Current-driven instabilities

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Previously

In the last lecture

- We looked in more detail at pressure-driven instabilities, and the basics of **interchange instabilities**.
  $\rightarrow$ This will lead onto ballooning modes next lecture
- In the process I outlined the derivation of the ideal MHD energy equation
- This lecture we'll look at another class of instabilities which are driven by parallel currents: kink and tearing modes.

Reminder: ideal MHD energy equation

Last time we looked at stabilising and destabilising effects through the energy equation:

$$\delta W_p =$$

$$\frac{1}{2}\int d^3x \left[\quad \frac{\left|\underline{B}_1\right|^2}{\mu_0} \quad \text{Field-line bending} \geq 0 \right.$$

$$+\frac{B^2}{\mu_0}\left|\nabla\cdot\underline{\xi}_\perp + 2\underline{\xi}_\perp\cdot\kappa\right|^2 \quad \text{Magnetic compression} \geq 0$$

$$+\gamma p_0\left|\nabla\cdot\underline{\xi}\right|^2 \quad \text{Plasma compression} \geq 0$$

$$-2\left(\underline{\xi}_\perp\cdot\nabla p\right)\left(\underline{\kappa}\cdot\underline{\xi}^*_\perp\right) \quad \text{Pressure/curvature drive, + or} -$$

$$\left. -\underline{B}_1\cdot\left(\underline{\xi}_\perp\times\underline{b}\right)j_{||} \right] \quad \text{Parallel current drive, + or} -$$

This time we'll look at the last parallel current ($j_{||}$) drive term

Reminder: ideal MHD energy equation

- Initially we're going to consider instabilities where the plasma pressure doesn't play a big role. This can be justified when the ratio $\boxed{\beta = \frac{\mu_0 p}{B^2}}$ of plasma pressure to magnetic pressure is small.
- We can usually assume that the most unstable mode will be the one which is incompressible, so drop those terms

This leaves:

$$\delta W_p = \frac{1}{2}\int d^3x \left[\frac{\left|\underline{B}_1\right|^2}{\mu_0} - \underline{B}_1 \cdot \left(\underline{\xi}_\perp \times \underline{b}\right) j_{||}\right]$$

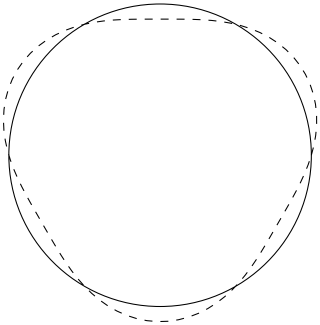
i.e. a balance between parallel current drive and bending field-lines

Instability overview

- Instabilities are often classified based on whether the boundary of the plasma moves:
  - **Internal** modes only move a region inside the plasma, and the boundary remains stationary
  - **External** modes involve motion of the plasma boundary, and so usually the whole plasma. These tend to be more dangerous to plasma stability
- There are two commonly used terms for current-driven instablities:
  - **Kink** modes are ideal MHD instabilities which involve a motion or deformation of flux surfaces, and apart from $m = 1$ (internal kink) require motion of the boundary (external kink)
  - **Tearing** instabilites are resistive, and involve the tearing or breaking of magnetic flux surfaces.

It is possible to use the energy equation to derive kink stability, but the calculations are pretty long. Instead, consider perturbations $\delta\psi$ perpendicular to flux surfaces

$$\delta\underline{B} = \underline{\hat{\phi}} \times \nabla\delta\psi \qquad \mu_0\delta J_\phi = \nabla^2\delta\psi$$



In this case the change in B field is

$$\delta B_r = -\frac{1}{r}\frac{\partial\delta\psi}{\partial\theta} \qquad \delta B_\theta = \frac{\partial\delta\psi}{\partial r}$$

We're going to assume a sinusoidal perturbation, so all quantities of the form:

$$\boxed{\delta\psi = \delta\hat{\psi}e^{i(m\theta - n\phi)}}$$

Flux surface perturbations

We can look at the form this perturbation should take by starting with the equilibrium

$$\boxed{\underline{J} \times \underline{B} = \nabla p}$$

If we're close to marginal stability, then the growth-rate is small and so the force imbalance is also small. Hence this equation still holds and we can linearise it as

$$\delta \underline{J} \times \underline{B}_0 + \underline{J}_0 \times \delta \underline{B} = \nabla \delta p$$

Taking the curl of this equation gets rid of the pressure term and so

$$\nabla \times [\delta \underline{J} \times \underline{B}_0 + \underline{J}_0 \times \delta \underline{B}] = 0$$

This can be expanded out, and assuming that B and δB change slowly along B (large aspect-ratio) becomes

$$(B \cdot \nabla)\, \delta \underline{J} + (\delta \underline{B} \cdot \nabla)\, \underline{J}_0 \simeq 0$$

Flux surface perturbations

Taking the toroidal component of this equation gives:

$$(B \cdot \nabla)\,\delta J_\phi + (\delta \underline{B} \cdot \nabla)\, J_\phi \simeq 0$$

If we assume large aspect-ratio (cylinder), then J_ϕ only depends on minor radius r and so

$$(\delta \underline{B} \cdot \nabla)\, \underline{J}_\phi = \delta B_r \cdot \frac{dJ_\phi}{dr} = -\frac{1}{r}\frac{\partial \delta\psi}{\partial\theta}\frac{dJ_\phi}{dr}$$

The first term is

$$(B \cdot \nabla)\,\delta J_\phi = \left[B_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + B_\phi \frac{1}{R}\frac{\partial}{\partial\phi}\right]\delta J_\phi$$

Since all perturbations are $\propto \exp\left[i\left(m\theta - n\phi\right)\right]$, $\frac{\partial}{\partial\theta} \rightarrow im$ and $\frac{\partial}{\partial\phi} \rightarrow -in$

$$i\left[\frac{B_\theta m}{r} - \frac{B_\phi n}{R}\right]\frac{1}{\mu_0}\nabla^2\delta\psi - i\frac{m}{r}\delta\psi\frac{dJ_\phi}{dr} = 0$$

Cylindrical Tearing Mode equation

This rearranges using $q = \frac{B_\phi r}{B_\theta R}$ to give the **Cylindrical Tearing Mode equation**

$$\nabla^2\delta\psi - \frac{\mu_0 \frac{dJ_\phi}{dr}}{B_\theta\left[1 - qn/m\right]}\delta\psi = 0$$

This equation gives the form of current-driven instabilities close to marginal stability and assumes:

- Close to marginal $\Rightarrow$ time derivatives and force imbalances are small. Plasma is incompressible.
- Large aspect ratio tokamak (cylindrical)
- No pressure effects, so low β plasmas

What happens when $q = m/n$? This is the resonance condition which minimises field-line bending

Cylindrical Tearing Mode equation

When $q = m/n$, the second derivative of $\delta\psi$ has goes to infinity. This corresponds to a perturbed current sheet

$$\delta J_\phi = \frac{1}{\mu_0}\nabla^2\delta\psi$$

In ideal MHD this is a finite perturbed current but localised only on the resonant surface, so the current density goes to infinity.

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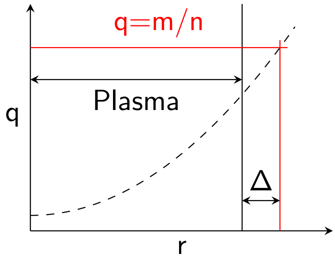
$\Rightarrow$ in ideal MHD, this magnetic surface must be outside the plasma (special exception is the $m = 1$ internal kink)



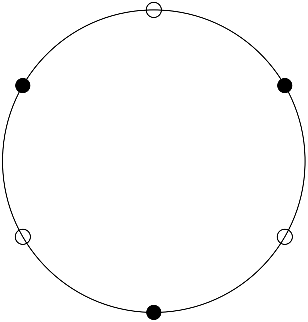
- The resonant q surface must be in the vacuum close to the plasma edge
- Since the plasma edge moves these are called **external kink** modes.
- Sensitive to boundary conditions and proximity of surface to plasma edge Δ
- Example is the **peeling mode**

Cylindrical Tearing Mode equation

In ideal MHD we are limited to either $m = 1$ internal kink, or external kinks with the resonant surface with $q = m/n$ just outside the plasma.

$$\delta J_\phi = \frac{1}{\mu_0}\nabla^2\delta\psi$$

If there is some resistivity, the current is spread over a finite region and δJ_ϕ can be finite



- On the surface where $q = m/n$ We have picked out particular field-lines and driven currents along them. This is called current **filamentation**.

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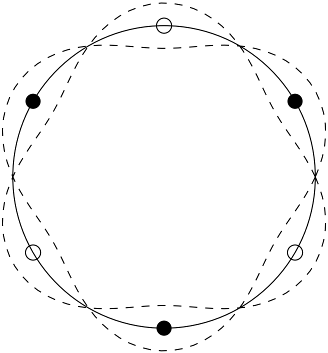
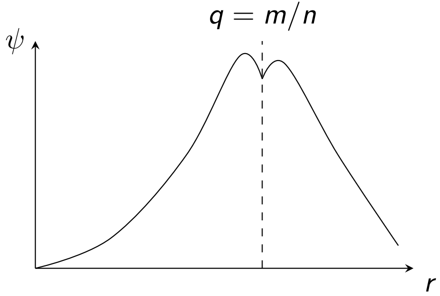
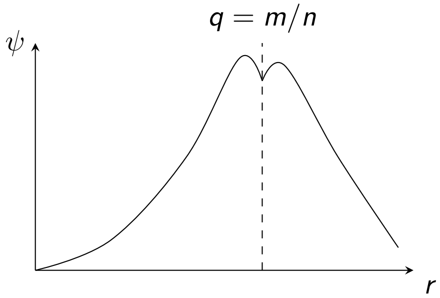


- On the surface where $q = m/n$ We have picked out particular field-lines and driven currents along them. This is called current **filamentation**.
- These perturbed currents produce a chain of magnetic islands
- This is called a **Tearing mode**
- Since this is a change in topology, it is forbidden in ideal MHD

Solving Cylindrical Tearing Mode equation



- The cylindrical Tearing mode breaks down at the resonant surface
- Instead, solve from inner and outer boundaries, matching the value of $\delta\psi$ at the resonant surface



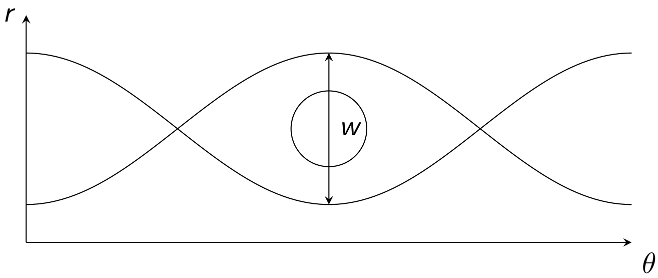
- The cylindrical Tearing mode breaks down at the resonant surface
- Instead, solve from inner and outer boundaries, matching the value of $\delta\psi$ at the resonant surface

- The jump in the gradient of $\delta\psi$ corresponds to currents on the resonant surface
- It is quantified by the **Tearing stability index**:

$$\boxed{\Delta' \equiv \left[\frac{1}{\delta\psi}\frac{\partial\delta\psi}{\partial r}\right]_-^+}$$

  where the limits are either side of the surface
- Δ' is related to the δW of a tearing mode and is $\boxed{\text{unstable if } \Delta' > 0}$

Mode width



- The growth of the island is limited by diffusion of the radial magnetic field which tears the flux surface into the island

$$\frac{\partial \delta B_r}{\partial t} \simeq \frac{\eta}{\mu_0}\nabla^2\delta B_r$$

- These modes tend to be narrow in r compared with the θ or ϕ directions

$$\Rightarrow \nabla^2 \simeq \frac{\partial^2}{\partial r^2} \qquad \frac{\partial \delta B_r}{\partial t} \simeq \frac{\eta}{\mu_0}\frac{\partial^2 \delta B_r}{\partial r^2}$$

Mode width

Now integrate this over the width of the island w:

$$\int_{-w/2}^{+w/2} \frac{\partial \delta B_r}{\partial t} dr \simeq \frac{\eta}{\mu_0} \int_{-w/2}^{+w/2} \frac{\partial^2 \delta B_r}{\partial r^2} dr$$

Since we are looking at a sinusoidal perturbation $\delta\psi \propto \exp\left[i\left(m\theta - n\phi\right)\right]$:

$$\delta B_r = -\frac{1}{r}\frac{\partial \delta\psi}{\partial \theta} \qquad \Rightarrow \delta B_r = -i\frac{m}{r}\delta\psi$$

Provided the island is quite small, $\delta\psi$ and so δB_r doesn't change much across w, only it's gradient. Therefore,

$$\int_{-w/2}^{+w/2} \frac{\partial \delta B_r}{\partial t} dr \simeq w\frac{\partial \delta B_r}{\partial t}$$

This is called the **constant ψ approximation**.

Rutherford equation

The second term is just the change in the gradient of δB_r from one side of the island to the other:

$$\frac{\eta}{\mu_0}\int_{-w/2}^{w/2}\frac{\partial^2\delta B_r}{\partial r^2}dr = \frac{\eta}{\mu_0}\left[\frac{\partial\delta B_r}{\partial r}\right]_{-w/2}^{+w/2}$$

Therefore,

$$\int_{-w/2}^{+w/2}\frac{\partial\delta B_r}{\partial t}dr \simeq \frac{\eta}{\mu_0}\left[\frac{\partial\delta B_r}{\partial r}\right]_{-w/2}^{+w/2}$$

Using $\delta B_r = -im\delta\psi/r$ and $B_r \propto w^2$

$$\Rightarrow \frac{dw}{dt} \simeq \frac{\eta}{2\mu_0}\left[\frac{1}{\delta\psi}\frac{\partial\delta\psi}{\partial r}\right]_{-w/2}^{+w/2}$$

This is the **Rutherford equation** (1973). This is $\dfrac{dw}{dt} \simeq \dfrac{\eta}{2\mu_0}\Delta'$ so if $\Delta' > 0$ then the mode will grow

Summary of current-driven modes

- $m = 1$ is a special case because it's a shift of the flux surfaces. **Internal kinks** Can occur when $q < 1$ in the plasma
- **External kinks** where the plasma boundary moves are possible when the resonant q surface is just outside the plasma
- In the presence of resistivity, flux surfaces can be torn through resistive diffusion
- This leads to **tearing modes** governed by Δ'
- The stability of all these modes depends on
  - Boundary conditions e.g. the wall of the vessel. This will be important for locked modes and Resistive Wall Modes
  - Plasma profiles: Solution of the CTM depends on J'_ϕ across entire profile, not just local (unlike e.g. Interchange modes).
  - Locations of resonant q surfaces
- Low m, n numbers are most favourable as they minimise field-line bending. Commonly observed are:
  - $q = 1 : \; m, n = 1, 1$ Sawteeth instabilities
  - $q = 2 : \; m, n = 2, 1$ tearing mode
  - $q = 3/2 : \; m, n = 3, 2$ tearing mode

Complications

- A more detailed analysis of tearing modes shows that only 2,1 modes should be unstable except in "pathological" cases.
- *But* 3,2 modes are common in high temperature plasmas. What is causing these?

- A more detailed analysis of tearing modes shows that only 2,1 modes should be unstable except in "pathological" cases.
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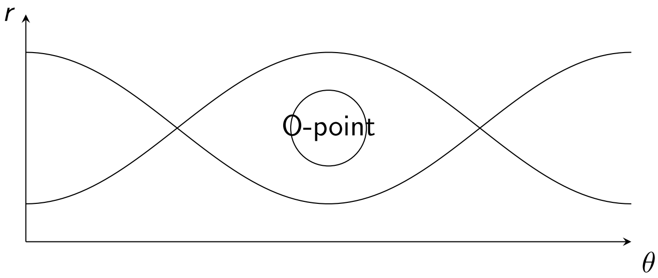
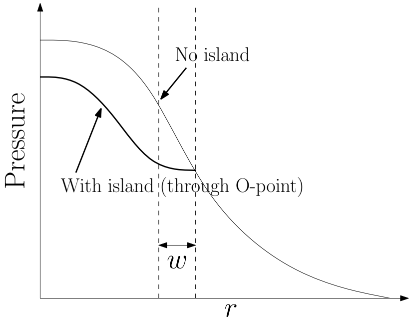
Look at what happens to the plasma with a tearing mode island



Transport of heat and particles is very fast along field-lines, so pressure will be equalised around the island flux surfaces.

Profile modification



- This means that if we plot the pressure through the O-point (widest point) of the island, we'll see a flat region
- This is bad news for confinement, as this part of the plasma is now providing little insulation of the core

Bootstrap current reminder

In lecture 5 we looked at Neoclassical currents. We saw that the bootstrap current is driven by a density or temperature gradient through collisions between trapped and passing particles. We derived the expression

$$\nu J_b \sim \nu/\epsilon J_t \qquad \Rightarrow J_b \sim \sqrt{\epsilon}\frac{T}{B_\theta}\frac{dn}{dr}$$

and in general the expression is a more complicated function of density and temperature. They are all approximately

$$J_b \sim \frac{\sqrt{\epsilon}}{B_\theta}\frac{dp}{dr}$$

Hence as the pressure gradient is modified, so is the bootstrap current. This introduces an interaction between plasma pressure and current-driven instabilities.

Modified Rutherford equation

Ohm's law is modified as the bootstap current needs no electric field to drive it

$$\delta E_{||} = \eta \left(\delta j_{||} - \delta j_{||b}\right)$$

This modifies the Rutherford equation to

$$\frac{dw}{dt} \simeq \frac{\eta}{2\mu_0}\left[\Delta' - \alpha \frac{\sqrt{\epsilon}}{w\left(r_s/a\right)}\frac{d\beta_p}{dr}\right]$$

where $\beta_p = \frac{\mu_0 p}{\left(\epsilon B_\theta\right)^2}$ is the poloidal beta.

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- α is an $O(1)$ constant, > 0 and so always destabilising
- Note that as $w \rightarrow 0$, $\frac{dw}{dt} \rightarrow \infty$
- The bootstrap current should drive all tearing modes unstable. Tokamaks shouldn't work at all!

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- There must be another effect which prevents this

Incomplete flattening

The most popular theory[1] is that small islands do not lead to flattening of the pressure profile.

- The distance $L_{||}$ along a field-line from one side of an island is bigger than the perpendicular distance across the island $L_\perp$

$$L_{||} \sim \frac{r_s^2}{\epsilon w} \gg L_\perp = w$$

where r_s is the minor radius of the magnetic island

- As the island gets smaller, $L_{||}$ gets longer.
- At a critical island size w_c parallel and perpendicular conduction are similar, and the pressure is **incompletely flattened**

$$\chi_{||}\nabla_{||}^2 T \sim \chi_\perp \nabla_\perp^2 T \qquad \rightarrow \frac{w_c}{r_s} \sim \left(\frac{\chi_\perp}{\epsilon^2 \chi_{||}}\right)^{1/4}$$

- Therefore, small islands do not affect the bootstrap current and there is a minimum size for islands to grow.

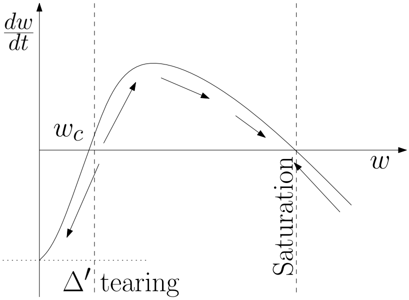
[1] R.Fitzpatrick, 1995

Modified Rutherford equation

Incomplete flattening modifies the Rutherford equation to something of the form:

$$\frac{dw}{dt} \simeq \frac{\eta}{2\mu_0}\left[\Delta' - \frac{\alpha}{r_s}\frac{\sqrt{\epsilon}\left(w/r_s\right)}{\left(w_c/r_s\right)^2 + \left(w/r_s\right)^2}\frac{d\beta_p}{dr}\right]$$

When w is small, this has the effect of limiting the effect of the bootstrap current on tearing modes. Including some additional nonlinear effects gives an equation of this form:



- At small w stability is determined by Δ' (usually stable)
- Above w_c profiles are flattened, mode grows
- At some amplitute the mode saturates

Summary

- Because this instability depends on neoclassical effects (bootstrap current) it's called a **Neoclassical Tearing Mode** (NTM)
- These instabilities are an important limit on performance of tokamaks. Lots more next time...
- Because of this, there's lots of work ongoing to detect and control them
- Note that they need a finite size "seed" to grow. This can emerge from background noise if the pressure gradient is high enough, but often will be triggered by another instability e.g. Sawtooth.
- There are other effects being investigated as mechanisms for island thresholds such as polarisation current