the velocity along the magnetic field, $v_{\perp}$ the speed around the magnetic field, and ϕ is the gyro-phase.

Average around an orbit

Averaging over gyro-angle ϕ (gyro-averaging) removes the dependence on ϕ, and reduces the number of dimensions to 5.

- Starting with a distribution of particles f, so that the number of particles within a small volume of phase space is

$$\delta n = f\left(\mathbf{x}, \mathbf{v}\right) \delta\mathbf{x}\delta\mathbf{v}$$

- We re-write this in terms of gyro-centre $\overline{\mathbf{x}}$ and gyro-angle ϕ

$$\delta n = \hat{f}\left(\overline{\mathbf{x}}, v_{||}, v_{\perp}, \phi\right) \delta\overline{\mathbf{x}}\delta v_{||}\delta v_{\perp}\delta\phi$$

- Integrate over gyro-phase

$$\overline{f}\left(\overline{\mathbf{x}}, v_{||}, v_{\perp}\right) = \frac{1}{2\pi}\oint \hat{f}\left(\overline{\mathbf{x}}, v_{||}, v_{\perp}, \phi\right) d\phi$$

$$\Rightarrow \delta n = \overline{f}\left(\overline{\mathbf{x}}, v_{||}, v_{\perp}\right) \delta\overline{\mathbf{x}}\delta v_{||}\delta v_{\perp}$$

Equation for $\overline{f}$

To derive an equation for gyro-averaged distribution function $\overline{f}$, take the Vlasov equation and gyro-average
$\rightarrow$ many many pages of maths.

A simple "derivation" is by analogy to the Vlasov equation:

$$\frac{d}{dt} f\left(\mathbf{x}, \mathbf{v}, t\right) = 0$$

Using the chain rule:

$$\frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

and finally putting in $\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}$ and the force to get $\frac{\partial \mathbf{v}}{\partial t}$ gives

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

Equation for $\overline{f}$

For the distribution of current loops we now have $\overline{f}\left(\overline{\mathbf{x}}, v_{||}, v_{\perp}\right)$

- We can choose to use the total kinetic energy K rather than the parallel velocity, and magnetic moment μ rather than perpendicular velocity

$$\rightarrow \overline{f}\left(\overline{\mathbf{x}}, K, \mu\right)$$

$$K = \frac{1}{2} m \left(v_{||}^2 + v_{\perp}^2\right) \qquad \mu = m v_{\perp}^2 / (2B)$$

NB: Not the only possible choice

- Now write down total derivative as before:

$$\frac{d}{dt} f\left(\mathbf{x}, \mathbf{v}, t\right) = 0 \qquad \Rightarrow \qquad \frac{d}{dt} \overline{f}\left(\overline{\mathbf{x}}, K, \mu, t\right) = 0$$

Equation for $\overline{f}$

From total derivative:

$$\frac{d}{dt}\overline{f}\left(\overline{\mathbf{x}}, K, \mu, t\right) = 0$$

Expand using chain rule

$$\frac{\partial \overline{f}}{\partial t} + \frac{\partial \overline{\mathbf{x}}}{\partial t} \cdot \frac{\partial \overline{f}}{\partial \overline{\mathbf{x}}} + \frac{\partial K}{\partial t}\frac{\partial f}{\partial K} + \frac{\partial \mu}{\partial t}\frac{\partial f}{\partial \mu} = 0$$

- $\frac{\partial}{\partial t}\overline{\mathbf{x}}$ is the motion of the gyro-center
- $\frac{\partial}{\partial t}K$ is the change in energy of the particle
- $\frac{\partial}{\partial t}\mu \simeq 0$ due to conservation of μ

Equation for $\overline{f}$

The motion of the gyro-center is the motion along the magnetic field and the drifts across the magnetic field:

$$\frac{\partial \overline{\mathbf{x}}}{\partial t} = \mathbf{v}_g = v_{||}\mathbf{b} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{1}{\Omega}\left[v_{||}^2 \mathbf{b} \times (\mathbf{b} \cdot \nabla)\,\mathbf{b} + \mu \mathbf{b} \times \nabla B\right]$$

The energy of a particle changes due to electric fields:

$$\frac{\partial K}{\partial t} = q\mathbf{v}_g \cdot \mathbf{E} + \mu \frac{\partial \mathbf{B}}{\partial t}$$

Putting this together gets us...

The Drift-Kinetic equation (DKE)

An equation for particle gyro-centers (current loops)

$$\frac{\partial \overline{f}}{\partial t} + \mathbf{v}_g \cdot \frac{\partial \overline{f}}{\partial \overline{\mathbf{x}}} + \left(q\mathbf{v}_g \cdot \mathbf{E} + \mu \frac{\partial \mathbf{B}}{\partial t} \right) \frac{\partial f}{\partial K} = 0$$

- These move along, and drift (relatively) slowly across, magnetic fields
- The fast gyro-frequency timescale has been removed, so time steps in a simulation can be much larger than for the Vlasov equation
- One velocity dimension removed, reducing the problem to 5D

But: This is not the gyro-kinetic equation!

From drift kinetics to gyro-kinetics

- We have neglected the finite size of the Larmor orbits, so assumed that the $\mathbf{E} \times \mathbf{B}$ is just given by the $\mathbf{E}$ field at the gyro-center position $\overline{\mathbf{x}}$
  $\rightarrow$ Need to average drift around the orbit
- We have not considered how to calculate the $\mathbf{E}$ and $\mathbf{B}$ fields
  $\rightarrow$ This is done using Poisson and Ampére laws. Calculation of electric field complicated: determined by polarisation, not charge separation

There are many subtleties in deriving and using gyro-kinetics, particularly for nonlinear calculations

Gyro-kinetic simulations

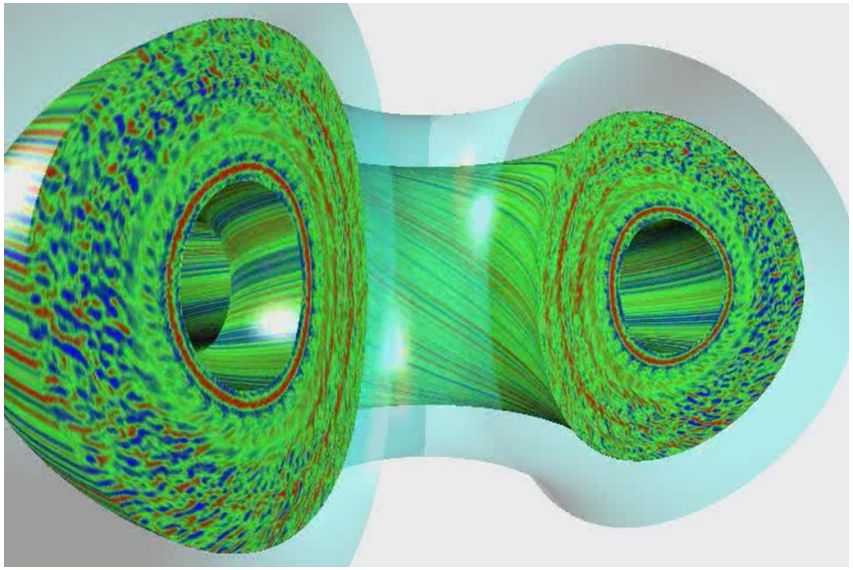
Whilst the details are more complicated, the principles of gyrokinetic PIC codes are the same as the 1D electrostatic code studied in Comp Lab:

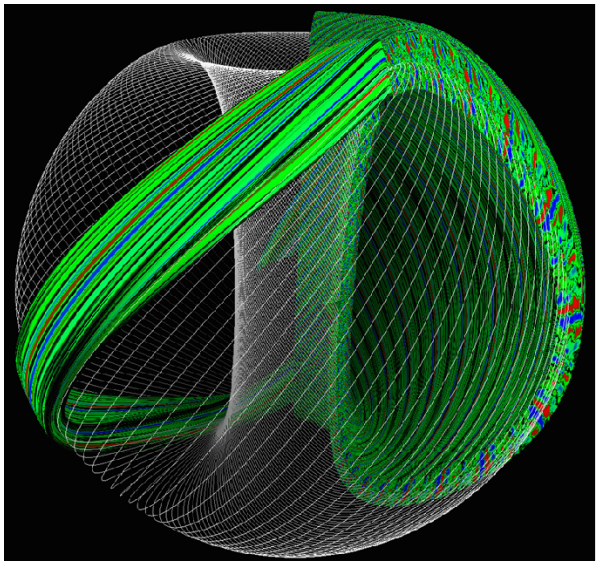
1. Gather electrons and ions to calculate gyro-center densities and velocities on grid cells
2. Solve for the electric (and magnetic) fields
3. Scatter the **E** and **B** fields on to the particles. This now involves averaging around a gyro-orbit, typically done by sampling several points on the orbit.
4. Calculate the particle drifts, and move the particles
5. Go to (1)

Many tricks have been developed to reduce the computational cost

Results - Turbulence

Simulations of DIII-D using GYRO (left), and of MAST using GS2 (right)

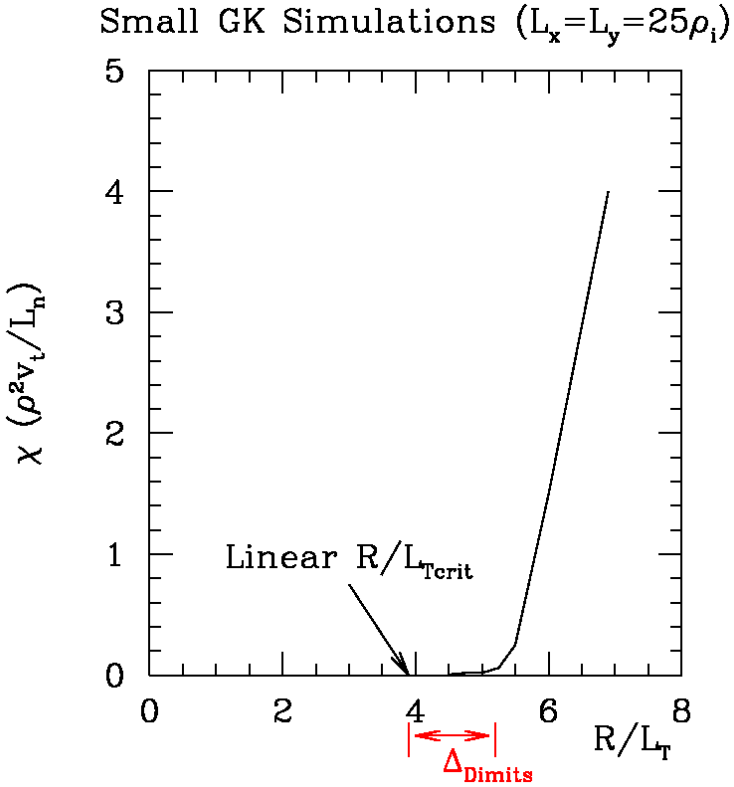


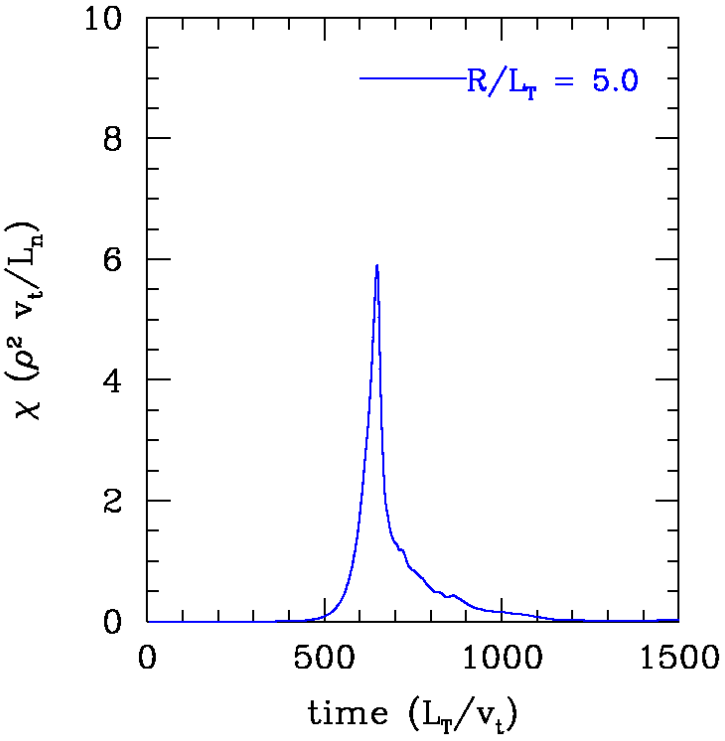


Figures: Waltz et al. Phys. Plasmas (2006), and Hammett PPPL (2002)

Results - Dimits shift

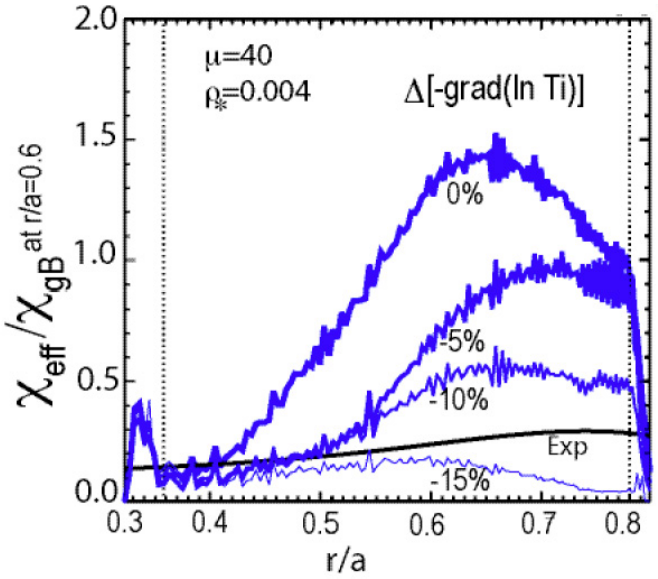
- The threshold temperature gradient for significant transport due to ITG turbulence is higher than linear theory predicts
- Non-linear simulations show an initial burst of turbulence, which then dies down to a low level
- Self-regulation of turbulence through generation of mean "zonal" flows

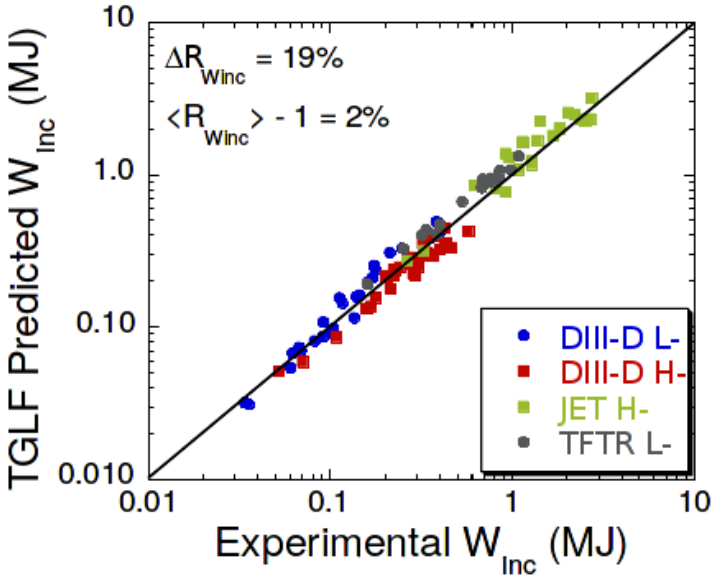




Results - Transport

- For core turbulence, gyro-kinetic codes can now get very close e.g. ITG threshold gradient within 5%
- Fluxes a strong function of gradient, so harder to predict
- Models like TGLF use fits to G-K simulations, and produce quite good results
- A "shortfall" is often observed near the edge, and the cause is still being debated





Summary

- There are many instabilities in plasmas which arise because the plasma is composed of different populations of particles, rather than a single homogenous fluid
- These instabilities are mainly electrostatic: magnetic fluctuations are present, and can be an important effect at high β, but are not essential.
- Drift-kinetics and gyro-kinetics average over gyro-motion, removing a fast timescale (cyclotron frequency) and a velocity dimension, making realistic 3D simulations possible

- Still lots of work needed, particularly in extending towards the plasma edge