MHD equilibrium

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Last time...

- Magnetic mirror effect leads to particle trapping in toroidal machines
- Trapped particles have "banana" orbits which lead to neoclassical transport
- This gives the minimum possible transport in a given configuration
- How is this magnetic configuration determined?

Self-consistent solution

- Up until now, we have considered the motion of individual particles in confinement devices, and assumed a background magnetic field

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- Up until now, we have considered the motion of individual particles in confinement devices, and assumed a background magnetic field
- In a tokamak, this field is produced in a large part by the plasma itself. We therefore need to find a self-consistent way to solve for both the magnetic field and the particles
- The particles are in a 6-dimensional phase space $f\left(\underline{x}, \underline{v}\right)$ (**Note:** even getting to here involves approximations)

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + q\left(\underline{E} + \underline{v} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{v}} = C\left(f\right)$$

plus the equations of electromagnetism (minus displacement current)

$$\begin{aligned} \nabla \cdot \underline{E} &= \rho/\epsilon_0 \qquad \nabla \cdot \underline{B} = 0 \\ \nabla \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \qquad \nabla \times \underline{B} = \mu_0 \underline{J} \end{aligned}$$

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- By averaging over Larmor orbits, gyro-kinetic equations can be derived which are (just) solvable on supercomputers
- Another approach useful for many situations is to take velocity moments

$$\langle f \rangle_n = \int \underline{v}^n f d\underline{v}$$

- The zeroth moment ($n = 0$) is density, first the average velocity, second the energy, third the heat flux, ...
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- By assuming the plasma is adiabatic (which links pressure and density) and adding the equations for electrons and ions we get a particularly useful set of equations: **ideal MHD**

Ideal MHD

Ideal MagnetoHydroDynamics (MHD) is a set of equations for the mass density ρ, velocity $\underline{v}$ and pressure P of a conducting fluid

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{v}\right) &= 0 && (1)\\
\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) &= -\nabla P + \underline{J}\times\underline{B} && (2)\\
\frac{\partial P}{\partial t} + \underline{v}\cdot\nabla P &= -\gamma P \nabla\cdot\underline{v} && (3)\\
\frac{\partial B}{\partial t} &= \nabla\times\left(\underline{v}\times\underline{B}\right) && (4)
\end{aligned}$$

Assumptions of ideal MHD

The main assumptions made in deriving ideal MHD are

- Quasineutrality $n_i \simeq n_e$
- Length scales $\gg$ Larmor radius
- Frequencies $\ll$ Cyclotron frequency
- No electron inertia ($m_e = 0$)
- Hall current neglected (no $\underline{j} \times \underline{B}$ term in Ohm's law)
- High collision rate (so nearly maxwellian)

- No dissipation: zero viscosity and resistivity
- No trapped particles, so no neoclassical effects

MHD equilibrium

- Find plasma configurations which are in **equilibrium** $\Rightarrow \frac{\partial}{\partial t} = 0$
- To simplify things, also assume a stationary equilibrium so $\underline{v} = 0$. This is reasonable if flow velocities are much less than the sound speed

This leaves us with just one expression which must be satisfied:

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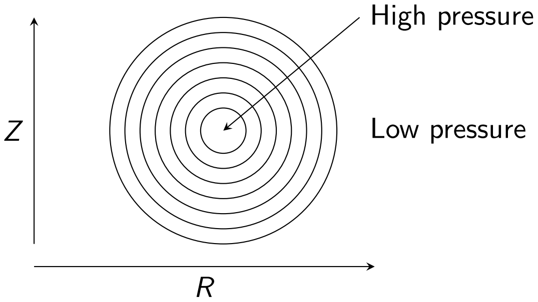
This equation has a few implications:

- $\underline{B} \cdot \nabla P = \underline{B} \cdot (\underline{J} \times \underline{B}) = 0$
  Magnetic field $\perp$ to pressure gradient
- $\underline{J} \cdot \nabla P = \underline{J} \cdot (\underline{J} \times \underline{B}) = 0$
  Current also $\perp$ to the pressure gradient

Pressure contours

- We want a configuration in which the pressure is high in the middle of the plasma, and low at the edge.
- Imagine a contour plot of the pressure:



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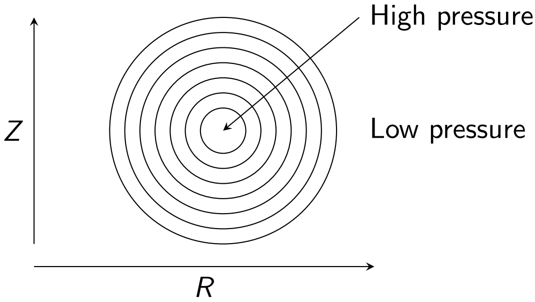

- $\underline{B} \cdot \nabla P = 0$ says that $\underline{B}$ must also lie on these surfaces
- These are of course the **flux surfaces** we saw in lecture 2

Magnetic pressure and tension

How does the magnetic field balance the plasma pressure? Use Ampére's law to eliminate $\underline{J}$

$$\frac{1}{\mu_0}\left(\nabla\times\underline{B}\right)\times\underline{B} = \nabla P$$

The following vector identity

$$\underline{B}\times\left(\nabla\times\underline{B}\right) = \frac{1}{2}\nabla B^2 - \left(\underline{B}\cdot\nabla\right)\underline{B}$$

then gives

$$\nabla\left(P + \frac{B^2}{2\mu_0}\right) - \frac{1}{\mu_0}\left(\underline{B}\cdot\nabla\right)\underline{B} = 0$$

Magnetic pressure and tension

After a bit of manipulation, this can be re-written as

$$\nabla_\perp \left(P + \frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\left(\underline{b}\cdot\nabla\right)\underline{b} = 0$$

where $\underline{b} = \underline{B}/B$ is the unit B vector, and $\nabla_\perp = \nabla - \underline{b}\left(\underline{b}\cdot\nabla\right)$ is the gradient perpendicular to $\underline{B}$.

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where $\underline{b} = \underline{B}/B$ is the unit B vector, and $\nabla_{\perp} = \nabla - \underline{b}\left(\underline{b}\cdot\nabla\right)$ is the gradient perpendicular to $\underline{B}$. This last term is the gradient of $\underline{b}$ in the direction of $\underline{b}$. This is the **curvature**

$$\boxed{\underline{\kappa} \equiv \left(\underline{b}\cdot\nabla\right)\underline{b} = -\underline{R}_C/R_C^2}$$

where $\underline{R}_C$ is the radius of curvature

$$\nabla_\perp \left(P + \boxed{\frac{B^2}{2\mu_0}}\right) - \frac{B^2}{\mu_0}\left(\underline{b}\cdot\nabla\right)\underline{b} = 0$$

- Magnetic fields exert a pressure on the plasma

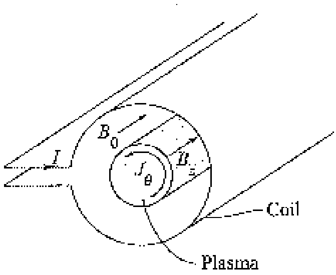

Figure : Θ-pinch. J.Friedberg, Plasma Physics and Fusion Energy

Magnetic pressure and tension

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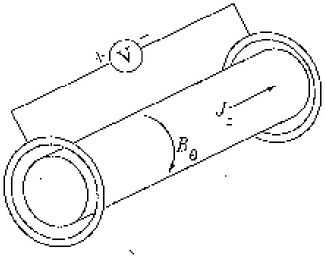
- Magnetic fields exert a pressure on the plasma
- They also have a tension which tries to straighten them



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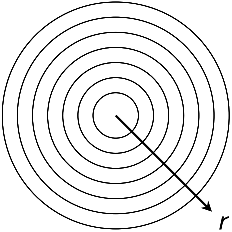
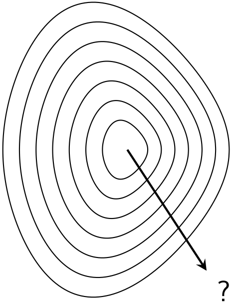
Calculating tokamak equilibria

How do we calculate a solution to $\underline{J} \times \underline{B} = \nabla P$ in a tokamak?

- First we need to introduce a way to label our flux surfaces





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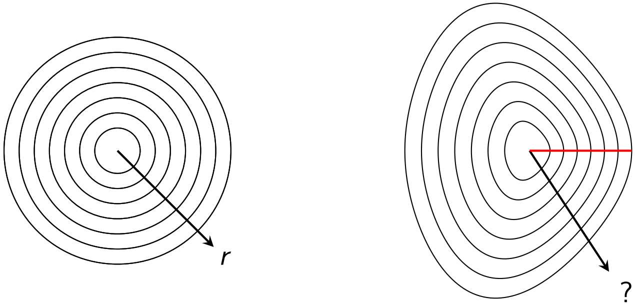

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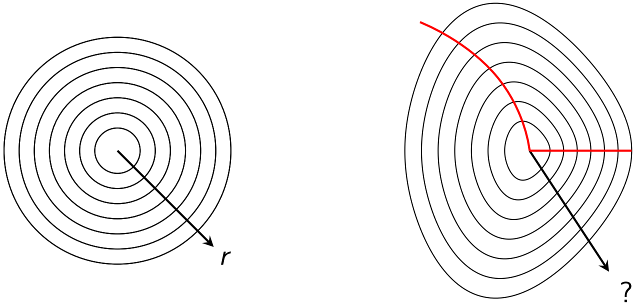

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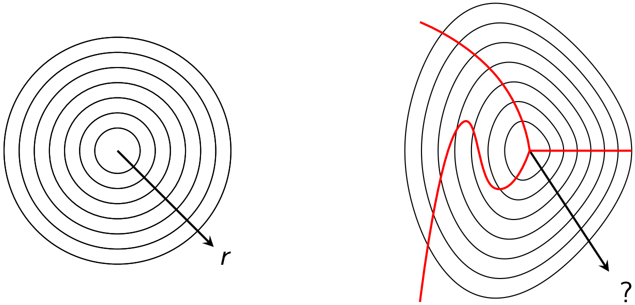

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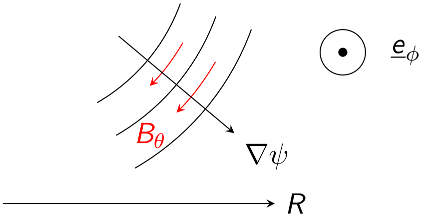
Magnetic field representation

Using ψ, we can write the poloidal field in a nice way:

$$d\psi = \underline{B} \cdot d\underline{S} = B_\theta \cdot Rdl \quad \Rightarrow \nabla\psi \propto B_\theta R$$

Note: Area $d\underline{S}$ is $2\pi R \times dl$ but the 2π is usually dropped from definition of ψ



The poloidal magnetic field is perpendicular to $\nabla\psi$ and $\underline{e}_\phi$ and can be written as

$$\underline{B}_\theta = \frac{1}{R}\nabla\psi \times \underline{e}_\phi = \nabla\psi \times \nabla\phi$$

Magnetic field representation

The toroidal magnetic field has two sources:

- external coils producing the vacuum field
- the poloidal current induced in the plasma which partly confines the plasma and acts to reduce the toroidal field

For a tokamak then it will be some function of R and Z:

$$\underline{B}_\phi = f\left(R, Z\right)\nabla\phi$$

Using Ampére's law to get the poloidal current gives:

$$\mu_0 \underline{j}_\theta = \nabla \times \left(f\left(R, Z\right)\nabla\phi\right) = -\nabla\phi \times \nabla f\left(R, Z\right)$$

Since $\underline{j} \cdot \nabla\psi = 0$, we get

$$\nabla\psi \cdot \left(-\nabla\phi \times \nabla f\left(R, Z\right)\right) = -\nabla\phi \cdot \left(\nabla f\left(R, Z\right) \times \nabla\psi\right) = 0$$

Hence $\nabla f\left(R, Z\right) \times \nabla\psi = 0$. The only way this can be true is if f is constant on flux surface i.e. $f = f\left(\psi\right)$

Grad-Shafranov equation

Combining the poloidal and toroidal components, we can write the total magnetic field as

$$\boxed{\underline{B} = f\left(\psi\right)\nabla\phi + \nabla\psi \times \nabla\phi}$$

This means that to describe our magnetic field everywhere we just need $\psi\left(R, Z\right)$ and $\boxed{f\left(\psi\right) = RB_\phi}$

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Taking the $\nabla\psi$ component of the force balance $\underline{J} \times \underline{B} = \nabla p$ and some manipulation gives the Grad-Shafranov equation:

$$R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial\psi}{\partial Z} = -\mu_0 R^2 \frac{\partial p(\psi)}{\partial\psi} - \mu_0^2 f(\psi)\frac{\partial f(\psi)}{\partial\psi}$$

This is a nonlinear PDE involving $\psi(R, Z)$, $f(\psi)$ and $p(\psi)$, and is used to design and interpret tokamak experiments

Solving the Grad-Shafranov equation

- This equation is a non-linear partial differential equation, and in general can't be solved analytically
- Instead, we need to find solutions numerically. Two main types of code
  - Forward codes, which calculate an equilibrium from $p(\psi)$ and $f(\psi)$ (or $q(\psi)$ or $j_{||}(\psi)$) and either the plasma boundary shape or coil currents
    Examples: SCENE, CORSICA, TEQ
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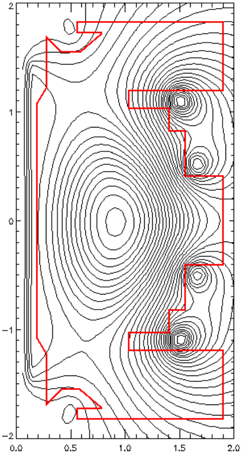
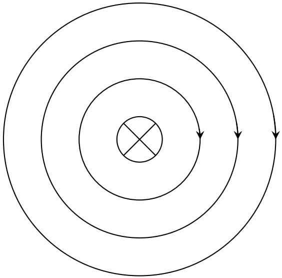
  - Interpretive codes, which take experimental measurements and work out the equilibrium
    Examples: EFIT, CLISTE
    $\Rightarrow$ Used by experimentalists to analyse experimental data

Typical MAST results

A typical contour plot of ψ from MAST (calculated using EFIT) looks something like:

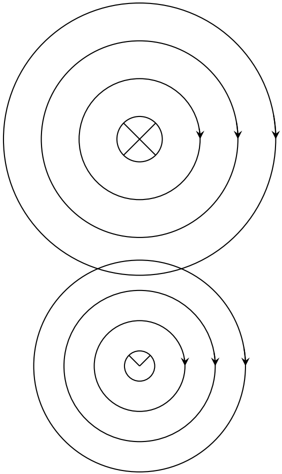


- Contours of ψ in black
- Simplified boundary in red
- Dense contours towards the right side are the poloidal field coils for plasma shaping and vertical stability
- Outside the core, plasma has a double null x-point configuration



- Poloidal field due to the plasma current

X-point equilibria



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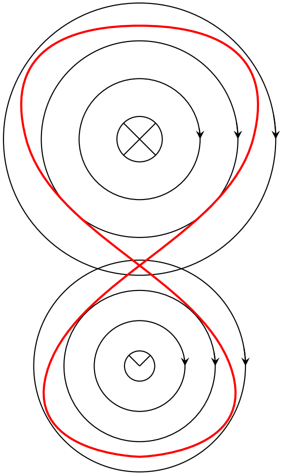

- Poloidal field due to the plasma current
- Add another coil carrying a current in the same direction
- At some point the poloidal field cancels, and forms an x-point
- The field line through the x-point is called a **separatrix**

X-point equilibria

All large tokamaks use additional coils to produce one or two x-points where the poloidal field cancels out

There are several reasons for this:

- Plasma which escapes the core is channelled along the divertor legs to specially armored regions which can handle the high heat load
- Separating the plasma from the surfaces reduces contamination of the plasma
- For reasons which aren't clear, turbulence can be suppressed near the edge of x-point configurations (H-mode).

Plasma shaping

One common parameterisation of plasma shape is[a]:

$$
\begin{aligned}
R &= R_0 - b + (a + b\cos\theta)\cos(\theta + \delta\sin\theta) \\
Z &= \kappa a \sin\theta
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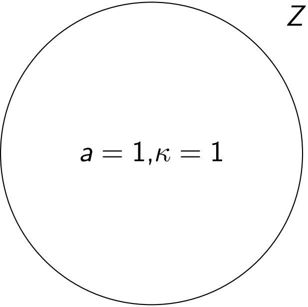

- Major radius R_0
- Minor radius a
- Ellipticity κ, which is 1 for a circle. Often see **elongation** $=$ ellipticity - 1 so zero for a circle

[a] J. Manickam, Nucl. Fusion **24** 595 (1984)

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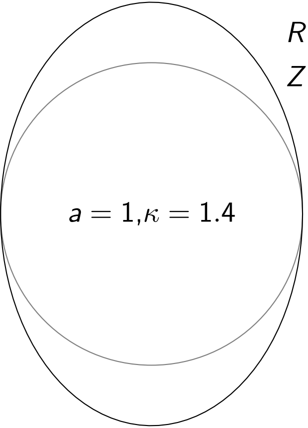

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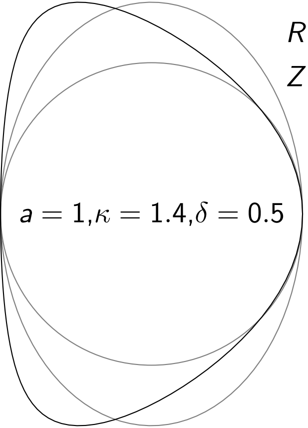

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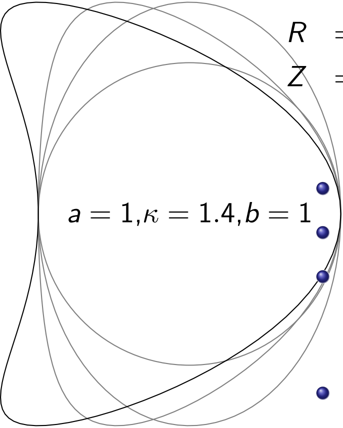

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- Indentation b ('bean' shape)

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Summary

- Ideal MHD is a simplified description of a plasma which describes many phenomena in plasmas
- Equilibrium is given by solutions to $\boxed{\underline{J} \times \underline{B} = \nabla P}$
- In axisymmetric configurations with flux surfaces, this can be simplified to the **Grad-Shafranov equation**
- Realistic configurations have x-points where the poloidal field goes to zero (but the toroidal field is not zero)
- Poloidal field coils are used to shape the plasma into configurations characterised by **elongation** and **triangularity**. This affects plasma performance and stability.