CONSTRUCTING HYPERBOLIC MANIFOLDS

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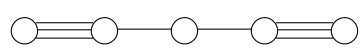
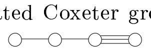
ABSTRACT. The Coxeter simplex with symbol  is a compact hyperbolic 4-simplex and the related Coxeter group Γ is a discrete subgroup of Isom($\mathbb{H}^4$). The Coxeter simplex with symbol  is a spherical 3-simplex, and the related Coxeter group G is the group of symmetries of the regular 120-cell. Using the geometry of the regular 120-cell, Davis [3] constructed an epimorphism $\Gamma \to G$ whose kernel K was torsion-free, thus obtaining a small volume compact hyperbolic 4-manifold $\mathbb{H}^4/K$.

In this paper we show how to obtain representations $\Gamma \to G$ of Coxeter groups Γ acting on $\mathbb{H}^n$ to certain classical groups G. We determine when the kernel K of such a homomorphism is torsion-free and thus $\mathbb{H}^n/K$ is a hyperbolic n-manifold. As an example, this is applied to the two groups described above, with G suitably interpreted as a classical group. Using this, further information on the quotient manifold is obtained.

1. Introduction

Let M^n be an n-dimensional hyperbolic manifold, that is, an n-dimensional Riemannian manifold of constant sectional curvature -1. Thus M^n is isometric to a quotient space $\mathbb{H}^n/K$ of $\mathbb{H}^n$ by the free action of a discrete group $K \cong \pi_1(M^n)$ of hyperbolic isometries.

This paper presents a method of constructing such groups K as the kernels of representations $\Gamma \to G$ of hyperbolic Coxeter groups Γ into finite classical groups. The homomorphisms arise by first representing Γ as a subgroup of the orthogonal group of a quadratic space over a number field k which preserves a lattice. Then reducing modulo a prime ideal in the ring of integers in the number field yields a representation into a finite classical group given as an orthogonal group of a quadratic space over a finite field.

Such kernels K will act freely if and only if they are torsion free. The volume of the resulting manifold $M^n = \mathbb{H}^n/K$ will be $N \times \mathrm{Vol}(P)$, where N is the order of the image in the finite classical group and P is the polyhedron defining the Coxeter group Γ. Starting from a suitable Coxeter group Γ, the method yields infinitely many examples of manifolds. There has been some interest lately in constructing small volume examples when $n \geq 4$ (see [10, 11]). In dimension 4, the compact Davis manifold [3] is constructed by a geometric technique using the existence of a regular compact 120-cell in $\mathbb{H}^4$, which has volume $26 \times 4\pi^2/3$. As an application of our method, we construct a compact 4-manifold M_0 of the same volume which turns out to be isometric to the Davis manifold. With the help of computational techniques, our method gives additional information, producing a presentation for the fundamental group from which we obtain that $H_1(M_0) = \mathbb{Z}^{24}$.

2. Finite representations of hyperbolic Coxeter groups

Consider an $(n+1)$-dimensional real space V equipped with a quadratic form q of signature $(n, 1)$. Thus with respect to an orthogonal basis,

$$q(\mathbf{x}) = -x_{n+1}^2 + \sum_{i=1}^{n} x_i^2. \tag{1}$$

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The quadratic form q determines a symmetric bilinear form

$$B(\mathbf{x},\mathbf{y}) := q(\mathbf{x}+\mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}), \tag{2}$$

with $B(\mathbf{x},\mathbf{x}) = 2q(\mathbf{x})$. The Lobachevski (or hyperboloid) model of $\mathbb{H}^n$ is the positive sheet of the sphere of unit imaginary radius in V (see [12]). Equivalently, we take the projection of the open cone $C = \{\mathbf{x} \in V \mid q(\mathbf{x}) < 0 \text{ and } x_{n+1} > 0\}$ with the induced form.

The isometries of the model are the positive $(n+1) \times (n+1)$ Lorentz matrices, that is, the orthogonal maps of V, with respect to q, that map C to itself.

A hyperplane in $\mathbb{H}^n$ is the image of the intersection with C of a Euclidean hyperplane in V. Each hyperplane is the projective image of the orthogonal complement $\mathbf{e}^{\perp} = \{\mathbf{x} \in C \mid B(\mathbf{x},\mathbf{e}) = 0\}$ of a vector $\mathbf{e}$ with $q(\mathbf{e}) > 0$. Such a vector is said to be space-like, and it is convenient to normalise so that $q(\mathbf{e}) = 1$. The map $r_{\mathbf{e}} : V \to V$ defined by

$$r_{\mathbf{e}}(\mathbf{x}) = \mathbf{x} - 2\frac{B(\mathbf{x},\mathbf{e})}{B(\mathbf{e},\mathbf{e})}\mathbf{e},$$

when restricted to $\mathbb{H}^n$, is the reflection in the hyperplane corresponding to $\mathbf{e}$.

A polyhedron P in $\mathbb{H}^n$ is the intersection of a finite collection of half-spaces, that is, the image of

$$\Lambda = \{\mathbf{x} \in C \mid B(\mathbf{x},\mathbf{e}_i) \leq 0, i = 1,2,\ldots,m\},$$

for some space-like vectors $\mathbf{e}_i$. The intersections of the hyperplanes $\mathbf{e}_i^{\perp}$ with P are the faces of the polyhedron. The dihedral angle θ_{ij} subtended by two intersecting faces of P is determined by $-2\cos\theta_{ij} = B(\mathbf{e}_i,\mathbf{e}_j)$. On the other hand, non-intersecting faces of P have a common perpendicular geodesic of length η_{ij}, where $-2\cosh\eta_{ij} = B(\mathbf{e}_i,\mathbf{e}_j)$.

All of this information is encoded in the Gram matrix $G(P)$ of P, an $m \times m$ matrix with (i,j)-th entry $a_{ij} = B(\mathbf{e}_i,\mathbf{e}_j)$. Let Γ be the group generated by the reflections $r_i := r_{\mathbf{e}_i}$ in the faces of P so that Γ is a subgroup of the isometry group $\mathrm{Isom}(\mathbb{H}^n)$. Moreover, Γ is discrete exactly when all the dihedral angles θ_{ij} of P are integer submultiples π/n_{ij} of π [12], and in this case, Γ is a hyperbolic Coxeter group. The polyhedron P is depicted by means of its Coxeter symbol, with a node for each face, two nodes joined by $n-2$ edges when the corresponding faces subtend a dihedral angle of π/n and other pairs of nodes joined by an edge labelled with the geodesic length between the faces. We use the Coxeter symbol to denote both P and the group Γ arising from it.

Using the Lobachevski model, such a hyperbolic Coxeter group Γ is a subgroup of $O(V,q)$. Using this, [13] gave necessary and sufficient conditions for such a group to be arithmetic. We adopt Vinberg's method to conveniently describe the groups Γ, although they need not be arithmetic.

We first give this method and some general notation which we will use throughout. Let M be a finite-dimensional space over a field F. Equipped with a quadratic form f which induces a symmetric bilinear form (as for instance in (1) and (2)), M is a quadratic space over F. The group $O(M,f)$ of orthogonal maps consists of linear transformations $\sigma : M \to M$ such that $f(\sigma(m)) = f(m)$ for all $m \in M$.

Consider the Gram matrix $G(P) = [a_{ij}]$, and for any $\{i_1,i_2,\ldots,i_r\} \subseteq \{1,2,\ldots,m\}$, define

$$b_{i_1 i_2 \cdots i_r} = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1},$$

and let $k = \mathbb{Q}(\{b_{i_1 i_2 \cdots i_r}\})$. Take the space-like vectors in V defined by

$$\mathbf{v}_{i_1 i_2 \cdots i_r} = a_{1 i_1} a_{i_1 i_2} \cdots a_{i_{r-1} i_r} \mathbf{e}_{i_r}.$$

Let M be the k-subspace of V spanned by the $\mathbf{v}_{i_1 i_2 \ldots i_r}$. A simple calculation gives

$$r_i(\mathbf{v}_{i_1 i_2 \cdots i_r}) = \mathbf{v}_{i_1 i_2 \cdots i_r} - \mathbf{v}_{i_1 i_2 \cdots i_r i}, \tag{3}$$

and

$$B(\mathbf{v}_{i_1 i_2 \cdots i_r}, \mathbf{v}_{j_1 j_2 \cdots j_s}) = b_{1 i_1 \cdots i_r j_s \cdots j_1}. \tag{4}$$

Thus, M is a quadratic space over k under the restriction of q, and from (3) and (4)

$$B(r_i(\mathbf{v}_{i_1 i_2 \cdots i_r}), r_i(\mathbf{v}_{j_1 j_2 \cdots j_s})) = B(\mathbf{v}_{i_1 i_2 \cdots i_r}, \mathbf{v}_{j_1 j_2 \cdots j_s}).$$

It follows that $\Gamma \to O(M, q)$.

Lemma 1. M is an $(n+1)$-dimensional space over $k = \mathbb{Q}(\{b_{i_1 i_2 \cdots i_r}\})$.

Proof. If P has finite volume then the vectors $\mathbf{e}_i$ span V and the Gram matrix is indecomposable [13]. So for each i, there is a $j \neq i$ such that $a_{ij} \neq 0$. Successively choose indices $1 = i_0, i_1, \ldots$ such that the i_k-th row contains a non-zero entry in the (i_{k+1})-st column, for $k \geq 1$. We can ensure that the i_k are distinct. For, if the only non-zero entries of the k-th row are those in the columns with indices $1, i_1, \ldots, i_k$, throw away i_k and go back to the (i_{k-1})-st row to rechoose a different column. Eventually, by discarding and moving backwards, we must be able to rechoose, in the i_j-th row, an index different from all the $i_{j+1}, \ldots, i_k$ discarded. Otherwise, $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ are orthogonal to the other basis vectors, contradicting indecomposability. In this way we must arrive at a sequence $1 = i_0, i_1, \ldots, i_{m-1}$ of length m. Hence, for any i, $\mathbf{e}_i = \mathbf{e}_{i_k}$ for some i_k, and $\mathbf{v}_{i_1 \cdots i_k} = a_{1 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_k} \mathbf{e}_{i_k}$ with coefficient non-zero. Thus, the vectors $\mathbf{v}_{i_1 i_2 \cdots i_r}$ span V over $\mathbb{R}$ and hence M is $(n+1)$-dimensional over $\mathbb{R}$. Now, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\}$ is an $\mathbb{R}$-basis for M and $\mathbf{v} = \sum x_i \mathbf{v}_i \in M$, then the system of equations $B(\mathbf{v}, \mathbf{v}_j) = \sum x_i B(\mathbf{v}_i, \mathbf{v}_j)$ has a unique solution, since the matrix with (i,j)-th entry $B(\mathbf{v}_i, \mathbf{v}_j)$ is invertible. But the solutions $x_i \in k$, since $B(\mathbf{u}, \mathbf{v}) \in k$ for all $\mathbf{u}, \mathbf{v} \in M$. Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\}$ is a k-basis for M. □

We make a number of simplifying assumptions which hold for many examples. Suppose that k is a number field and let $\mathcal{O}$ denote the ring of integers in k. Suppose furthermore that all $b_{i_1 \cdots i_r} \in \mathcal{O}$. Finally let N be the $\mathcal{O}$-lattice in M spanned by the elements $\mathbf{v}_{i_1 \cdots i_r}$ and assume that N is a free $\mathcal{O}$-lattice. This will hold, in particular, when $\mathcal{O}$ is a principal ideal domain.

By (3) above, N is invariant under Γ so that

$$\Gamma \subset O(N, q) := \{\sigma \in O(M, q) \mid \sigma(N) = N\}. \tag{5}$$

With the restriction of q, N is a quadratic module over $\mathcal{O}$. If $\mathfrak{P}$ is any prime ideal in $\mathcal{O}$, let $\bar{k} = \mathcal{O}/\mathfrak{P}$. Reducing modulo $\mathfrak{P}$, we obtain a quadratic space $\bar{N}$ over $\bar{k}$ with respect to $\bar{q}$ and an induced map $\Gamma \to O(\bar{N}, \bar{q})$.

The groups $O(\bar{N}, \bar{q})$ are essentially the finite classical groups referred to earlier. However, the quadratic space $(\bar{N}, \bar{q})$ may not be a regular quadratic space, in which case we must factor out the radical to obtain a regular quadratic space (see Section 4 below). This will occur if the discriminant of $\bar{N}$ is zero. Since the discriminant of $\bar{N}$ is the image in $\bar{k}$ of the discriminant of N this will only occur for finitely many prime ideals $\mathfrak{P}$.

We now attend to the matter of when the kernel of a representation of Γ is torsion-free. In certain circumstances, this can be decided arithmetically using a small variation of a result of Minkowski (see for example [6, page 176]).

Lemma 2. Let k be a quadratic number field, whose ring of integers $\mathcal{O}$ is a principal ideal domain. Let p be a rational prime. Let $\alpha \in \mathcal{O}$ be such that $\alpha \not| 2$, and, if 3 is ramified in the extension $k \mid \mathbb{Q}$, then $\alpha \not| 3$. If $A \in GL(n, \mathcal{O})$ is such that $A^p = I$ and $A \equiv I (\text{mod } \alpha)$, then $A = I$.

Proof. Suppose $A \neq I$ so that $A = I + \alpha E$ where $E \in M_n(\mathcal{O})$ and we can take the g.c.d. of the entries of E to be 1. From $(I + \alpha E)^p = I$ we have

$$pE + \frac{p(p-1)}{2} \alpha E^2 \equiv 0 \, (\text{mod } \alpha^2).$$

Thus $pE \equiv 0 \pmod{\alpha}$ and so $p \equiv 0 \pmod{\alpha}$. Since $\alpha \not| 2$, p is odd. Suppose p is unramified in the extension $k \mid \mathbb{Q}$, then either $p = \alpha$ or $p = \alpha\alpha'$ with $\alpha' \in \mathcal{O}$ and $(\alpha, \alpha') = 1$. So $pE \equiv 0 \pmod{\alpha^2}$ so $E \equiv 0 \pmod{\alpha}$. This is a contradiction.

Now suppose that p is ramified. Then $p = u\alpha^2$ where $u \in \mathcal{O}^*$ and by assumption $p \neq 3$. Expanding as above, but to three terms, gives

$$u\alpha^2 E + u\alpha^2 \frac{p-1}{2}\alpha E^2 + u\alpha^2 \frac{(p-1)(p-2)}{6}\alpha^2 E^3 \equiv 0 \pmod{\alpha^3}.$$

This yields the contradiction $E \equiv 0 \pmod{\alpha}$. □

Corollary 1. If $\alpha \not| 2$ and if 3 is ramified in $k \mid \mathbb{Q}$, $\alpha \not| 3$, then the kernel of the mapping on $GL(n, \mathcal{O})$ induced by reduction (mod α) is torsion-free.

More generally, a geometrical argument allows us to determine when *any* representation has torsion-free kernel, albeit by expending a little more effort. Suppose $v \in P$ is a vertex of the polyhedron P, and Γ_v is the stabiliser in Γ of v. For P of finite volume, v is either in $\mathbb{H}^n$ or on the boundary, and v is called finite or ideal respectively. We have the following "folk-lore" result,

Lemma 3. Suppose Γ is a discrete group generated by reflections in the faces of some polyhedron P in n-dimensional Euclidean space $\mathbb{E}^n$ or n-dimensional hyperbolic space $\mathbb{H}^n$. If $\gamma \in \Gamma$ is a torsion element, then for some vertex $v \in P$, γ is Γ-conjugate to an element of Γ_v.

Notice that in the situation described in the lemma, a Coxeter symbol for Γ_v is obtained in the following way: take the sub-symbol of Γ with nodes (and their mutually incident edges) corresponding to faces of P containing v. For brevity's sake, when we say torsion element from now on, we will mean *non-trivial* torsion element.

Corollary 2. The kernel of a representation $\Gamma \to G$ is torsion-free exactly when every torsion element of every vertex stabiliser Γ_v has the same order as its image in G.

At this point the situation bifurcates into two cases: if $v \in P$ is a finite vertex, then Γ_v is isomorphic to a discrete group acting on the $(n-1)$-sphere S^{n-1} centered on v, hence is finite. Thus, the conditions of the corollary are satisfied exactly when Γ_v and its image in G have the same order.

If v is ideal, then consider a horosphere Σ based at v, and restrict the action of Γ_v to Σ. Then Σ is isometric to an $(n-1)$-dimensional Euclidean space $\mathbb{E}^{n-1}$, and Γ_v acts on it discretely with fundamental region P', the intersection with Σ of P. Any torsion element of Γ_v is then Γ_v-conjugate by Lemma 3 to the stabiliser (in Γ_v!) of a vertex v' of P'. Write $\Gamma_{v,v'}$ for this stabiliser, and observe that it is isomorphic to a discrete group acting on the $(n-2)$-sphere S^{n-2} in $\mathbb{E}^{n-1}$, centered on v', and hence is also finite. The conditions of the corollary are satisfied exactly when for each $v' \in P'$, the group $\Gamma_{v,v'}$ and its image in G have the same order.

Summarising,

Proposition 1. Suppose Γ is a hyperbolic Coxeter group generated by reflections in the faces of a polyhedron P as above. For each finite vertex v of P, take the stabiliser Γ_v. For each ideal vertex, take the stabilisers $\Gamma_{v,v'}$ for each vertex v' of the Euclidean polyhedron P'. Then kernel$(\Gamma \to G)$ is torsion-free if and only if each such Γ_v and $\Gamma_{v,v'}$ has the same order as its image in G.

It is an elementary process to verify the conditions of the proposition. For, each vertex stabiliser is a finite spherical reflection group of some lower dimension, hence from the well-known list (see [5], Section 2.11 for their orders). To find the orders of their images in G, the computational algebra package Magma is enlisted.

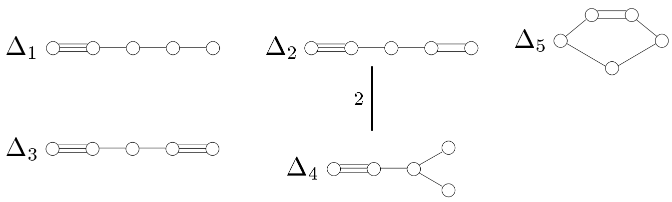


FIGURE 1

3. POLYHEDRA IN $\mathbb{H}^n$

Let P be a polyhedron in $\mathbb{H}^n$, thus the image of

$$\Lambda = \{\mathbf{x} \in C \mid B(\mathbf{x}, \mathbf{e}_i) \leq 0, i = 1, 2, \ldots, m\},$$

for some space-like vectors $\mathbf{e}_i$. On occasion, a connected union of several copies of P will yield another polyhedron of interest. In particular, we may want to glue copies of P onto its faces using some of the reflections r_i as glueing maps.

Lemma 4. If r_i is a reflection in a face of P, then

$$\Lambda \cup r_i(\Lambda) = \Lambda' := \{\mathbf{x} \in C \mid B(\mathbf{x}, \mathbf{e}_j) \text{ and } B(\mathbf{x}, r_i(\mathbf{e}_j)) \leq 0, \text{ for all } j \neq i\}.$$

Proof. If $\mathbf{x} \in \Lambda'$, then either $B(\mathbf{x}, \mathbf{e}_i) \leq 0$, in which case $\mathbf{x} \in \Lambda$, or $B(\mathbf{x}, \mathbf{e}_i) > 0$, in which case $B(\mathbf{x}, r_i(\mathbf{e}_i)) \leq 0$, hence $\mathbf{x} \in r_i(\Lambda)$. Conversely, if $\mathbf{x} \in \Lambda$, then $B(\mathbf{x}, \mathbf{e}_j) \leq 0$ for all j. If $j \neq i$, then

$$B(\mathbf{x}, r_i(\mathbf{e}_j)) = B(\mathbf{x}, \mathbf{e}_j) - B(\mathbf{x}, \mathbf{e}_i)B(\mathbf{e}_i, \mathbf{e}_j) \leq 0,$$

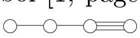
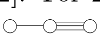
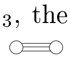
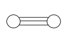
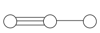
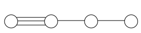
since all three terms are ≤ 0. A similar argument deals with the $\mathbf{x} \in r_i(\Lambda)$. □

We illustrate the lemma by considering the situation in four dimensions. In particular, if P is a compact simplex it has Coxeter symbol one of the five depicted in Figure 1 (see [5], Section 6.9). In fact, and this explains the idiosyncratic numbering, $\text{Vol}(\Delta_i) < \text{Vol}(\Delta_j)$ if and only if $i < j$ (see [7]). Suppose the nodes of Δ_2, read from left to right, correspond to hyperplanes $\mathbf{e}_i^\perp$ for $i = 1, \ldots, 5$. If $r_5 = r_{\mathbf{e}_5}$, we have

$$\Delta_2 \cup r_5(\Delta_2) = \{\mathbf{x} \in C \,|\, B(\mathbf{x}, \mathbf{e}_i) \leq 0, i = 1, \ldots, 4, \text{ and } B(\mathbf{x}, r_5(\mathbf{e}_4)) \leq 0\},$$

since $r_5(\mathbf{e}_i) = \mathbf{e}_i$ for $i = 1, 2$ and 3. Now, $B(\mathbf{e}_3, r_5(\mathbf{e}_4)) = -2\cos\pi/3$ and $B(\mathbf{e}_4, r_5(\mathbf{e}_4)) = -2\cos\pi/2$, so $\Delta_2 \cup r_5(\Delta_2)$ is a simplex with Coxeter symbol Δ_4. Thus, if Γ_i is the group generated by the reflections in the faces of Δ_i, we have that Γ_4 has index two in Γ_2. By comparing the volumes of the simplices using the results of [7], the only other possible inclusions are Γ_4 and Γ_3 as subgroups of indices 17 and 26 respectively in Γ_1. But a low index subgroups procedure in MAGMA shows that Γ_1 has no subgroups of these indices. Thus Figure 1 is a complete picture of the possible inclusions.

4. AN EXAMPLE

In this section, we apply our method in dimension 4 starting with the Coxeter simplex Δ_3 and related group Γ_3 described above. If P is a finite volume Coxeter polyhedron in $\mathbb{H}^4$, then $\text{vol}(P) = \chi(P)4\pi^2/3$ where $\chi(P)$ is the Euler characteristic of P (see [4]), which coincides with the Euler characteristic of the associated group. This is readily computed from the Coxeter symbol [1, page 250], [2]. For Δ_3, the Euler characteristic is 26/14400. The vertex stabilisers are , $\mathbb{Z}_2 \times$ ,  $\times$ ,  $\times \mathbb{Z}_2$ and , having orders $14400, 240, 100, 240$ and 14400 (see [5], Section 2.11). Thus the minimum index any torsion free subgroup of Γ_3 can have is 14400, and we show that there is a normal torsion free subgroup of precisely this index. The corresponding manifold then has Euler characteristic 26, making it the same volume as the Davis manifold [3]. Indeed, it has been shown in [9] that Γ_3 has a unique torsion-free normal subgroup of index 14400. It follows that this manifold is the Davis manifold.

Note that

$$G(\Delta_3) = \begin{pmatrix} 2 & -2\cos\pi/5 & 0 & 0 & 0 \\ -2\cos\pi/5 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -2\cos\pi/5 \\ 0 & 0 & 0 & -2\cos\pi/5 & 2 \end{pmatrix},$$

so $k = \mathbb{Q}(\sqrt{5})$ and M is a 5-dimensional space over $\mathbb{Q}(\sqrt{5})$. Notice that all the a_{ij}, and hence the $b_{i_1 i_2 \cdots i_r}$, are algebraic integers. Now, $\mathbf{v}_2 = -(1+\sqrt{5})/2\ \mathbf{e}_2$, $\mathbf{v}_{21} = (3+\sqrt{5})/2\ \mathbf{e}_1$, $\mathbf{v}_{23} = (1+\sqrt{5})/2\ \mathbf{e}_3$, $\mathbf{v}_{234} = -(1+\sqrt{5})/2\ \mathbf{e}_4$, $\mathbf{v}_{2345} = (3+\sqrt{5})/2\ \mathbf{e}_5$, and since these coefficients are all units in $\mathcal{O}$, we have $N = \mathcal{O}$-span of $\{\mathbf{e}_1, \ldots, \mathbf{e}_5\}$. The criterion of [13] show that Γ_3 is arithmetic.

Let $\mathfrak{P} = \sqrt{5}\mathcal{O}$, so that $\bar{k}$ is the field $\mathbb{F}_5$ of five elements. By Lemma 2, the kernel of $\Gamma \to O(\bar{N}, \bar{q})$ is torsion free. Since $(1+\sqrt{5})/2 \equiv -2 \pmod{\mathfrak{P}}$, we have that

$$\bar{q}(\mathbf{x}) = x_1^2 + 2x_1x_2 + x_2^2 - x_2x_3 + x_3^2 - x_3x_4 + x_4^2 + 2x_4x_5 + x_5^2.$$

We use the same letters $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_5}\}$ for the basis of $\bar{N}$. The images of the generating reflections of Γ_3 are then 5×5 matrices with entries in $\mathbb{F}_5$. The computational system MAGMA then shows that the group they generate has order 14400 so that the kernel has index 14400 in Γ_3 as required.

The index 14400 is too large to allow MAGMA to implement the Reidemeister-Schreier process to obtain a presentation for K. However, closer examination of the image group allows this process to be implemented by splitting into two steps. We will deal with this now.

The bilinear form $\bar{B}$ on $\bar{N}$ is degenerate and there is a one-dimensional radical $\bar{N}^\perp$ spanned by $\mathbf{v}_0 = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4 - \mathbf{e}_5$. Thus $\bar{N} = W \oplus \bar{N}^\perp$. If $\mathbf{w} \in W$ and $\sigma \in O(\bar{N}, \bar{q})$, then $\sigma(\mathbf{w}) = \mathbf{w}' + t\mathbf{v}_0$ where $\mathbf{w}' \in W$ and $t \in \mathbb{F}_5$. The induced mapping $\bar{\sigma}$ defined by $\bar{\sigma}(\mathbf{w}) = \mathbf{w}'$ is easily seen to be an orthogonal map on W and we obtain a representation $\Gamma_3 \to O(W, \bar{q})$. We now identify $O(W, \bar{q})$ as one of the classical finite groups using the notation in [8]. Let $\mathbf{g}_i, \mathbf{h}_i \in W$, for $i = 1, 2$ be defined by $\mathbf{g}_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{h}_1 = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{g}_2 = \mathbf{e}_1 + 2\mathbf{e}_5$ and $\mathbf{h}_2 = -\mathbf{e}_1 + 2\mathbf{e}_5$. Then $\bar{q}(\mathbf{g}_i) = \bar{q}(\mathbf{h}_i) = 0$ and $\bar{B}(\mathbf{g}_i, \mathbf{h}_j) = \delta_{ij}$. Thus $O(W, \bar{q}) \cong O_4^+(5)$. There is a chain of subgroups

$$1 \overset{2}{\subset} Z \overset{3600}{\subset} \Omega_4^+(5) \overset{2}{\subset} SO_4^+(5) \overset{2}{\subset} O_4^+(5),$$

where Z is the largest normal soluble subgroup of $\Omega_4^+(5)$ and

$$\Omega_4^+(5) \cong \frac{SL(2,5) \times SL(2,5)}{\langle(-I, -I)\rangle}. \tag{6}$$

The image of Γ_3 is isomorphic to a subgroup of index 2 in $O_4^+(5)$, different from $SO_4^+(5)$ and the orientation-preserving subgroup Γ_3^+ maps onto $\Omega_4^+(5)$. The target group has a normal subgroup of index 60 with quotient isomorphic to $PSL(2,5)$ and hence so does Γ_3^+. Using MAGMA we find a presentation for this subgroup K_1 with three generators and nine relations. The group K is then the kernel of the induced map from K_1 onto $SL(2,5)$. Again using MAGMA , we obtain a presentation for K on 24 generators and several pages of relations. The abelianisation of K is $\mathbb{Z}^{24}$. This agrees with the homology calculations in [9].

This calculation is readily carried out once the images of the generators of Γ_3^+ are identified with pairs of matrices. We sketch the method of obtaining this description.

Let V be a two dimensional space over $\mathbb{F}_5$ with symplectic form f defined with respect to a basis $\mathbf{n}_1, \mathbf{n}_2$ by

$$f(\sum x_i\mathbf{n}_i, \sum y_i\mathbf{n}_i) = x_1y_2 - x_2y_1.$$

Let $U = V \otimes V$ and define g on U by

$$g(\mathbf{v}_1 \otimes \mathbf{v}_2, \mathbf{w}_1 \otimes \mathbf{w}_2) = f(\mathbf{v}_1, \mathbf{w}_1)f(\mathbf{v}_2, \mathbf{w}_2).$$

Then g is a symmetric bilinear form on U and $O(U, g) \cong O_4^+(5)$. Note that $SL(2,5) \times SL(2,5)$ acts on U by

$$(\sigma, \tau)(\mathbf{v} \otimes \mathbf{w}) = \sigma(\mathbf{v}) \otimes \tau(\mathbf{w}),$$

and this action preserves g with $(-I, -I)$ acting trivially. This describes the group $\Omega_4^+(5)$. Additionally, the mapping $\rho : U \to U$ given by $\rho(\mathbf{v} \otimes \mathbf{w}) = \mathbf{w} \otimes \mathbf{v}$ also lies in $O(U, g)$ and has determinant -1. Let H be the subgroup generated by $\Omega_4^+(5)$ and ρ.

We identify the image of Γ_3 with H, by first identifying U and W by the linear isometry induced by

$$\mathbf{n}_1 \otimes \mathbf{n}_1 \mapsto \mathbf{g}_1, \mathbf{n}_1 \otimes \mathbf{n}_2 \mapsto \mathbf{g}_2, \mathbf{n}_2 \otimes \mathbf{n}_1 \mapsto -\mathbf{h}_2, \mathbf{n}_2 \otimes \mathbf{n}_2 \mapsto \mathbf{h}_1.$$

It is now easy to check that the image of r_5 is ρ. Recall that Γ_3^+ is generated by $x = r_5r_4, y = r_5r_3, z = r_5r_2, w = r_5r_1$. Now determine the images of x, y, z, w as pairs of matrices in $SL(2,5) \times SL(2,5)$.

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