Symmetry and partial symmetry

Contents : 1. Motivation
2. Reflection groups
3. hyperplane arrangements
4. Inverse monoids
5. Reflection monoids

Lecture 1 : Motivation

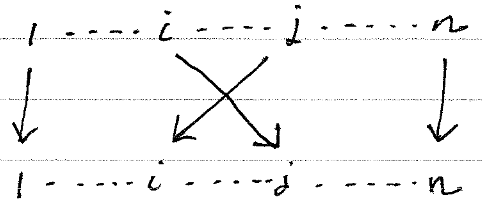
(i).
• **Symmetry** : X set , $\mathfrak{S}_X$ symmetric group on X
ie: group of all bijections $X \to X$
(under composition)
"measures symmetry" of X.

$X = \{1, \ldots, n\}$, write $\mathfrak{S}_n$ for $\mathfrak{S}_X$.

recall : any $\pi \in \mathfrak{S}_n$ is a product of transpositions,

ie: permutations (i, j).

$\mathfrak{S}_n$ generated by the transpositions



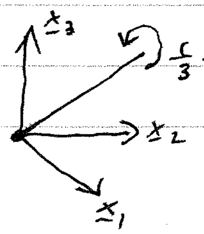
(ii). Another version: $V = \mathbb{R}^n$ with usual basis $\{\underline{x}_1, \ldots, \underline{x}_n\}$.

$GL(V)$ = group of invertible linear maps $V \to V$.

$\pi \in \mathfrak{S}_n \rightsquigarrow$ invertible linear map s.t $\underline{x}_i \longmapsto \underline{x}_{i\pi}$

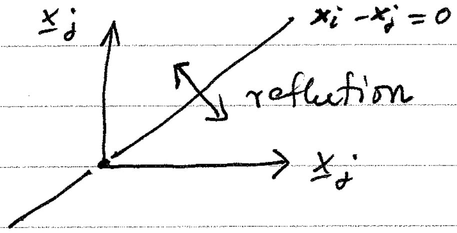
embeds $\mathfrak{S}_n \subset GL(V)$ (permuting coordinates)

Eg : $\pi = (1, 2, 3) \in \mathfrak{S}_3 \rightsquigarrow$

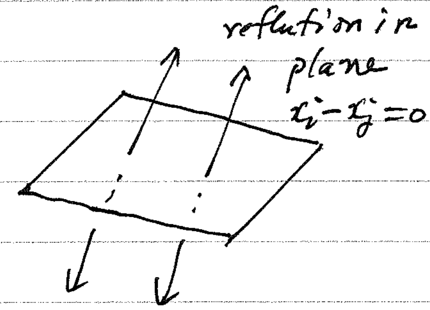


$\frac{1}{3}$ turn rotation of $\mathbb{R}^3$

Eg: $\pi = (i,j) \in \mathfrak{S}_n$



ie: $\pi \rightsquigarrow$



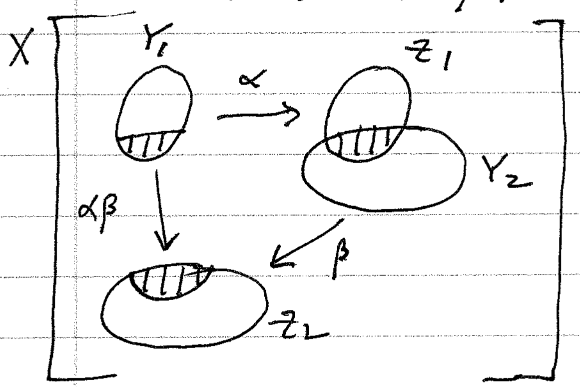
conclusion: $\mathfrak{S}_n \subset GL(V)$ generated by reflections, ie: is a reflection group (see lecture 2).

(iii). Yet another version: $\mathfrak{S}_n$ is the Weyl group of the linear algebraic group $GL_n\mathbb{R}$.

• Partial symmetry : (i). X set and $\mathcal{I}_X$ symmetric inverse monoid on X
= inverse monoid partial bijections of X.

partial bijection a bijection $Y \to Z$ for $Y, Z \subset X$.

(in particular "full" bijections $X \to X$ and "empty" bijection $0 : \emptyset \to \emptyset$).



$\alpha\beta : (Z_1 \cap Y_2)\alpha^{-1} \longrightarrow (Z_1 \cap Y_2)\beta$

or, $(Z_1 = Y_1\alpha)$

$\alpha\beta : Y_1 \cap Y_2\alpha^{-1} \longrightarrow Y_1(\alpha\beta) \cap Y_2\beta$.

gives $\mathcal{I}_X$ structure of a monoid: $\begin{cases} \text{associative} \\ \text{identity } id : X \to X \\ \text{no inverses in general} \end{cases}$

<u>inverse</u> monoid : there are "local" inverses

$$X\left[\begin{tikzcd} Y \arrow[r, shift right, "\alpha"'] & Z \arrow[l, shift right, "\beta"'] \end{tikzcd}\right]$$

for $\alpha \in \mathcal{I}_X$ there is $\beta \in \mathcal{I}_X$
with $\alpha\beta = id_Y$
$\beta\alpha = id_Z$

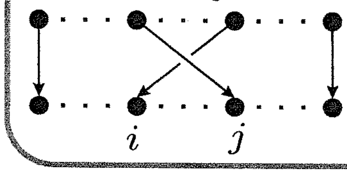
(ii). G_n a reflection group ... $\mathcal{I}_n$ a reflection monoid?
(yes : see lecture 5)

(iii). G_n a Weyl group .... $\mathcal{I}_n$ the <u>Renner monoid</u> of the linear algebraic <u>monoid</u> $M_n\mathbb{R}$ (= all $n\times n$ $\mathbb{R}$-matrices).

Permutation groups

bijections $X \to X$

$X = \{1, 2, \dots, n\}$

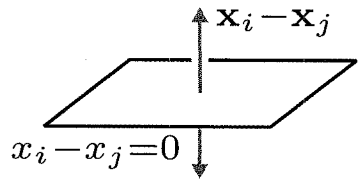


$\rightsquigarrow (i, j)$

Reflection groups

V Euclidean space

basis $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$



$\langle \text{reflection } s_{ij} \rangle \subset \mathrm{GL}(V)$

$s_{ij} \rightsquigarrow (i, j)$

Symmetric $\boxed{\mathfrak{S}_n}$ group

Weyl groups

$\mathbb{G} = \mathbf{GL}_n(\mathbb{F})$

$T = \mathbf{D}_n^*(\mathbb{F}) \subset \mathbf{GL}_n(\mathbb{F})$ torus

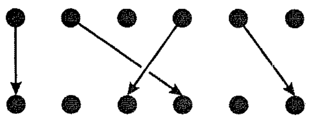
$W = N_{\mathbb{G}}(T)/T$ permutation matrices

$$\begin{array}{cc} & \begin{array}{ccccccccccc} & & & i & & & & j & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array} & \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & & & 1 & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & 1 & & & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \end{array} \rightsquigarrow (i, j)$$

Partial permutations

bijections $X \supset Y \to Y' \subset X$

$X = \{1, 2, \ldots, n\}$



Reflection monoids

?

Symmetric

$\mathscr{I}_n$

inverse monoid

Renner monoids

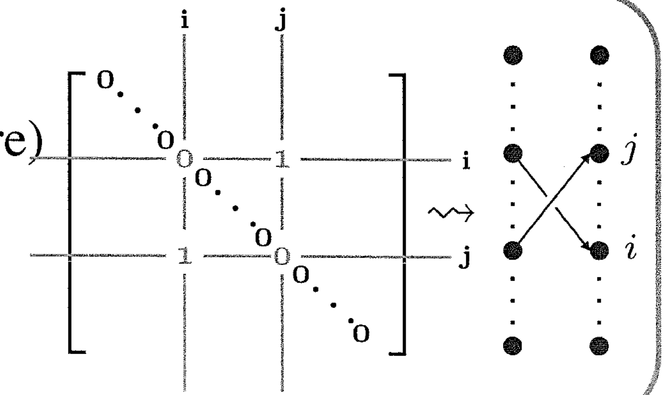
$\mathbb{G} = \mathrm{GL}_n(\mathbb{F}) \subset \mathrm{M}_n(\mathbb{F}) = \mathbb{M}$

$T = \mathrm{D}_n^*(\mathbb{F}) \subset \mathrm{D}_n(\mathbb{F}) = \overline{T}$ (Zariski closure)

$W = N_{\mathbb{G}}(T)/T \subset \overline{N_{\mathbb{G}}(T)}/T$

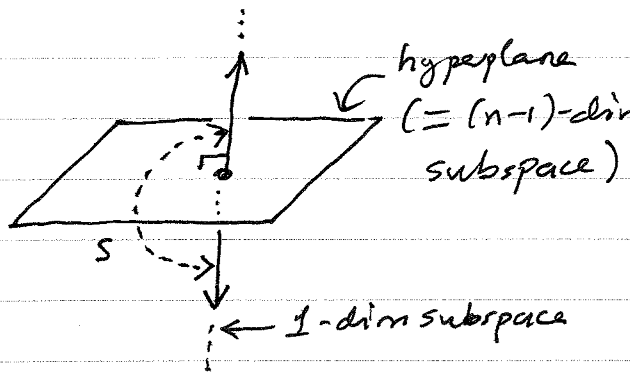
partial permutation matrices

=Rook monoid



Lecture 2 : Reflection groups

- What is a reflection? $V = n$-dim. $\mathbb{R}$-space



$\overbrace{\mathbb{R}\oplus\cdots\oplus\mathbb{R}}^{\times 1}\oplus\overset{\times(-1)}{\mathbb{R}}$ ... a real reflection.

-1 = the primitive n-th root of 1 in $\mathbb{R}$.

k any field, then a k-reflection : $\overbrace{k\oplus\cdots\oplus k}^{\times 1}\oplus\overset{\times\zeta}{k}$ $\;\zeta \leftarrow$ primitive nth root of 1 in k

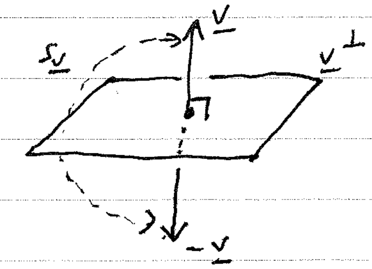
If V a k-space, $GL(V)$ = gp. invertible linear maps of V

then a k-reflection gp := subgp. of $GL(V)$ generated by k-reflections.

In these lectures, we will restrict to finite, real reflection gps.

- throughout, V a real space with basis $\{\underline{x}_1, \ldots, \underline{x}_n\}$ and inner product $(\underline{x}_i, \underline{x}_j) = \delta_{ij}$ (i.e. V Euclidean).

$\underline{v} \in V$, $s_{\underline{v}}$:= linear map $V \to V$ fixing $\underline{v}^\perp$ pt.-wise and sending $\underline{v} \mapsto -\underline{v}$ ($\underline{v}^\perp$ a hyperplane)

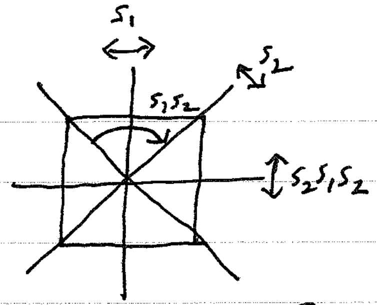


(Ex: $s_{\underline{v}} : \underline{u} \mapsto \underline{u} - 2\dfrac{(\underline{u}, \underline{v})}{(\underline{v}, \underline{v})}\underline{v}$).

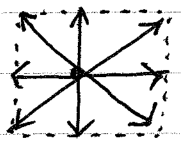
$W \subset GL(V)$ finite is a reflection gp $\overset{\text{def}}{\iff} W = \langle s_{\underline{v}_1}, \ldots, s_{\underline{v}_k} \rangle$.

• **Eg**: $V=\mathbb{R}^2$

$W=$ symmetries of square



$W=\langle s_1,s_2\rangle$

$\Phi=$  the $\pm \underline{v}$ with $s_{\underline{v}}\in W$

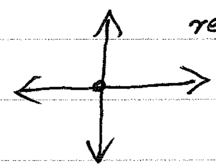
$$(*)\begin{cases}\Phi \text{ finite set non-zero vectors}\\ \underline{v}\in\Phi \Rightarrow \mathbb{R}\underline{v}\cap\Phi=\pm\underline{v}\\ \underline{v}\in\Phi\Rightarrow \Phi s_{\underline{v}}=\Phi\end{cases}$$

• A $\Phi\subset V$ satisfying (*) a rootsystem (the $\underline{v}\in\Phi$ a roots)

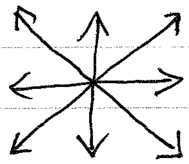
$$W(\Phi)=\langle s_{\underline{v}}\mid \underline{v}\in\Phi\rangle\subset GL(V) \text{ a finite ref. gp.}$$

rootsystems $\longleftrightarrow$ finite $\mathbb{R}$-ref. gps.
(combinatorics) (gp. theory)

Φ reducible iff $V=V_1\perp V_2$ with $\Phi_i\subset V_i$; irreducible otherwise.
$\Phi=\Phi_1\cup\Phi_2$



reducible



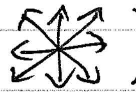
irred.

(Ex: Φ reducible $\Rightarrow W(\Phi)\cong W(\Phi_1)\times W(\Phi_2)$).

• the irred. rootsystems have been "classified":

A_{n-1} $(n\geq 2)$, B_n $(n\geq 2)$, D_n $(n\geq 4)$, $I_2(m)$ $(m\geq 3)$

$H_3, H_4, F_4, E_6, E_7, E_8$.

• Eg: $B_n = \{\pm x_i \pm x_j\} \cup \{\pm x_i\}$ ($B_2 =$ )

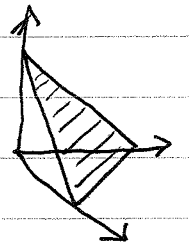
$W(B_n) \cong$ symmetries of n-cube / n-octahedron.

• Eg: $A_{n-1} = \{ x_i - x_j \}$ $s_{x_i - x_j} : x_i - x_j \mapsto x_j - x_i$

$\rightsquigarrow \quad (i,j) \in \mathfrak{S}_n$

$W(A_{n-1}) \cong \mathfrak{S}_n \subset GL(V)$ under permutation action

$x_i \mapsto x_{i\pi}$



$=$ symmetries of $(n-1)$-simplex $\subset \mathbb{R}^n$.

$= \{ \sum \lambda_i x_i \mid \sum \lambda_i = 1 \text{ and } \lambda_i \geq 0 \}$

• Eg: $W(D_n) \cong$ symmetries of n-dim. cross polytope.

• Eg: $W(H_3) \cong$ ——" —— icosahedron/dodecahedron

$W(H_4) \cong$ ——" —— 120/600-cell

$W(F_4) \cong$ ——" —— 24-cell

} 4-dim. Platonic solids

Lecture 3 : Hyperplane arrangements

Throughout: V Euclidean with orthonormal basis $\{\underline{x}_1, \dots, \underline{x}_n\}$.

- A real arrangement $\mathcal{A}$ is a finite set of linear hyperplanes in V.

describing hyperplanes: (i). $\underline{v} \in V$ and hyperplane

$H := \underline{v}^\perp = \{\underline{u} \in V \mid (\underline{u}, \underline{v}) = 0\}$, or (ii). coordinate maps

$x_i : V \to \mathbb{R}$ with $x_i(\sum t_i \underline{x}_i) = t_i$. If $\underline{v} = \sum a_i \underline{x}_i$ then

$\underline{v}^\perp$ the kernel of map $a_1x_1 + \cdots + a_nx_n : V \to \mathbb{R}$. (ie: Cartesian equation)

- Eg: the Boolean arrangement $\mathcal{A} = \{\underline{x}_1^\perp, \dots, \underline{x}_n^\perp\}$ or

$\{$hyperplanes with equations $x_1 = 0, x_2 = 0, \dots, x_n = 0\}$.

- Eg: braid arrangement $\mathcal{A} = \{$hyperplanes $x_i - x_j = 0$ for all $i \neq j\}$.

recall (Lecture 2) the root system $A_{n-1} = \{\underline{x}_i - \underline{x}_j \ (i \neq j)\} \subset V$

with reflection gp. $W(A_{n-1}) \cong \mathfrak{S}_n$

Thus, braid arrangement = the reflecting hyperplanes of this ref. gp.

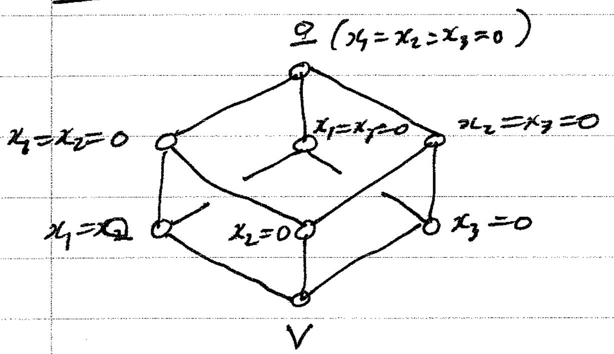
In general, a reflection arrangement = $\{$reflecting hyperplanes of some reflection gp$\}$.

- Combinatorics (of hyperplane arrangements).

intersection lattice $L(\mathcal{A})$:= all possible intersections of elements of $\mathcal{A}$ partially ordered by reverse inclusion.

— a poset. (together with V itself).

Eg: Boolean arrangement ($n=3$): $\mathcal{A} = \{x_1=0, x_2=0, x_3=0\}$

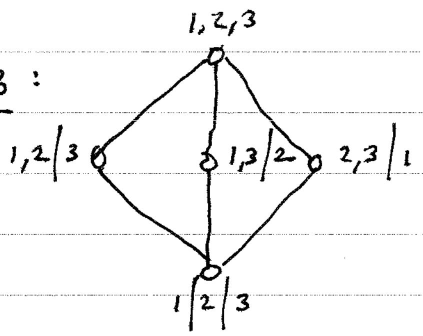


$\cong$ (as a poset) poset of subsets of $\{1,2,3\}$, ie: a Boolean algebra.

Eg: braid arrangement: let $I = \{1, \dots, n\}$. A partition of I is a $\Lambda = \{\Lambda_1, \dots, \Lambda_p\}$ (blocks) with $\Lambda_i \subset I$, $\Lambda_i \cap \Lambda_j = \phi$ and $I = \bigcup \Lambda_i$.

If $\Lambda' = \{\Lambda'_1, \dots, \Lambda'_q\}$ then define $\Lambda \le \Lambda' \overset{def}{\iff}$ for each Λ_i there is a Λ'_j with $\Lambda_i \subset \Lambda'_j$. Write $\Pi(n)$ for set of partitions of I; the set $\Pi(n)$ with partial order $\le$ is a poset called the partition lattice.

$n=3$:



Ex: there is a bijective map $\Lambda \mapsto X(\Lambda)$ from $\Pi(n) \to L(\mathcal{A})$ such that $\Lambda \le \Lambda' \iff X(\Lambda) \supseteq X(\Lambda')$.

ie: an isomorphism of posets.

def^n: for any $L(\mathcal{A})$ the Möbius f$^{\underline{ction}}$ $\mu : L(\mathcal{A}) \to \mathbb{R}$ is

$$\mu(X) = \begin{cases} 1, & \text{if } X = V \\ -\sum_{Y \supset X} \mu(Y), & \text{if } X \neq V \end{cases}$$

and the Poincaré polynomial is

$$\pi(\mathcal{A},t) := \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{codim} X}$$

<u>beautiful result 1</u> (Zaslavsky): $\pi(\mathcal{A},1) = N^{o}$ regions, i.e.:

N^{o} of connected components in $V - \bigcup_{X \in \mathcal{A}} X$.

• <u>Topology</u> (of hyperplane arragements)

$\mathcal{A} \subset V$ hyperplane arrangement $\overset{\text{complexify}}{\rightsquigarrow}$ $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, $\mathcal{A}_{\mathbb{C}} = \{ X \otimes_{\mathbb{R}} \mathbb{C} \mid X \in \mathcal{A} \}$

<u>Ex</u> (amusing): $V \setminus X$ disconnected
$V_{\mathbb{C}} \setminus (X \otimes \mathbb{C})$ connected!

Form space $M_{\mathcal{A}} := V_{\mathbb{C}} - \bigcup_{\mathcal{A}_{\mathbb{C}}} (X \otimes_{\mathbb{R}} \mathbb{C})$, the complement of the hyperplanes in $V_{\mathbb{C}}$.

The <u>Poincaré</u> polynomial of $M_{\mathcal{A}}$ defined to be $\operatorname{Poin}(M_{\mathcal{A}},t) := \sum_{k \geq 0} \operatorname{rk} H^k(M_{\mathcal{A}},\mathbb{Z})\, t^k$.

<u>beautiful result 2</u> (Arnold): $\mathcal{A}$ = braid arrangement

$$\Rightarrow \operatorname{Poin}(M_{\mathcal{A}},t) = (1+t)(1+2t)\cdots(1+(n-1)t).$$

<u>beautiful result 3</u> (Orlik-Solomon): for any $\mathcal{A}$,

$$\operatorname{Poin}(M_{\mathcal{A}},t) = \pi(\mathcal{A},t).$$

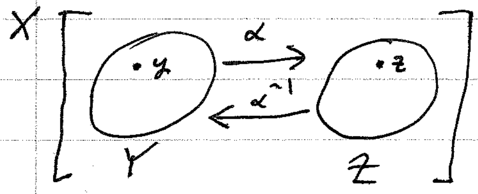
Lecture 4 : Inverse monoids

• $X=\{1,\dots,n\}$. Recall symmetric inverse monoid I_n has elements partial bijections $Y\xrightarrow{\alpha}Z$ $(Y,Z\subseteq X)$ under composition of partial maps.

Write $\operatorname{dom}\alpha=Y$, $\operatorname{im}\alpha=Z$.

I_n monoid: $\begin{cases}\text{composition of partial maps is associative}\\ id: X\to X \text{ with } id\cdot\alpha=\alpha=\alpha\cdot id \text{ for all } \alpha\end{cases}$

• $Y\subseteq X$, $id_Y: Y\to Y$ partial identity; if $\alpha\in I_n$ and $Y=\operatorname{dom}\alpha$ then $id_Y\cdot\alpha=\alpha$ (but, $id_Y\cdot\beta\neq\beta$ in general).



For $\alpha\in I_n$ there is $\alpha^{-1}\in I_n$ defined by
$\alpha^{-1}(z)=y \Leftrightarrow z\in Z$ and $\alpha(y)=z$
i.e.: partial bij. with $\operatorname{dom}\alpha^{-1}=\operatorname{im}\alpha$
$\operatorname{im}\alpha^{-1}=\operatorname{dom}\alpha$

have $\alpha\alpha^{-1}=id_Y$, $\alpha^{-1}\alpha=id_Z$ (note $\alpha\alpha^{-1}\neq\alpha^{-1}\alpha$)

$\Rightarrow \alpha\alpha^{-1}\alpha=\alpha$ and $\alpha^{-1}\alpha\alpha^{-1}=\alpha^{-1}$

I_n inverse monoid: a monoid M s.t. for all $a\in M$ there is a unique $a^{-1}\in M$ with $aa^{-1}a=a$ and $a^{-1}aa^{-1}=a^{-1}$.

!! Caution: in an inverse monoid we have $(a^{-1})^{-1}=a$ and $(ab)^{-1}=b^{-1}a^{-1}$; we do not have $aa^{-1}=a^{-1}a$ or $ab=ac\Rightarrow b=c$ (as we do in a group).

• In I_n we have the bijections $\alpha: X\to X$, i.e.: S_n is a subgp. of I_n

consisting of precisely those $\alpha \in \mathcal{I}_n$ with $\alpha\alpha^{-1} = id = \alpha^{-1}\alpha$.

For M inverse monoid, the <u>units</u> $G = G(M)$ are those $a \in M$ with

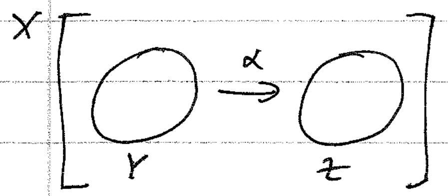
$aa^{-1} = id = a^{-1}a$.

- $Y \subset X \Rightarrow id_Y \cdot id_Y \, (= id_Y^2) = id_Y$. For M (inverse) monoid the <u>idempotents</u> $E = E(M)$ are those $e \in M$ s.t $e^2 = e$.

(in $\mathcal{I}_n$ this includes the empty map $0_X : \emptyset \to \emptyset$)

The units act on the idempotents: in $\mathcal{I}_n$, if $\alpha \in G_n$ then $\alpha^{-1} \cdot id_Y \cdot \alpha = id_{Y\alpha}$, and in general, for $g \in G$, $e \in E$ have $g^{-1}eg \in E$.

- $\alpha \in \mathcal{I}_n$ with $Y = \operatorname{dom}\alpha \Rightarrow$ there is $\hat{\alpha} \in G_n$ with $\hat{\alpha}|_Y = \alpha$.



($\hat{\alpha} = \alpha$ on Y and is any bijection $X \setminus Y \to X \setminus Z$)

Thus, every $\alpha \in \mathcal{I}_n$ a <u>restriction of a unit</u> (<u>warning</u>: $\hat{\alpha}$ not unique).

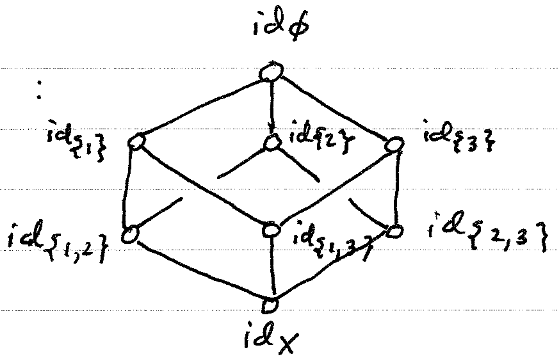
Put another way: $\alpha = id_Y \cdot \hat{\alpha} \in E(\mathcal{I}_n)G(\mathcal{I}_n)$.

In general, a monoid is <u>factorizable</u> $\overset{def.}{\Longleftrightarrow} M = EG$.

- <u>Partially order</u> $E(\mathcal{I}_n)$ by $id_Y \leq id_Z \Leftrightarrow Y \supseteq Z \Leftrightarrow id_Y id_Z = id_Z$

(for general M, partially order $E(M)$ by $e \leq f \Leftrightarrow ef = f$)

$E(\mathcal{I}_3)$:



(Boolean algebra)

<u>general principle</u>: much of the structure of M determined by the <u>group</u> $G(M)$, the <u>poset</u> $E(M)$ and the action of G on E.

- All of the above holds for: k field, V vector space over k

$ML(V)$ = vector space isoms. $Y \to Z$ (Y, Z subspaces of V) under composition of partial maps.

Lecture 5: Reflection monoids

- recall (lecture 4): V vector space, $Y, Z \subset V$ subspaces, then an isomorphism $Y \to Z$ a partial (linear) isomorphism; $ML(V)$ (inverse) monoid of partial isoms. under composition of partial maps.

- notation: $g \in GL(V)$ (ie: $V \xrightarrow{g} V$ full isom.) and $Y \subset V$ subspace, write g_Y for partial isom. $Y \xrightarrow{g|_Y} Yg$. In particular, $s \in GL(V)$ a reflection $\rightsquigarrow s_Y$ a partial reflection. Recall $g_Y, h_Z \in ML(V)$

$$g_Y h_Z = (gh)_{Y \cap Zg^{-1}}.$$

- $W \subset GL(V)$ a group. A set $\mathcal{S}$ of subspaces of V is a system in V for W $\overset{def}{\iff}$ (1). $V \in \mathcal{S}$, (2). $\mathcal{S}W = \mathcal{S}$ (ie: $X \in \mathcal{S}, g \in W \Rightarrow Xg \in \mathcal{S}$)

(3). $X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S}$.

- Eg: V Euclidean, orthonormal basis $\{\underline{x}_1, \dots, \underline{x}_n\}$, recall root system $A_{n-1} = \{\underline{x}_i - \underline{x}_j \mid i \neq j\}$ and reflection group $W(A_{n-1})$ $\cong \mathfrak{S}_n \subset GL(V)$ under permutation action ($\underline{x}_i \overset{\pi}{\longmapsto} \underline{x}_{i\pi}$).
(write $g(\pi) \longmapsto \pi$ under this isom.)

Let $I = \{1, \dots, n\}$ and for $J \subseteq I$, let $X(J) = \bigoplus_J \mathbb{R}\underline{x}_j \subset V$ and

$\mathcal{S}_1 = \{X(J) \mid J \subseteq I\}$, $(X(\emptyset) = \underline{0})$.

Then (1). $V = X(I)$, (2). $X(J)g(\pi) = X(J\pi)$ and (3). $X(J_1) \cap X(J_2) = X(J_1 \cap J_2)$. $\Rightarrow \mathcal{S}_1$ system in V for $W(A_{n-1})$.

• Eg: same W as above; $\mathcal{A}$ = braid arrangement (Lecture 3) $= \{$hyperplanes $x_i - x_j = 0\}$ = reflecting hyperplanes of $W(A_{n-1})$. Let $\mathcal{S}_2 = L(\mathcal{A})$ intersection lattice. Recall that there is an isomorphism of posets $\Pi(n) \to \mathcal{S}_2$ (partitions), written $\Lambda = \{\Lambda_1, \dots, \Lambda_p\} \mapsto X(\Lambda) \in \mathcal{S}_2$. It turns out that $X(\Lambda)g(\pi) = X(\Lambda\pi)$ with $\Lambda\pi = \{\Lambda_1\pi, \dots, \Lambda_p\pi\}$

$\Rightarrow \mathcal{S}_2$ a system for $W(A_{n-1})$.

• Eg: in general, if $W \subset GL(V)$ a reflection gp., $\mathcal{A}$ = reflecting hyperplanes of W and $\mathcal{S} = L(\mathcal{A})$, then let $X \in \mathcal{A}$, $g \in W$ and s = reflection in X

$\Rightarrow g^{-1}sg = s'$ = reflection in $Xg \Rightarrow Xg \in \mathcal{A} \Rightarrow \mathcal{A}W = \mathcal{A} \Rightarrow \mathcal{S}W = \mathcal{S}$.

Thus, $\mathcal{S}$ a system in V for W.

• $W \subset GL(V)$, $\mathcal{S}$ system in V for W. Let

$$M(W, \mathcal{S}) := \{g_Y \mid g \in W, Y \in \mathcal{S}\} \subset ML(V).$$

Lect. 4

Theorem: TFAE: (1). $M \subset ML(V)$ factorizable inverse monoid generated by partial reflections, (2). there is a reflection group

$W \subset GL(V)$ and a system $\mathcal{S}$ for W in V with $M = M(W, \mathcal{S})$.

A <u>reflection monoid</u> is an M satisfying (1) or (2).

- <u>Eg</u>: $W = W(A_{n-1}) \cong \mathfrak{S}_n$ (Lect. 1); writing $M(A_{n-1}, \mathcal{S}_1)$ for $M(W, \mathcal{S}_1)$, we have $M(A_{n-1}, \mathcal{S}_1) \cong \mathcal{I}_n$.

- <u>Theorem</u>: $W \subset GL(V)$ finite and system $\mathcal{S}$ finite. Then

$$|M(W, \mathcal{S})| = \sum_{X \in \mathcal{S}} [W : W_X]$$

($W_X :=$ isotropy gp. of $X = \{g \in W \mid \underline{v}g = \underline{v}$ for all $\underline{v} \in X\}$).

- <u>Eg</u>: $W = W(A_{n-1}) \cong \mathfrak{S}_n$; $\mathcal{S}_2$ system above. If $X = X(\Lambda)$ for $\Lambda = \{\Lambda_1, \ldots, \Lambda_p\}$ then $W_X \cong \mathfrak{S}_{\Lambda_1} \times \cdots \times \mathfrak{S}_{\Lambda_p}$ and

$$|M(A_{n-1}, \mathcal{S}_2)| = \sum_{\Lambda} [\mathfrak{S}_n : \mathfrak{S}_{\Lambda_1} \times \cdots \times \mathfrak{S}_{\Lambda_p}].$$

- <u>In general</u>: if $W = W(\Phi)$ then a $\Psi \subset \Phi$ satisfying the axioms for a root system (see Lecture 2) is a <u>sub-root system</u>. A <u>reflection subgroup</u> is a $W(\Psi)$ for Ψ a sub-root system. Then,

<u>Theorem</u>: $W \subset GL(V)$ finite and $\mathcal{S}_2 = L(\mathcal{A})$ the intersection lattice of the reflecting hyperplanes. Then $M(W, \mathcal{S}_2)$ has order the sum of the indices of the reflection subgroups.