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3-manifolds from Platonic solids

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Abstract

The problem of classifying, up to isometry, the orientable spherical and hyperbolic 3-manifolds that arise by identifying the faces of a Platonic solid is formulated in the language of Coxeter groups. This allows us to complete the classification begun by Best [Canad. J. Math. 23 (1971) 451], Lorimer [Pacific J. Math. 156 (1992) 329], Richardson and Rubinstein [Hyperbolic manifolds from a regular polyhedron, Preprint].
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1. Introduction

One of the first examples of a compact orientable hyperbolic 3-manifold arose from the identification of the faces of a solid hyperbolic dodecahedron [31]. In the intervening years, much more has been said about such manifolds. Yet the classical question of which spherical or hyperbolic manifolds arise by identifying the faces of a Platonic solid has a surprisingly incomplete solution.

In this paper the problem is formulated in terms of classifying certain subgroups of rank four Coxeter groups. This approach is implicit in [16,26] and follows an earlier, oft quoted but flawed attempt in [4]. The manifolds we obtain can be found scattered in the literature, arising from various constructions. The reformulation here has two advantages:

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it provides a unified construction, and more importantly, completely answers for the first time the question of whether the manifolds are distinct.

2. Platonic solids and Coxeter groups

Let $X = S^3, \mathbb{E}^3$ or $\mathbb{H}^3$, and suppose $\Delta \subset X$ is a finite volume Coxeter simplex (see [14]) with symbol,

$$\circ\overset{p}{\text{———}}\circ\overset{q}{\text{———}}\circ\overset{r}{\text{———}}\circ \tag{1}$$

Each node of the symbol corresponds to a face of Δ, which in turn has a vertex of Δ opposite it. Call this the vertex corresponding to the node. Let $\Gamma = \{p, q, r\}$ be the Coxeter group generated by reflections of X in the faces of Δ, and for any vertex, edge or face of Δ, say $*$, let Γ_* be its stabiliser in Γ. In particular, if v is a vertex of Δ, then Γ_v is also Coxeter group, its symbol obtained from (1) by deleting the node corresponding to v together with its incident edges.

Let v be the vertex of Δ corresponding to the left-most node of (1). Then,

$$\Sigma = \bigcup_{\gamma \in \Gamma_v} \gamma(\Delta), \tag{2}$$

is a solid with r-gonal faces, q meeting at each vertex, and dihedral angle (that is, angle subtended by adjacent faces) $2\pi/p$. Similarly for the last node of (1), from which we obtain a solid Σ' with p-gonal faces, q meeting at each vertex and dihedral angle $2\pi/r$. The two tessellations of X by congruent copies of Σ and Σ' that result from successive reflections in their faces are dual to one another, and both have automorphism group Γ.

On the other hand, suppose we have a Platonic solid in X. By this we mean a finite volume polytope P with the combinatorial type of a Platonic solid, and all side lengths equal, as well as interior face angles and dihedral angles. For face identifications of P to yield an X-manifold, the dihedral angle must be a submultiple of 2π, say $2\pi/p$. Barycentric subdivision of P then gives a Coxeter simplex with symbol (1), and P can be recovered in the form (2) using the vertex v of the simplex lying at the center of P. Thus, the problem of obtaining manifolds from a general Platonic solid P reduces to consideration of the Σ obtained at (2).

All Coxeter simplices in X of the form (1) are known and listed in Sections 2.4 and 6.9 of [14]. For $X = S^3$ we have,

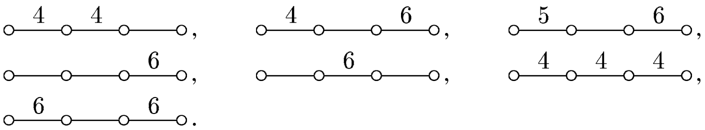
$$\circ\text{———}\circ\text{———}\circ\text{———}\circ, \qquad \circ\overset{4}{\text{———}}\circ\text{———}\circ\text{———}\circ, \qquad \circ\text{———}\circ\overset{4}{\text{———}}\circ\text{———}\circ,$$
$$\circ\text{———}\circ\text{———}\circ\overset{5}{\text{———}}\circ;$$

for $X = \mathbb{E}^3$ we get

$$\circ\overset{4}{\text{———}}\circ\text{———}\circ\overset{4}{\text{———}}\circ;$$

and for $X = \mathbb{H}^3$,

$$\circ\overset{4}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ, \qquad \circ\text{———}\circ\overset{5}{\text{———}}\circ\text{———}\circ, \qquad \circ\overset{5}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ,$$



In the spherical case, the tessellations of S^3 by copies of Σ or Σ' give the six 4-dimensional regular solids [12]. In another incarnation, the first three give Γ that are the Weyl groups of the simple Lie algebras of type $A_4 = \mathfrak{sl}_5(\mathbb{C})$, $B_4 = \mathfrak{so}_9(\mathbb{C})$ and F_4. The hyperbolic Γ give Σ and Σ' of finite volume: the first three compact, the others non-compact, with their vertices lying on the boundary $\partial\mathbb{H}^3$ of hyperbolic space.

We get a total of six spherical, one Euclidean and eight hyperbolic Platonic solids from these groups: spherical tetrahedra with dihedral angles $2\pi/3, 2\pi/4$ and $2\pi/5$, a cube with angle $2\pi/3$, an octahedron with angle $2\pi/3$ and a dodecahedron with angle $2\pi/3$; a compact hyperbolic cube, icosahedron and two dodecahedra with angles $2\pi/5, 2\pi/3, 2\pi/4$ and $2\pi/5$; finally, a non-compact but finite volume hyperbolic cube, octahedron, dodecahedron and tetrahedron with dihedral angles $2\pi/6, 2\pi/4, 2\pi/6$ and $2\pi/6$, respectively. The Euclidean solid is of course the familiar cube.

3. Constructing the manifolds

Any X-manifold (see [28, §3.3]) arises as a quotient X/K by a group K acting properly discontinuously and without fixed points on X. When $X = \mathbb{E}^n$ or $\mathbb{H}^n$, the isometries of X with fixed points are precisely those of finite order, and this allows a simple algebraic formulation of the problem (Theorem 1 below). Alternatively, recourse to a more geometric view yields Theorem 2, which holds for all geometries, and is classical (see, for instance, [25, §10.1]). The statements in the remainder of the paper will be given in terms of the solid Σ, those for Σ' being entirely analogous.

Establishing first some notation, let $\mathfrak{S}_m$ be the symmetric group of degree m. If Λ is a subgroup of $\mathfrak{S}_m$, let Λ_i be the stabiliser of i in the action of Λ on $\{1, \dots, m\}$. For any group G, let $\mathcal{T}(G)$ be a subset that contains *at least one* representative from each conjugacy class in G of elements of finite prime order.

Theorem 1. *Let $X = \mathbb{E}^n$ or $\mathbb{H}^n$ for $n \geqslant 2$; Γ a group acting properly discontinuously by isometries on X with fundamental region P; F a finite subgroup of Γ of order m and*

$$\Sigma = \bigcup_{\gamma \in F} \gamma(P).$$

An X-manifold M arises from the identification of points on the boundary of Σ if and only if there is a homomorphism $\varepsilon : \Gamma \to \mathfrak{S}_m$, such that

(1) *if $\Lambda = \varepsilon(\Gamma)$, then Λ acts transitively on $\{1, \dots, m\}$, and*
(2) *for all $\gamma \in \mathcal{T}(\Gamma)$, the permutation $\varepsilon(\gamma)$ fixes no point of $\{1, \dots, m\}$.*

Moreover, if $i \in \{1, \dots, m\}$, then $\pi_1(M) \cong \varepsilon^{-1}(\Lambda_i)$.

Proof. An X-manifold M arises by identifying points on $\partial\Sigma$ if and only if $M \cong X/K$ for some torsion free group K having Σ as a fundamental region (and then $\pi_1(M) \cong K$). Firstly, K has fundamental region Σ if and only if it is a subgroup of Γ with F a transversal (that is, a complete and non-redundant set of coset representatives). Equivalently, $K \cap F = \{1\}$ and $KF = \Gamma$. Since F is a finite subgroup the first will hold when K is torsion free, and thus the second as well if and only if the index of K in Γ equals the order of the subgroup F. Thus if K is torsion free, we require only that it has index m in Γ.

Certainly, K is a subgroup of index m in Γ if and only if there is a homomorphism $\varepsilon : \Gamma \to \mathfrak{S}_m$ with transitive image Λ (so that for any $i \in \{1, \ldots, m\}$ we then have $\varepsilon^{-1}(\Lambda_i)$ is conjugate in Γ to K). The subgroup is torsion free if and only if it intersects trivially the conjugacy class of each $\gamma \in \mathcal{T}(\Gamma)$, which in turns happens precisely when $\varepsilon(\gamma)$ has no fixed points among the $\{1, \ldots, m\}$. $\square$

We will be applying Theorem 1 with F the stabiliser Γ_v. In an arbitrary Coxeter group Γ, any torsion element is conjugate to an element of a finite parabolic subgroup (see [6, Exercise V.4.2] or [10, Theorem 4], [13]). This subgroup is a finite reflection group whose conjugacy classes can be enumerated (including representatives) by the results of [8]. For the group with symbol (1), or in fact for any 3-dimensional hyperbolic Coxeter group, it is particularly easy to find a $\mathcal{T}(\Gamma)$: one need only take the generating reflections and the powers of their pairwise products that have prime order. To see this, conjugate the fixed point of the torsion element so that it lies on the boundary of the Coxeter simplex Δ.

A more geometric version of Theorem 1 can be formulated. To simplify notation we do so only for $n = 3$. Suppose we have a subgroup K of Γ for which Γ_v is a transversal. Let S be a face of Σ. In the tessellation of X by congruent copies of Σ there is a unique copy Σ_S of Σ with $\Sigma \cap \Sigma_S = S$. Since Σ forms a fundamental region for K, there is a unique element $\gamma_S \in K$ sending Σ to Σ_S, and hence there is a unique face S' of Σ with $\gamma_S(S') = S$. The collection of isometries $\{\gamma_S\}_{S\in\Sigma}$ yield a side-pairing of Σ as in [25, Section 10.1]. The following follows immediately from Theorems 10.1.2 and 10.1.3 of [25].

Theorem 2. *Let $X = S^3, \mathbb{E}^3$ or $\mathbb{H}^3$. An X-manifold M arises from the identification of the faces of* (2) *if and only if Γ has a subgroup K of orientation preserving isometries such that*

(1) *Γ_v forms a transversal in Γ for K;*
(2) *if $\{\gamma_S\}$ are the resulting side pairings of Σ, then γ_S fixes no point of S'; and*
(3) *for $x \in \Sigma$, let $[x]$ denote the points of Σ identified with it under the side pairing. If x lies in the interior of an edge of Σ, then $[x]$ has cardinality p.*

So we merely require that the faces of Σ are identified in pairs and the edges in groups of p. The identifications can be described algebraically as follows: since Γ acts transitively on the k-cells ($k = 0, 1, 2, 3$) of the tessellation of X by Σ, the faces of Σ are in one to one correspondence with the cosets $(\Gamma_f)\gamma$, where f is the common face of Σ and Δ, and $\gamma \in \Gamma_v$. Two faces $(\Gamma_f)\gamma_1$ and $(\Gamma_f)\gamma_2$ are identified by K exactly when $(\Gamma_f)\gamma_1 k = (\Gamma_f)\gamma_2$

for some $k \in K$. Similarly for the edge identifications, where one takes cosets of Γ_e for e the common edge of Δ and Σ.

When $X = S^n$ or $\mathbb{H}^n$, two X-manifolds M_1 and M_2 are the same if and only if there is an X-isometry between them (when $X = \mathbb{E}^n$ similarities are also allowed). Equivalently, if $M_i = X/K_i$, then the K_i are conjugate in the group of isometries of X. In some cases this can be considerably strengthened:

Theorem 3. *Let Γ be a maximal, non-arithmetic, irreducible lattice in $G = \operatorname{Isom}\mathbb{H}^n$, and K_i, $i = 1, 2$, torsion-free subgroups of finite index in Γ such that $\gamma^{-1}K_1\gamma \neq K_2$ for every $\gamma \in \Gamma$. Then the manifolds $M_i = \mathbb{H}^n/K_i$ are non-isometric.*

For basic definitions and results regarding lattices in semisimple Lie groups, see [30]. Arithmetic is meant here in the sense of [29].

Proof. Γ is a non-arithmetic irreducible lattice in the semisimple Lie group $G \cong PO_{1,n}(\mathbb{R})$, hence so is $\Gamma^\circ = \Gamma \cap G^\circ$ in the connected component G° of the identity. By a theorem of Margulis ([18], see [30, Theorem 6.17]), the commensurator $\operatorname{Comm}_{G^\circ}(\Gamma^\circ) = \{g \in G^\circ : g^{-1}\Gamma^\circ g, \Gamma^\circ \text{ are commensurable}\}$ is not dense, hence discrete in G° ([5], see [30, Lemma 6.14]). Thus, $\operatorname{Comm}_G(\Gamma)$ is discrete in G, and by maximality, $\operatorname{Comm}_G(\Gamma) = \Gamma$. For the M_i to be isometric we would require a $g \in G$ with $g^{-1}K_1 g = K_2$. But then such a $g \in \operatorname{Comm}_G(\Gamma)$ hence $g \in \Gamma$, and no such exists by assumption. □

In particular, the hyperbolic Coxeter group with symbol,

$$\circ\overset{5}{\text{———}}\circ\text{———}\circ\overset{6}{\text{———}}\circ, \qquad (3)$$

is non-arithmetic by the results of [29]. In [1] the six cofinite discrete subgroups of $G \cong PO_{1,3}(\mathbb{R})$ having the smallest covolume are enumerated: they are all commensurable with the Bianchi groups $PGL_2\mathcal{O}_1$ or $PGL_2\mathcal{O}_3$, where $\mathcal{O}_d$ is the ring of integers in the number field $\mathbb{Q}(\sqrt{-d})$. Thus the group Γ with symbol (3) is maximal, for if not, then by comparing volumes it is contained as a subgroup of finite index in one of the six above. This cannot be, for these six are arithmetic. We will thus be able to apply Theorem 3 to Γ in Section 4.

4. The manifolds

Of the fourteen Platonic solids listed at the end of Section 2, four can immediately be removed from consideration using Theorem 2: the number of edges of the Σ is not divisible by p, so they will never give manifolds. Of those that remain, the spherical dodecahedron with dihedral angle $2\pi/3$ was handled in [16] with results listed in Table 1 (the notation is described below). The first of the two manifolds is the Poincaré homology sphere. The compact hyperbolic dodecahedron and icosahedron with angles $2\pi/5$ and $2\pi/3$ were investigated in [26] with the results in Table 2. The first eight manifolds

Table 1
The spherical manifolds arising from a dodecahedron with dihedral angle $2\pi/3$, [16]

N	F	E	π_1	H_1
1	abcdefefbcda	a(-+)b(-+)c(-+)d(-+)e(-+)f(-+)g(-+) h(-+)i(-+)j(-+)idjefagbhcghijfeabcd	120	0
2	abcdefbdcfea	a(-+)b(-+)c(-+)d(-+)e(-+)f(++)g(++) h(++)i(++)j(++)ajcgbfeidhfhgjieabcd	120	0(15)

come from the dodecahedron, the others from the icosahedron.[2] The first is the Weber–Seifert space [31]. This leaves the spherical $\{3,3,3\}, \{4,3,3\}, \{3,4,3\}$ and hyperbolic $\{4,4,3\}, \{4,3,6\}, \{5,3,6\}, \{3,3,6\}$.

As the reader may have gathered by now, the only practical way the techniques of the previous section can be implemented is computationally. We use Sims's low index subgroups algorithm as implemented in Magma [7] to find the homomorphisms required by Theorem 1 when $X = \mathbb{H}^3$. For the spherical manifolds, we use Theorem 2. In any case, we obtain a complete list of the K, subgroups of the various Γ, satisfying the conditions of the two theorems. We only seek orientable manifolds, so require that the generators of K are words of even length in the generators for Γ (although it is worth noting that all spherical 3-manifolds are orientable, and a computer search has determined that all closed hyperbolic Platonic manifolds are orientable [23]). It is a consequence of the low index subgroups algorithm that the K we obtain will be *non-conjugate in Γ*, although not necessarily so in G, the full isometry group of X.

The results are listed in Tables 3–5 which we will discuss in some detail presently. First we describe the notation for Tables 1–5. The column headed N indexes the manifolds M_i carrying the indicated geometric structure. The columns F and E give the face and edge identifications in the form of an encoded string of letters and $\pm$ signs to be read in conjunction with Figs. 1 and 2. The ith and jth faces are paired when the ith and jth positions of the string in column F are occupied by the same letter. Similarly for the edge identifications, where a string of $\pm$'s after a letter indicates whether the corresponding edge is identified with subsequent ones with the orientations matching or reversed. For example, the manifold M_{18} arising from the dodecahedron $\{5,3,6\}$ has edge identifications

```
a(+---+)b(+--++)bc(++--++)d(---+-)
  bcae(+-+--)ceadddbeacedcaabecbed,
```

where the `e`'s indicate that edges $9, 11, 17, 20, 26$ and 29 are identified, and the `e(+-+--)` says how edge 9 is identified with edges $11, 17, 20, 26$ and 29: namely, with edge 11 so that the identifications match, with edge 17 so they are reversed, with edge 20 so they match, and so on. From the data in these two columns one may reconstruct the side

[2] While there are pairs in Table 2 with the same first homology, algebraic arguments are provided in [26] that show that the list is non-redundant (this is to be contrasted with the list in [4] which contains isometric pairs). Generally this involves consideration of quotients of terms in the derived series for $K = \pi_1(M)$, for instance, K'/K''.

Table 2
The compact hyperbolic manifolds arising from a dodecahedron with dihedral angle $2\pi/5$ and an icosahedron with angle $2\pi/3$ [26]

N	F	E	H_1
1	abcdefefbcda	a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+) cdeabf(++++)afbfcfdfecdeabdeabc	0555
2	abcdefdefbca	a(++++)b(++++)c(++++)d(++++)e(++++) abcdebf(++++)cfdfefafcdeabbcdea	0555
3	abcdefdefbca	a(+-++)b(-+++)c(---+)d(++-+)e(+-++) debaf(+-++)bcfafefcdcfedabeabcd	033
4	abccadeefbfd	a(++--)ab(-+++)ac(-+-+)d(-+++)bab e(+++-)ef(--+-)bfdcaecdfffddcbece	057
5	abcdefebfdca	a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+) edacbf(++++)cfefbfafdbdaeceabcd	0355
6	abcdeffbdeca	a(++++)b(++++)c(++++)d(++++)e(++++) adbcecf(++++)efdfbfafeacdbdeabc	03355
7	abcdebedffca	a(+-++)b(+-++)c(---+)d(-+-+)e(-+++) cedaef(--+-)afdfbfcfebdcbacdeab	03(16)
8	abbcadefecfd	a(+++-)b(++-+)c(--++)ad(-++-)a e(+-+-)dbbeaecf(+--+)acfceffdedbdbfc	0(29)
9	abcbdaefghihdefjgcji	a(-+)b(+-)c(--)d(-+)e(-+)deabf(++) g(+-)h(-+)i(+-)iaccj(++)jhdebfgfghij	0(11)(11)
10	abcdebfceghhiijjfgda	a(-+)b(-+)c(-+)d(--)e(++)cf(--)ea g(--)ebh(+-)gi(++)dj(+-)fghhdiifjjabc	09
11	abcdefbdgehiijjhfgca	a(++)b(++)c(++)d(++)e(+-)cdf(+-)ad g(+-)bfh(-+)gi(+-)ej(-+)ijgjhehifabc	0229
12	abcdaefdgfhihcjjbige	a(++)b(+-)bc(+-)d(--)e(+-)baf(--) g(+-)efgh(++)ghci(+-)dj(-+)jjdeiicahf	057
13	abcdabefghcijidfjghe	a(++)ab(-+)c(++)d(++)e(--)bacf(+-) g(+-)h(+-)ei(-+)j(++)djfidhgihebgjfc	0(29)
14	abcdaebdfghicjehjfgi	a(++)b(+-)bc(--)d(-+)e(++)bacdef(+-) g(--)h(+-)di(-+)aj(--)ijfehgcighjf	0(29)

Table 3
The spherical manifolds arising from a tetrahedron with dihedral angle $2\pi/3$, a cube with angle $2\pi/3$, and an octahedron with angle $2\pi/3$

N	F	E	π_1	H_1
1	abab	a(--)b(--)aabb	5	05
2	ababcc	a(++)b(+-)aac(+-)bcd(+-)bcdd	8	08
3	abcbca	a(++)b(--)c(+-)cd(--)bdabdac	8	022
4	abcacbdd	a(++)b(+-)c(+-)ad(++)cbdacdb	24	026
5	abcacdbd	a(++)b(-+)c(++)ad(-+)cbcaddb	24	08
6	abcdcdab	a(++)b(++)c(++)d(++)bcdadabc	24	03

pairing transformations[3] $\{\gamma_S\}_{s\in\Sigma}$ of Theorem 2. In particular, the vertex identifications can be obtained in the spherical case; in the hyperbolic there are no vertices! (They lie on the boundary of hyperbolic space in these non-compact examples.)

[3] It is traditional to provide just these, with the vertices labelled rather than the faces and edges. The advantage of our more cumbersome notation, is that it can be presented using less space.

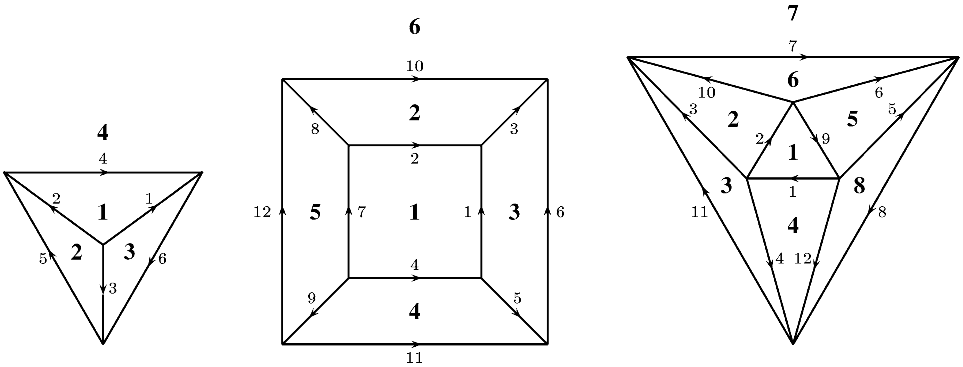


Fig. 1.

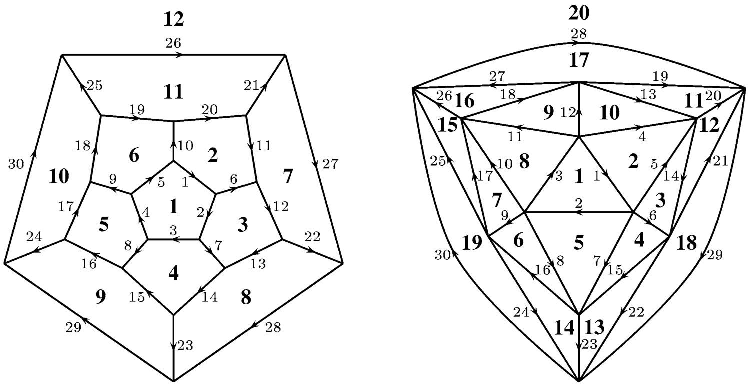


Fig. 2.

The next column in Tables 1 and 3 gives the orders of the fundamental groups. By Theorem 2, part 1, each has order the index in Γ of Γ_v, which in turn is equal to $|\Gamma|$ divided by the number of symmetries of the cell Σ. The orders of the spherical Coxeter groups listed in Section 2 are, from left to right, 120, 384, 1152 and 14400. Table 5 also has a column C that gives the number of cusps of the manifold. The final column gives the first homology $\mathbb{Z}^a \oplus \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$ (obtained by abelianising the K_i) in the form of a sequence `abcde`. Any of b to e that are zero are omitted, and brackets are used in Tables 1 and 2 to distinguish double digits.

Table 3 gives the spherical results, which can also be found in [9]. Manifold M_1 comes from the tetrahedron $\{3,3,3\}$, M_2, M_3 from the cube $\{4,3,3\}$ and M_4, M_5, M_6 from the octahedron $\{3,4,3\}$. Manifold M_3 is Montesinos's quaternionic space [22, page 120] and M_6 is his octahedral space [22, page 117]. No two of the manifolds are isometric, as can be seen by comparing the π_1 and H_1 columns.

For completeness we have included the results arising from the Euclidean cube $\{4,3,4\}$ in Table 4. Unfortunately, our methods are not able to distinguish between the manifolds

Table 4
The Euclidean manifolds arising from a cube with dihedral angle $\pi/2$

N	F	E	H_1
7	abacbc	a(+++)b(+++)aac(+++)bccbcba	13
8	abbcca	a(-+-)ab(--+)c(-+-)bacbbacc	122
9	abccba	a(-+-)ab(--+)c(+--)bccbbcaa	044
10	abcbca	a(+++)b(+++)c(+++)bcaaccbba	3
11	abcbca	a(+++)b(+++)c(-+-)cbaacbbca	12
12	abcbca	a(-+-)b(+--)c(+++)bcaaccbba	122

Table 5
The non-compact, finite volume hyperbolic manifolds arising from an octahedron with dihedral angle $2\pi/4$, a cube with angle $2\pi/6$, and a dodecahedron with angle $2\pi/6$

N	F	E	C	H_1
13	ababcdcd	a(---)aaab(---)c(+-+)bccbcb	2	2
14	abacbdcd	a(-+-)b(+--)babbaac(---)ccc	2	2
15	ababcc	a(++---)b(-+-+-)aabbbbaaba	2	22
16	ababcc	a(++--+)b(+++--)aabbbabbaa	1	124
17	abcbca	a(+-+-+)b(----+)bbabaabaab	2	22
18	abacbddceeff	a(+---+)b(+--++)bc(++--++)d(---+-) bcae(+-+--)ceadddbeacedcaabecbed	1	12
19	abacdcdbefef	a(--+-+)b(+-+--)bc(+----)d(--+-+) bcacdcadddae(----+)badcaceeeebeb	2	22
20	abacdbdcefef	a(-+-++)b(+--++)bc(+--+-)d(+---+) bcaadcadddce(----+)bcdacaeeeebeb	2	22
21	abcacdedeffb	a(++---)b(--+++)abbc(+-++-)bbc d(--++-)dadbadde(---+-)ceecceeadedc	1	122
22	abcacdedfebf	a(+--++)b(--+++)abbc(+-+++)bbc d(-+++-)e(+-+--)edbadedeadcceeaedac	1	222
23	abcacdedfefb	a(+-++-)b(--+++)abbc(++++-)bbc d(-+-+-)e(+-+++)edbadecdacdceeaecad	1	226
24	abcacdedeffb	a(+----)b(--+++)abbc(++-+-)bbc d(-++-+)ae(---+-)dbadecdecdceeeacad	2	222
25	abcacbdeedff	a(+-+--)b(+-+++)ac(+--++)d(-++-+) e(--+-+)dbdedbccaebeadceecbabacd	2	26
26	abcacdebdeff	a(+--++)b(+-+++)ac(-+-+-)d(-++--) e(+-+--)dbdecbccaebeacdeedbabadc	2	22
27	abcbdefdcfae	a(-++--)b(--+++)ac(+++--)d(--+--) ccbdbdae(+++-+)daeeebaceccadebbd	1	122

M_8 and M_{12}. Indeed, by [24], there is a similarity between the two. Manifold M_{10} is the 3-torus.

Table 5 gives the hyperbolic results. Manifolds M_{13} and M_{14} come from the octahedron $\{4, 4, 3\}$, M_{15}, M_{16} and M_{17} from the cube $\{4, 3, 6\}$ and M_i, $i = 18$ to 27, from the dodecahedron $\{5, 3, 6\}$. Manifold M_{14} is the Whitehead link complement [28, Section 3.3] and M_{15} the complement in $\mathbb{R}P^3$ of a two component link [3] (there is a small typographical error in Fig. 9 of [3], where the orientation on the bottom right-most a labelled edge should be reversed). The tetrahedron in $\{3, 3, 6\}$ gave no orientable manifolds, although the non-orientable Gieseking manifold of 1911 is known to arise from it (as indeed do non-orientable examples from $\{4,4,3\}$, $\{4,3,6\}$ and $\{5,3,6\}$, see [23].) There are also a number of examples in the literature of knot and link complements arising from the identification of the faces of *two* regular solids, see [2,11,15,19,27,28,32].

Manifolds M_{13} and M_{14} are non-isometric, despite having the same first homology, for, using low index subgroups in Magma again, K_{13} has five conjugacy classes of index 3 subgroups while K_{14} has six, so these two groups cannot be conjugate in G. For the same reason, M_{15} and M_{17} are distinct.[4] Now the group $\Gamma = \{4,3,6\}$ is arithmetic by [29], and thus the subgroups K_{15} and K_{17} are too. On the other hand, by the comments at the end of Section 3, K_{19}, K_{20} and K_{21} are non-arithmetic, so cannot be conjugate to K_{15} and K_{17}. Hence M_{15} and M_{17} are not isometric to any of M_{19}, M_{20} or M_{26}. This conclusion can also be reached via a volume argument.

Finally, there are a number of pairs with the same first homology among the M_i for $i = 18$ to 27. Clearly M_{16} and M_{18} must be non-isometric, for they have a different number of cusps. In fact, *all ten* are distinct: that the corresponding K_i are non-conjugate in the Γ with symbol (3) is a consequence of the low index subgroups algorithm; now apply Theorem 3.

These manifolds have also been constructed by non-algebraic methods in [21,23,24], but the techniques there do not show that M_{18}–M_{27} are non-isometric.

As a final note, the results summarised in this paper allow us to fill-in some of the blanks in the table on page 202 of Milnor's paper [20]. The ten Coxeter simplices in $\mathbb{H}^3$ listed in Section 2 are given, together with the smallest index of a torsion free subgroup of the corresponding Γ (and hence the volume of the resulting manifold). There are four Γ for which the index of this subgroup is stated as unknown, namely $\Gamma = \{3,5,3\}$, $\{4,3,5\}$, $\{5,3,6\}$ and $\{6,3,6\}$ (our notation). Milnor also states that, "I do not know whether this subgroup is essentially unique". (He also conjectures that there are exactly six commensurability classes of hyperbolic groups with the symbol (1). This is indeed the case—see, for example, Sections 13.1 and 13.2 of [17].)

For $\Gamma = \{3,5,3\}$ and $\{5,3,6\}$ the index of this subgroup, by [26] and this paper, is 120. Moreover, Tables 2 and 5 give that there are at least six and ten conjugacy classes, in $\mathrm{Isom}^+(\mathbb{H}^n)$, of index 120 torsion free subgroups in these respective groups.

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[4] It is interesting to speculate whether, like M_{15}, the manifold M_{17} is also a link complement in $\mathbb{R}P^3$, and if so, whether the two manifolds can be distinguished using the links.

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