Gyrokinetics and tokamak turbulence simulations

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Summary of last lecture

- Turbulence is chaotic, apparently random fluid motion which results in enhanced diffusion / transport
- Usually thought of in terms of **eddies** of size L, or wave-number $k = 2\pi/L$
- Large eddies break up into smaller eddies, transferring energy from large to small scales (low to high k). This process is known as a **cascade**.
- Large eddies are less affected by dissipation (viscosity) than small eddies. This is quantified by the **Reynolds number** $Re = uL/\nu$, which is ~ 1 for the smallest eddies

Turbulence in plasmas

- Plasmas have many more waves and instabilities than fluids, making turbulent dynamics more complex
- Strong magnetic fields tend to elongate structures, leading to essentially 2D dynamics in some cases
- Finite sized Larmor orbits provide a limit to how small eddies can become, reducing the resolution required

- Plasma turbulence can be divided into two categories:
  - High-β (small magnetic field). Essentially fluid turbulence modified by presence of Alfvén waves, so generally electromagnetic. Space and astrophysical plasmas.
  - Low-β (large magnetic field). The magnetic field strongly constrains the motion of the plasma, and the resulting turbulence tends to be electrostatic e.g. drift waves

$\rightarrow$ Here we're interested in the low-β case ($\beta \sim 1\%$)

- Transport of heat and particles in tokamaks is typically "anomalous", meaning above neoclassical
- Little progress can be made analytically, so computer simulations used to study and predict transport levels
- These require sophisticated plasma models (gyro-kinetics) and supercomputers

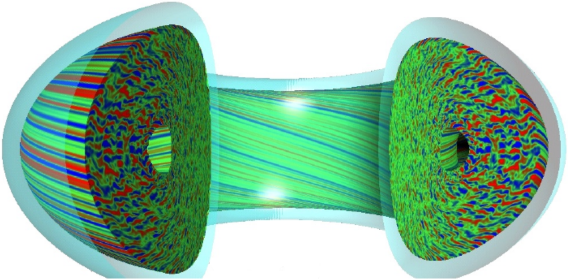


Image: GYRO simulation of DIII-D by J.Candy, Waltz

Simulation of turbulence

Like fluid turbulence, plasma turbulence has a **closure problem**:

- The equations (classical mechanics + electromagnetism) are well known, but cannot in general be solved
- Average quantities are well behaved, but form an infinite set of equations, due to the nonlinear advection term $\mathbf{v} \cdot \nabla \mathbf{v}$

How can transport be calculated?

- Since the 1970's computers have been used to study turbulence (e.g. Orszag & Patterson 1972)
- Computational Fluid Dynamics (CFD) is now a huge area of research
- Plasma simulations are at the stage where they can be compared to experiment, sometimes successfully

Outline

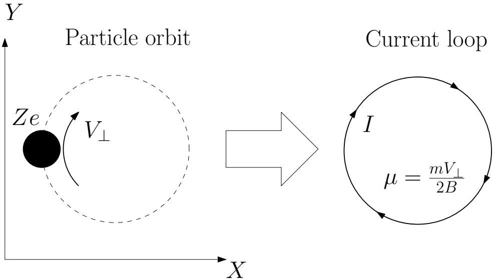
Problem: The full 6-D Vlasov equation is too difficult to solve in most situations of interest, but the plasma core is not collisional enough for a fluid (MHD-like) model to be valid

How is plasma turbulence calculated?

- **Gyrokinetics**: Remove fast timescales and reduce number of dimensions
- **Numerical tricks**: Speed up calculations by many orders of magnitude
- **High Performance Computing**: Algorithms needed to parallelise efficiently across thousands of processors

Gyrokinetics

- Recall that the Vlasov equation describes a collection of particles, each with a position **x** and velocity **v**. Both are 3D, so this is a 6D problem.
- In a strong field, these particles are gyrating quickly ($\sim$ GHz) around the magnetic field, much faster than the turbulence we want to calculate ($\sim$ 100 kHz)
- We can think of these particles as small current loops



Gyrokinetics describes the dynamics of these current loops

(Short) History of Gyrokinetics

One of the major achievements in plasma theory

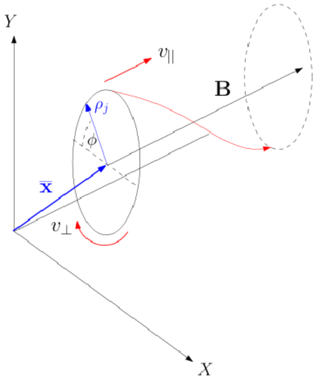
- Early work on linear theory e.g. J.B. Taylor, R.J. Hastie (1968), Rutherford and Frieman (1968), P.Catto (1978)
- Nonlinear theory: Frieman & Chen (1982), and first simulations: W. W. Lee (1983)
- Hamiltonian formulation: R.G.Littlejohn (1979,1982), Dublin et al. (1983) ensures conservation of energy
- Modern gyrokinetics uses sophisticated mathematics of differential geometry and field theories

Here I will give only a brief outline of the basic versions

Consider a particle at position $\mathbf{x}$ with velocity $\mathbf{v}$

- We need 6 numbers to describe the position of this particle in phase space
- We're free to choose what coordinates to use:

$$(\mathbf{x}, \mathbf{v}) \rightarrow \left(\overline{\mathbf{x}}, v_{||}, v_\perp, \phi\right)$$



where $\overline{\mathbf{x}}$ is the middle of the orbit, $v_{||}$ is the velocity along the magnetic field, $v_\perp$ the speed around the magnetic field, and ϕ is the gyro-phase.

Average around an orbit

Averaging over gyro-angle ϕ (gyro-averaging) removes the dependence on ϕ, and reduces the number of dimensions to 5.

- Starting with a distribution of particles f, so that the number of particles within a small volume of phase space is

$$\delta n = f\left(\mathbf{x}, \mathbf{v}\right) \delta\mathbf{x}\delta\mathbf{v}$$

- We re-write this in terms of gyro-centre $\overline{\mathbf{x}}$ and gyro-angle ϕ

$$\delta n = \hat{f}\left(\overline{\mathbf{x}}, v_{||}, v_\perp, \phi\right) \delta\overline{\mathbf{x}}\delta v_{||}\delta v_\perp\delta\phi$$

- Integrate over gyro-phase

$$\overline{f}\left(\overline{\mathbf{x}}, v_{||}, v_\perp\right) = \frac{1}{2\pi}\oint \hat{f}\left(\overline{\mathbf{x}}, v_{||}, v_\perp, \phi\right) d\phi$$

$$\Rightarrow \delta n = \overline{f}\left(\overline{\mathbf{x}}, v_{||}, v_\perp\right) \delta\overline{\mathbf{x}}\delta v_{||}\delta v_\perp$$

Equation for $\overline{f}$

To derive an equation for gyro-averaged distribution function $\overline{f}$, take the Vlasov equation and gyro-average
$\rightarrow$ many many pages of maths.

A simple "derivation" is by analogy to the Vlasov equation:

$$\frac{d}{dt} f\left(\mathbf{x}, \mathbf{v}, t\right) = 0$$

Using the chain rule:

$$\frac{\partial f}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

and finally putting in $\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}$ and the force to get $\frac{\partial \mathbf{v}}{\partial t}$ gives

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

Equation for $\overline{f}$

For the distribution of current loops we now have $\overline{f}\left(\overline{\mathbf{x}}, v_{||}, v_\perp\right)$

- We can choose to use the total kinetic energy K rather than the parallel velocity, and magnetic moment μ rather than perpendicular velocity

$$\rightarrow \overline{f}\left(\overline{\mathbf{x}}, K, \mu\right)$$

$$K = \frac{1}{2}m\left(v_{||}^2 + v_\perp^2\right) \qquad \mu = mv_\perp^2 / \left(2B\right)$$

NB: Not the only possible choice

- Now write down total derivative as before:

$$\frac{d}{dt}f\left(\mathbf{x}, \mathbf{v}, t\right) = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\overline{f}\left(\overline{\mathbf{x}}, K, \mu, t\right) = 0$$

Equation for $\overline{f}$

From total derivative:

$$\frac{d}{dt}\overline{f}\left(\overline{\mathbf{x}}, K, \mu, t\right) = 0$$

Expand using chain rule

$$\frac{\partial\overline{f}}{\partial t} + \frac{\partial\overline{\mathbf{x}}}{\partial t}\cdot\frac{\partial\overline{f}}{\partial\overline{\mathbf{x}}} + \frac{\partial K}{\partial t}\frac{\partial f}{\partial K} + \frac{\partial\mu}{\partial t}\frac{\partial f}{\partial\mu} = 0$$

- $\frac{\partial}{\partial t}\overline{\mathbf{x}}$ is the motion of the gyro-center
- $\frac{\partial}{\partial t}K$ is the change in energy of the particle
- $\frac{\partial}{\partial t}\mu \simeq 0$ due to conservation of μ

Equation for $\overline{f}$

The motion of the gyro-center is the motion along the magnetic field and the drifts across the magnetic field:

$$\frac{\partial \overline{\mathbf{x}}}{\partial t} = \mathbf{v}_g = v_{||}\mathbf{b} + \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{1}{\Omega}\left[v_{||}^2\mathbf{b}\times\left(\mathbf{b}\cdot\nabla\right)\mathbf{b} + \mu\mathbf{b}\times\nabla B\right]$$

The energy of a particle changes due to electric fields:

$$\frac{\partial K}{\partial t} = q\mathbf{v}_g\cdot\mathbf{E} + \mu\frac{\partial \mathbf{B}}{\partial t}$$

Putting this together gets us...

The Drift-Kinetic equation (DKE)

An equation for particle gyro-centers (current loops)

$$\frac{\partial \overline{f}}{\partial t} + \mathbf{v}_g \cdot \frac{\partial \overline{f}}{\partial \overline{\mathbf{x}}} + \left(q\mathbf{v}_g \cdot \mathbf{E} + \mu\frac{\partial \mathbf{B}}{\partial t}\right)\frac{\partial f}{\partial K} = 0$$

- These move along, and drift (relatively) slowly across, magnetic fields
- The fast gyro-frequency timescale has been removed, so time steps in a simulation can be much larger than for the Vlasov equation
- One velocity dimension removed, reducing the problem to 5D

But: This is not the gyro-kinetic equation!

From drift kinetics to gyro-kinetics

- We have neglected the finite size of the Larmor orbits, so assumed that the $\mathbf{E} \times \mathbf{B}$ is just given by the $\mathbf{E}$ field at the gyro-center position $\bar{\mathbf{x}}$
  $\rightarrow$ Need to average drift around the orbit
- We have not considered how to calculate the $\mathbf{E}$ and $\mathbf{B}$ fields
  $\rightarrow$ This is done using Poisson and Ampére laws. Calculation of electric field complicated: determined by polarisation, not charge separation

There are many subtleties in deriving and using gyro-kinetics, particular nonlinear calculations

Gyro-kinetic simulations

Whilst the details are more complicated, the principles of gyrokinetic PIC codes are the same as the 1D electrostatic code studied in Comp Lab:

1. Gather electrons and ions to calculate gyro-center densities and velocities on grid cells
2. Solve for the electric (and magnetic) fields
3. Scatter the **E** and **B** fields on to the particles. This now involves averaging around a gyro-orbit, typically done by sampling several points on the orbit.
4. Calculate the particle drifts, and move the particles
5. Go to (1)

Many tricks have been developed to reduce the computational cost

Some notable G-K codes

Altogether, algorithmic and theoretical advances over the past ~ 25 years have sped up Gyro-Kinetic simulations by $\sim 10^{25}$. Compare to Moore's law over the same period $\sim 10^5$ speedup.
– G.Hammett, APS 2007

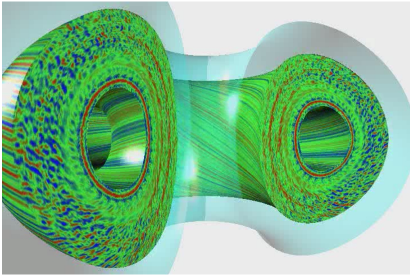
Common ways to classify codes:

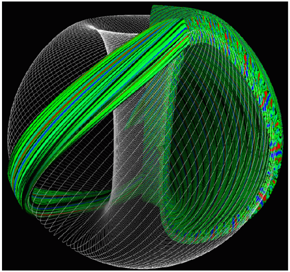
- Continuum / PIC
- Global / Flux tube
- Delta-f / Full-f

GS2 (Dorland & Kotschenreuther): Delta-f, Cont. Flux
GKW () : Delta-f, Cont. Flux
GENE (Jenko) : Cont. Flux
GYRO (Candy & Waltz) : Cont. Global
GEM (Parker & Chen) : Delta-f, PIC, Global
GTC (Z.Lin) : PIC, Global
XGC (C.S.Chang) : PIC, Global

Results - Turbulence

Simulations of DIII-D using GYRO (left), and of MAST using GS2 (right)

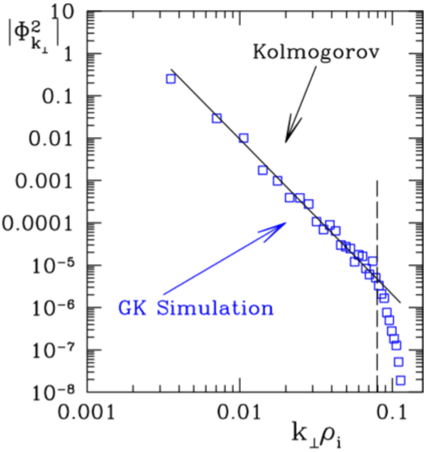




Figures: Waltz et al. Phys. Plasmas (2006), and Hammett PPPL (2002)

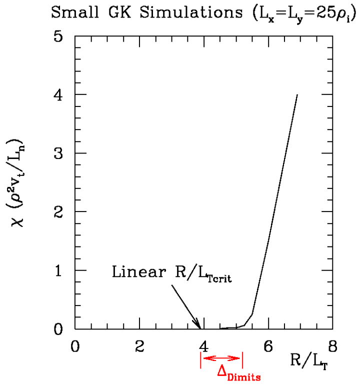
High-β, homogenous stirred plasma turbulence shows Kolmogorov scaling. Here $\beta = 8$.

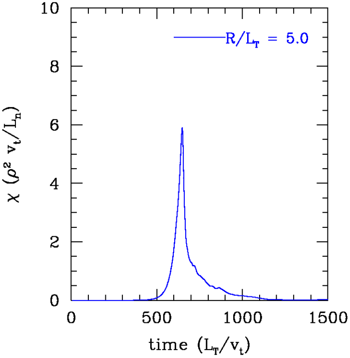
From W.Dorland, S.C.Cowley, G.W.Hammett, E.Quataert EPS 2002



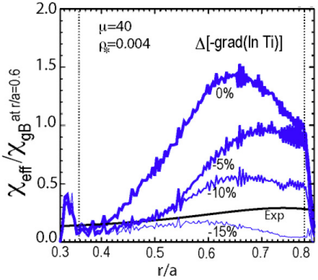
Results - Dimits shift

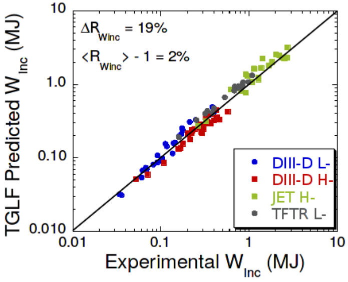
- The threshold temperature gradient for significant transport due to ITG turbulence is higher than linear theory predicts
- Non-linear simulations show an initial burst of turbulence, which then dies down to a low level
- Self-regulation of turbulence through generation of mean “zonal” flows





- For core turbulence, gyro-kinetic codes can now get very close e.g. ITG threshold gradient within 5%
- Fluxes a strong function of gradient, so harder to predict
- Models like TGLF use fits to G-K simulations, and produce quite good results
- A "shortfall" is often observed near the edge, and the cause is still being debated





Summary

- Drift-kinetics and gyro-kinetics average over gyro-motion
- This removes a fast timescale (cyclotron frequency) and a velocity dimension, making realistic 3D simulations possible
- Gyro-kinetics can treat fluctuations of the same size as the Larmor orbit, whereas drift-kinetics cannot, and assumes orbits are small
- Major theoretical and computational improvements have been made since the early '80s
- Simulations can now reproduce experiments with reasonable accuracy

- Still lots of work needed, particularly in extending towards the plasma edge
- Calculation are still far from routine, and several more orders of magnitude speedup are needed