Pressure-driven instabilities

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Previously

In the last lecture we looked at the basics of plasma instabilities

- Analysed in the same way as waves:
  - Expand equations into equilibrium and perturbed quantities
  - **Linearise** by keeping only terms linear in a perturbed quantity
  - Linear equations of the form $\frac{\partial \underline{y}}{\partial t} = \underline{\underline{F}} \cdot \underline{y}$ have solutions $\propto \exp\left(-i\omega t\right)$
  - For instabilities, ω has an imaginary component usually called γ which is the **growth rate**.
- Examples are the Sausage and Kink instabilities in z-pinches
- Driven unstable by plasma pressure and current
- Can be stabilised by introducing additional magnetic fields as bending field-lines act to restore the equilibrium
- An application is the Kruskal-Shafranov condition and $q > 1$ for stability in tokamaks.

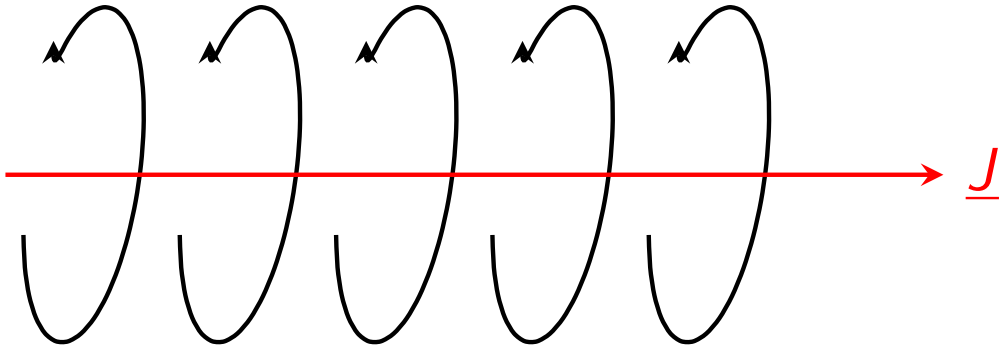
Analysis of plasma stability

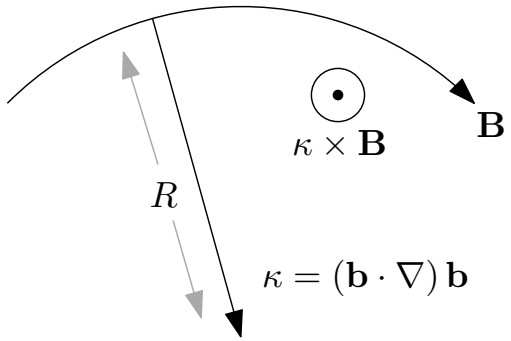
There are usually several ways to study a plasma instability:

- Linearising fluid equations
- Sometimes just considering particle orbits and drifts is enough
- Calculate the energy available and the energy needed to overcome stabilising effects

Single-particle picture of sausage instability

In a Z-pinch the magnetic field is curved around the axis:





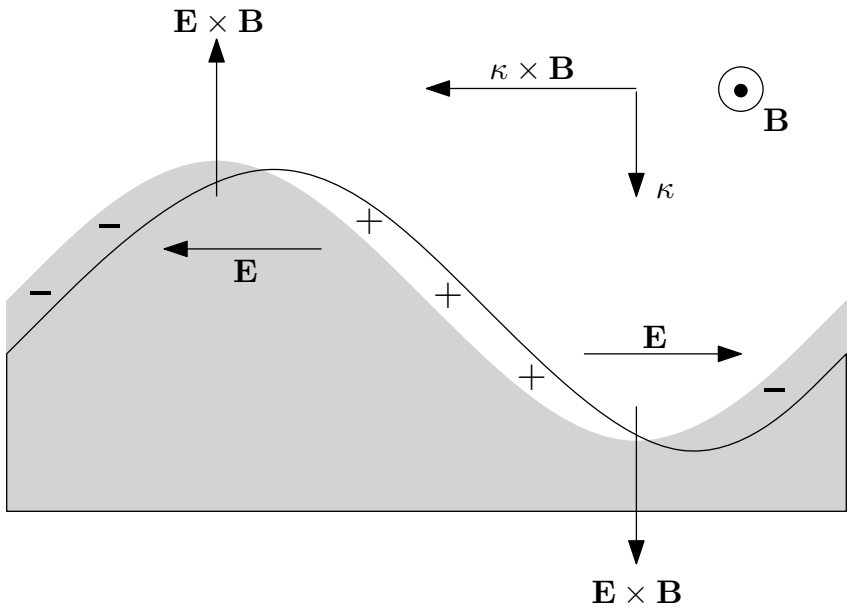
In curved magnetic fields particles drift:

$$\underline{v}_R = \frac{v_{\parallel}^2}{\Omega} \frac{\underline{R}_C \times \underline{B}}{R_C^2 B} = -\frac{v_{\parallel}^2}{\Omega} \underline{\kappa} \times \underline{b}$$

Since $\Omega = qB/m$, the sign is different for electrons and ions. For a z-pinch the drift is along the axis.

Single-particle picture of sausage instability

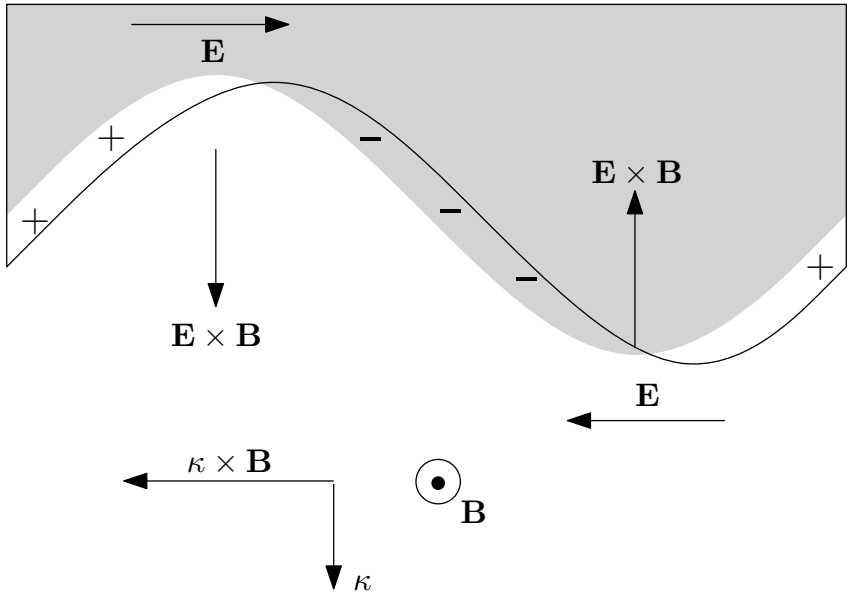
- The curvature drift is in opposite directions for electrons and ions along the axis of the plasma (z or ϕ direction)
- If there is a perturbation to the plasma, then this leads to charge separation and electric field in the z direction
- $\underline{E} \times \underline{B}$ drift is then in the radial direction



For a z-pinch, this $\underline{E} \times \underline{B}$ drift is such that it enhances the original perturbation $\Rightarrow$ instability.

Good and bad curvature

If we reverse the direction of the curvature, so put the plasma on the outside of the z-pinch (reverse pressure gradient), then we get:



Here the $\underline{E} \times \underline{B}$ acts to reduce the perturbation $\Rightarrow$ stable.

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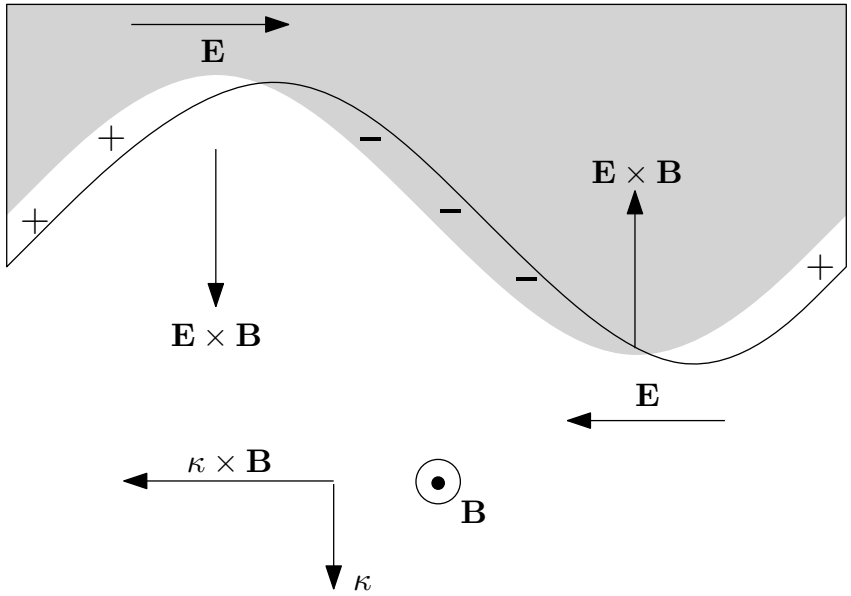

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Plasma stability depends on the relative directions of the magnetic field curvature and the pressure gradient. These are known as **good curvature** and **bad curvature**.

Good and bad curvature

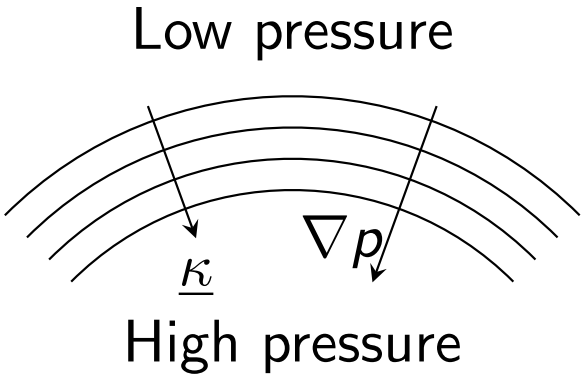


Figure : Bad curvature: $\kappa \cdot \nabla p > 0$

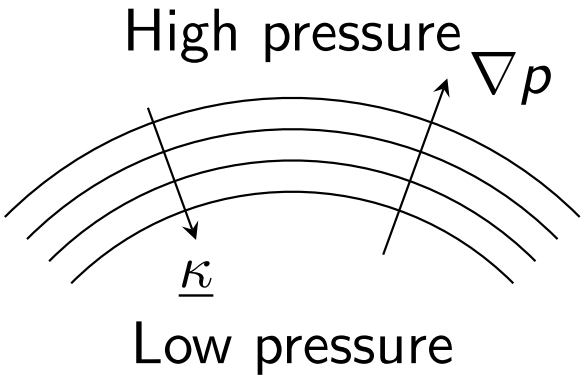
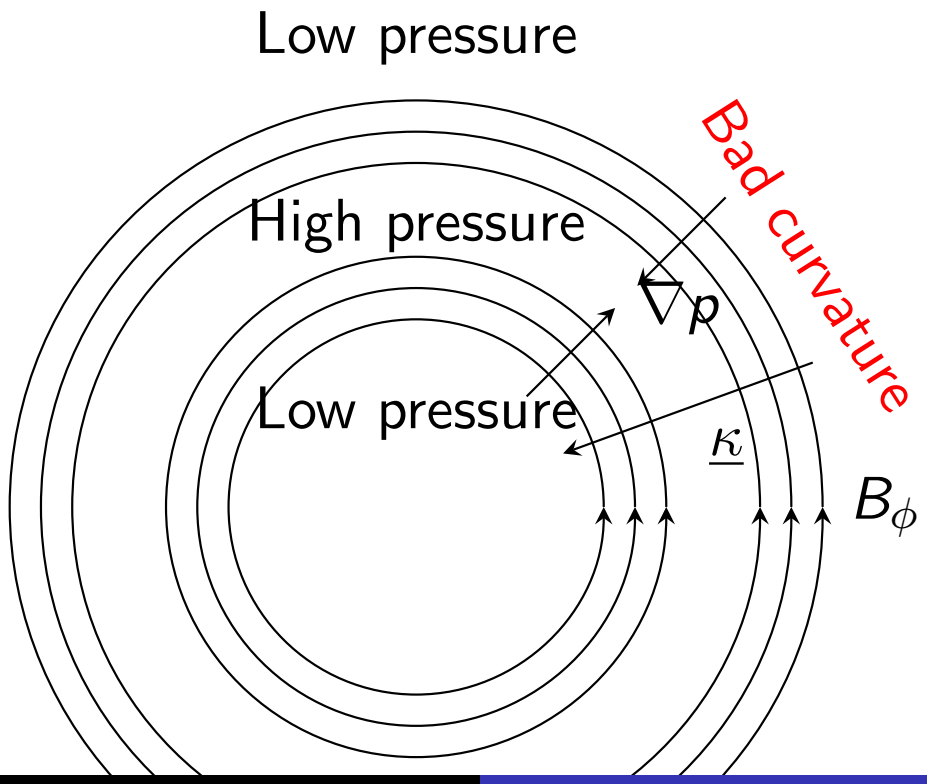


Figure : Good curvature: $\kappa \cdot \nabla p < 0$

- The relative directions of curvature κ and pressure gradient has an important effect on plasma stability
- Instabilities driven by bad curvature are analogous to Rayleigh-Taylor fluid instability and are called **Interchange instabilities**

Good and bad curvature in tokamaks

- In a tokamak, the toroidal field is curved and the pressure is highest in the core
  $\Rightarrow$ on the **outboard side** (large R) the curvature is bad, whilst on the **inboard side** (small R) the curvature is good
- Many tokamak instabilities have maximum amplitudes on the outboard side, called **ballooning** type modes



Energy and plasma stability

- To calculate whether a plasma is unstable we need to consider both sources of instability such as pressure gradients, but also stabilising effects
- Last lecture we saw that bending field-lines produced a force $\underline{F}$ which opposed the motion i.e $\underline{F} \cdot \underline{v} < 0$
- This means that the instability is having to do work to bend the field-lines
- To be unstable, the energy available has to be greater than the energy needed to overcome this force

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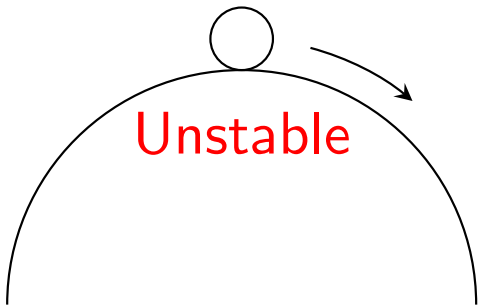
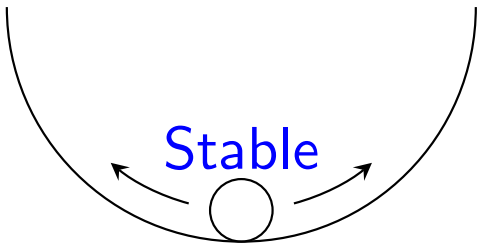
Analogous to a ball on a hill





Energy and plasma stability

The difference between stable and unstable situations is the change in potential energy δW due to a small perturbation

- $\delta W < 0$ **Unstable**: potential energy converted to kinetic energy
- $\delta W > 0$ **Stable**: kinetic to potential, then oscillates
- $\delta W = 0$ **Marginal**: Like a ball on a flat surface

A plasma is stable if $\delta W > 0$ for **all** possible perturbations, and unstable if **any** perturbation results in $\delta W < 0$

To calculate δW we'll use the ideal MHD equations...

Ideal MHD linearisation

To calculate the change in energy from a small perturbation we need to first linearise the equations:

$$n = n_0 + \epsilon n_1 \qquad \underline{v} = \underline{v}_0 + \epsilon \underline{v}_1 \qquad \ldots$$

which after substituting into the ideal MHD equations, and assuming a stationary equilibrium $\underline{v}_0 = 0$ gives:

$$\begin{aligned}
\frac{\partial}{\partial t} n_1 &= -n_0 \nabla \cdot \underline{v}_1 - \underline{v}_1 \cdot \nabla n_0 \\
\frac{\partial}{\partial t} \underline{v}_1 &= \frac{1}{m_i n_0} \left[-\nabla p_1 + \frac{1}{\mu_0} (\nabla \times \underline{B}_1) \times \underline{B}_0 + \frac{1}{\mu_0} (\nabla \times \underline{B}_0) \times \underline{B}_1 \right] \\
\frac{\partial}{\partial t} p_1 &= -\gamma p_0 \nabla \cdot \underline{v}_1 - \underline{v}_1 \cdot \nabla p_0 \\
\frac{\partial}{\partial t} \underline{B}_1 &= \nabla \times (\underline{v}_1 \times \underline{B}_0)
\end{aligned}$$

Ideal MHD dispacement

In ideal MHD all perturbed quantities n_1, $\underline{v}_1$, p_1, and $\underline{B}_1$ can be written in terms of a single **displacement** $\underline{\xi}(\underline{x})$ which is the distance the fluid has moved from equilibrium.

- The velocity is $\underline{v}_1 = \dfrac{\partial \underline{\xi}}{\partial t}$
- Substitute this into the other equations:

$$\frac{\partial}{\partial t} n_1 = -n_0 \nabla \cdot \frac{\partial \underline{\xi}}{\partial t} - \frac{\partial \underline{\xi}}{\partial t} \cdot \nabla n_0$$

  Since the equilibrium quantities do not depend on time, this can be integrated trivially:

$$\Rightarrow n_1 = -n_0 \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla n_0$$

- Similarly:

$$\begin{aligned} p_1 &= -p_0 \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla p_0 \\ \underline{B}_1 &= \nabla \times \left(\underline{\xi}_1 \times \underline{B}_0 \right) \end{aligned}$$

Note: This only works for ideal MHD: resistivity breaks this

Ideal MHD normal modes

Substituting into the equation for velocity gives:

$$m_i n_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \nabla \underbrace{\left(\underline{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{\xi}\right)}_{-p_1} + \frac{1}{\mu_0} \left(\nabla \times \underline{B}_1\right) \times \underline{B}_0 + \frac{1}{\mu_0} \left(\nabla \times \underline{B}_0\right) \times \underline{B}$$

This is a linear operator

$$m_i n_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = F\left(\underline{\xi}\right) \qquad \text{Fancy form of } m\underline{a} = \underline{F}$$

Solutions are linear $\underline{\xi}\left(\underline{x}, t\right) = \underline{\xi}\left(\underline{x}\right) e^{-i\omega t}$ and include the shear Alfvén and magnetosonic waves.
Hence we can write an eigenvalue equation with eigenfunction $\underline{\xi}$

$$-m_i n_0 \omega^2 \underline{\xi} = F\left(\underline{\xi}\right)$$

Ideal MHD energy principle

To calculate the work done on the plasma, we can start with the power: force $\times$ velocity

$$\frac{d}{dt}\delta W = -\int d^3x F\left(\underline{\xi}\right) \cdot \frac{\partial \underline{\xi}}{\partial t}$$

Integrate this by parts: $u = F\left(\underline{\xi}\right)$, $\frac{d\underline{V}}{dt} = \frac{\partial \underline{\xi}}{\partial t}$

$$\Rightarrow \delta W = -\left[\int d^3x F\left(\underline{\xi}\right) \cdot \underline{\xi}\right]_{t=0}^{t} + \int dt \int d^3x \underline{\xi} \cdot \frac{\partial F\left(\underline{\xi}\right)}{\partial t}$$

By writing $\frac{\partial F\left(\underline{\xi}\right)}{\partial t} = \frac{\partial F\left(\underline{\xi}\right)}{\partial \underline{\xi}} \cdot \frac{\partial \underline{\xi}}{\partial t}$, this becomes:

$$\delta W = -\left[\int d^3x F\left(\underline{\xi}\right) \cdot \underline{\xi}\right]_{t=0}^{t} + \int dt \int d^3x \underline{\xi} \cdot \left(\frac{\partial F\left(\underline{\xi}\right)}{\partial \underline{\xi}} \cdot \frac{\partial \underline{\xi}}{\partial t}\right)$$

Ideal MHD energy principle

Because ideal MHD is Hermitian,

$$\underline{\xi} \cdot \left(\frac{\partial F\left(\underline{\xi}\right)}{\partial \underline{\xi}} \cdot \frac{\partial \underline{\xi}}{\partial t} \right) = \left(\underline{\xi} \cdot \frac{\partial F\left(\underline{\xi}\right)}{\partial \underline{\xi}} \right) \cdot \frac{\partial \underline{\xi}}{\partial t} = F\left(\underline{\xi}\right) \cdot \frac{\partial \underline{\xi}}{\partial t}$$

and so this becomes:

$$\delta W = -\left[\int d^3x F\left(\underline{\xi}\right) \cdot \underline{\xi} \right]_{t=0}^{t} + \underbrace{\int dt \int d^3x F\left(\underline{\xi}\right) \cdot \frac{\partial \underline{\xi}}{\partial t}}_{=-\delta W}$$

Hence

$$\boxed{\delta W = -\frac{1}{2} \int d^3x F\left(\underline{\xi}\right) \cdot \underline{\xi}}$$

Ideal MHD energy equation (standard form)

Substituting the expression for $F\left(\underline{\xi}\right)$ from earlier:

$$F\left(\underline{\xi}\right) = \nabla\left(\underline{\xi}\cdot\nabla p_0 + \gamma p_0 \nabla\cdot\underline{\xi}\right) + \frac{1}{\mu_0}\left(\nabla\times\underline{B}_1\right)\times\underline{B}_0 + \frac{1}{\mu_0}\left(\nabla\times\underline{B}_0\right)\times\underline{B}_1$$

then integrating over plasma and vacuum regions gives the following **standard form** of the **ideal MHD energy equation**:

$$\begin{aligned}
\delta W_p &= \frac{1}{2}\int d^3x \left[\frac{\left|\underline{B}_1\right|^2}{\mu_0} + \gamma p\left|\nabla\cdot\underline{\xi}\right|^2 - \underline{\xi}^*\cdot\left(\underline{J}_0\times\underline{B}_1\right)\right. \\
&\qquad \left. + \left(\underline{\xi}_\perp\cdot\nabla p\right)\left(\nabla\cdot\underline{\xi}_\perp^*\right)\right] \qquad \text{Plasma terms} \\
\delta W_V &= \frac{1}{2}\int d^3x \frac{\left|\underline{B}_1\right|^2}{\mu_0} \qquad \text{Vacuum (always stabilising)} \\
\delta W_S &= -\frac{1}{2}\oint\left[\gamma p\nabla\cdot\underline{\xi} - \frac{\underline{B}_0\cdot\underline{B}_1}{\mu_0}\right]\underline{\xi}^*\cdot d\underline{S} \qquad \text{Surface terms}
\end{aligned}$$

Ideal MHD energy equation (intuitive form)

The plasma contribution δW_p can be rearranged:

$$\delta W_p = \frac{1}{2}\int d^3x \Bigg[\frac{|\underline{B}_1|^2}{\mu_0} \quad \text{Field-line bending} \geq 0$$

$$+\frac{B^2}{\mu_0}\left|\nabla\cdot\underline{\xi}_\perp + 2\underline{\xi}_\perp\cdot\kappa\right|^2 \quad \text{Magnetic compression} \geq 0$$

$$+\gamma p_0\left|\nabla\cdot\underline{\xi}\right|^2 \quad \text{Plasma compression} \geq 0$$

$$-2\left(\underline{\xi}_\perp\cdot\nabla p\right)\left(\underline{\kappa}\cdot\underline{\xi}_\perp^*\right) \quad \text{Pressure/curvature drive, } + \text{ or } -$$

$$-\underline{B}_1\cdot\left(\underline{\xi}_\perp\times\underline{b}\right)j_{||}\Bigg] \quad \text{Parallel current drive, } + \text{ or } -$$

This is a very useful form of the energy equation because it makes clear the balance between destabilising and stabilising effects

Instability drives

The first three terms in this equation are always ≥ 0 and so are stabilising, but the last two can be positive (stabilising) or negative (destabilising):

- $-2\left(\underline{\xi}_{\perp}\cdot\nabla p\right)\left(\underline{\kappa}\cdot\underline{\xi}_{\perp}^{*}\right)$ depends on ∇p and κ: if $\nabla p\cdot\kappa>0$ then this is destabilising. Instabilities driven by this term are often called pressure-driven
  $\rightarrow$ This is the interchange instability drive we saw earlier.
- $-\underline{B}_1\cdot\left(\underline{\xi}_{\perp}\times\underline{b}\right)j_{||}$ depends on the parallel current $j_{||}$ and leads to parallel current-driven kink modes

Compression

The term $\gamma p_0 \left|\nabla \cdot \underline{\xi}\right|^2$ represents compression of plasma

- The only place in this equation where the parallel displacement $\xi_{||} \equiv \underline{b} \cdot \underline{\xi}$ enters explicitly is this term
- Therefore, we can choose $\epsilon_{||}$ to minimise $\nabla \cdot \underline{\xi}$
- For a fluid or plasma motion parallel to $\underline{B}$,

$$|\nabla \cdot \underline{v}| \sim M_S^2 \frac{v}{L}$$

  close to marginal stability where L is a typical length and M_S is the Mach number
- Perpendicular to the field, a similar expression applies, but with the Alfvénic Mach number:

$$|\nabla \cdot \underline{v}| \sim M_A^2 \frac{v}{L}$$

$\Rightarrow$ Close to marginal stability, plasma instabilities tend to be incompressible

Field-line bending

$\frac{|\underline{B}_1|^2}{\mu_0}$ is the energy which goes into bending field-lines, and is always stabilising. The perturbed magnetic field $\underline{B}_1$ is given by

$$\underline{B}_1 = \nabla \times \left(\underline{\xi}_1 \times \underline{B}_0 \right) = \underline{\xi} \underbrace{\left(\nabla \cdot \underline{B}_0 \right)}_{=0} - \underline{B}_0 \left(\nabla \cdot \underline{\xi} \right) + \left(\underline{B}_0 \cdot \nabla \right) \underline{\xi} - \left(\underline{\xi} \cdot \nabla \right) \underline{B}_0$$

Assuming we're already minimising the compression, neglect the $\nabla \cdot \underline{\xi}$ term.

$$\Rightarrow \text{look for modes which minimise } \left(\underline{B}_0 \cdot \nabla \right) \underline{\xi} - \left(\underline{\xi} \cdot \nabla \right) \underline{B}_0$$

Field-line bending

Trying to minimise $(\underline{B}_0 \cdot \nabla)\,\underline{\xi} - (\underline{\xi} \cdot \nabla)\,\underline{B}_0$
Consider a perturbation of the form

$$\underline{\xi}\,(r,\theta,\phi) = \hat{\underline{\xi}}\,(r)\,e^{i(m\theta - n\phi)}$$

The first of these terms $(\underline{B}_0 \cdot \nabla)\,\underline{\xi}$ can be written in a cylinder (large aspect-ratio tokamak) as:

$$(\underline{B}_0 \cdot \nabla)\,\underline{\xi} = \left[\frac{B_\theta}{r}\frac{\partial}{\partial\theta} + \frac{B_\phi}{R}\frac{\partial}{\partial\phi}\right]\underline{\xi} = i\left[m\frac{B_\theta}{r} - n\frac{B_\phi}{R}\right]\underline{\xi}$$

rearranging:

$$(\underline{B}_0 \cdot \nabla)\,\underline{\xi} = i\frac{B_\phi}{R}\left(\frac{m}{q} - n\right) \quad \text{where} \quad \boxed{q = \frac{rB_\phi}{RB_\theta}}$$

This is minimised when $q \simeq m/n$ so instabilities tend to localise around resonant surfaces

Summary of interchange modes

In cylindical plasmas, pressure-driven modes which are constant along $\underline{B}_0$ are stabilised by magnetic shear $s = \frac{r}{q}\frac{dq}{dr}$. The **Suydam criterion** says that they are stable if

$$\mu_0 \frac{2r^2}{B_\theta^2} \frac{1}{s^2} \kappa \cdot \nabla p < 1/4$$

In tokamaks the equivalent is the **Mercier criterion**

$$D = -\mu_0 \frac{2r}{B^2} \frac{1}{s^2} \frac{dp}{dr} \left(1 - q^2\right) < 1/4$$

If $q > 1$ then the good curvature region tends to win and interchange modes are stable

Summary

- Pressure driven instabilities can be destabilised when $\kappa \cdot \nabla p > 0$ (bad curvature regions)
- Plasma compression is always stabilising, so tends to be minimised close to marginal stability
- To minimise parallel field bending, modes tend to be localised around resonant surfaces $q = m/n$
- Interchange modes are constant along B to minimise field-line bending, but are usually stable in tokamaks. Exceptions are in the SOL and if $q < 1$.
- In tokamaks, ballooning modes have some variation along B so that they can maximise their amplitude in the bad curvature region
- We'll cover these in more detail later when considering performance limits...