Symmetry and partial symmetry

Contents: 1. Motivation
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5. Reflection monoids

Lecture 1: Motivation

(i) Symmetry: \( X \) set, \( G \) symmetric group on \( X \)

\( \sigma \): group of all bijections \( X \to X \)

(under composition)

"measures symmetry" of \( X \).

\( X = \{e, \ldots, n\} \), write \( G_n \) for \( G_X \).

recall: any \( \pi \in G_n \) is a product of transpositions,

\( \sigma \): permutations \( (i, j) \).

\( G_n \) generated by the transpositions

(ii). Another version: \( V = \mathbb{R}^n \) with usual basis \( \{e_1, \ldots, e_n\} \).

\( GL(V) = \) group of invertible linear maps \( V \to V \).

\( \pi \in G_n \) maps \( x_i \mapsto x_{\pi(i)} \)

embeds \( G_n \subseteq GL(V) \) (permuting coordinates)

\( E_3 : \pi = (1, 2, 3) \in G_3 \)

\( \frac{\pi}{3} \) turn rotation of \( \mathbb{R}^3 \),
Example: $\pi = (i, j) \in S_n$

\[ x_i - x_j = 0 \]

Reflection in plane $z_i - z_j = 0$

Conclusion: $G_n \subset GL(V)$ generated by reflections, $\bar{a}$ is a reflection group (see lecture 2).

(iii). Yet another version: $G_n$ is the Weyl group of the linear algebraic group $GL_n(\mathbb{R})$.

A Partial symmetry: (i) $X$ set and $I_X$ symmetric inverse monoid on $X$.

\[ I_X = \text{inverses monoid partial bijection of } X. \]

Partial bijection $a$ bijection $Y \rightarrow Z$ for $Y, Z \subseteq X$.

(On particular, "full" bijections $X \rightarrow X$ and "empty" bijection $0 : \emptyset \rightarrow \emptyset$).

\[ \alpha \beta : (Z \cap Y_2)^+ \rightarrow (Z \cap Y_2) \beta \]

\[ \alpha \gamma : (Z_1 = Y_1 \cap) \]

\[ \alpha \beta : Y_1 \cap Y_2, \rightarrow Y_1 (\alpha \beta) \cap Y_2 \beta. \]

Gives $I_X$ structure of a monoid:

\[ \begin{cases} \text{associative} \\ \text{identity id : } X \rightarrow X \\ \text{no inverses in general} \end{cases} \]
inverse monoid: There are "local" inverses:

\[
X \begin{bmatrix} O & \alpha \\ Y & \beta \end{bmatrix} \xrightarrow{\gamma} X
\]

for any \( X \) then is \( \beta \in I_X \)

with \( \alpha \beta = \text{id}_Y \)

\( \beta \alpha = \text{id}_X \)

(iii). On a reflection group ... In a reflection monoid?

(yes: see lecture 5)

(iii). On a Weyl group ... In the Ranner monoid of the

linear algebraic monoid

\( \text{GL}_n \mathbb{R} = \text{all } n \times n \mathbb{R} \text{-matrices} \)
Permutation groups

bijectsions \( X \rightarrow X \)
\( X = \{1, 2, \ldots, n\} \)

Symmetric group \( S_n \)

Reflection groups

Euclidean space \( V \)
basis \( \{x_1, \ldots, x_n\} \)

Weyl groups

\[ G = GL_n(\mathbb{F}) \]
\[ T = D_n^*(\mathbb{F}) \subset GL_n(\mathbb{F}) \text{ torus} \]
\[ W = N_G(T)/T \text{ permutation matrices} \]

Reflection monoids
Partial permutations

Bijections $X \supset Y \rightarrow Y' \subset X$

$X = \{1, 2, \ldots, n\}$

Symmetric inverse monoid

Renner monoids

$G = \text{GL}_n(F) \subset M_n(F) = M$

$T = D_n^*(F) \subset D_n(F) = \overline{T}$ (Zariski closure)

$W = N_G(T)/T \subset \overline{N_G(T)}/T$

Partial permutation matrices

= Rook monoid

Reflection monoids
Lecture 2: Reflection groups

- What is a reflection? \( V = \text{n-dim. IR-space} \)

- A real reflection:
  - \( I = \text{the primitive } n^{th} \text{ root of } 1 \text{ in } \mathbb{R} \)
  - \( x \mapsto \text{primitive } n^{th} \text{ root of } 1 \text{ in } \mathbb{R} \)

- In any field, \( \text{a } k\text{-reflection: } k \oplus \cdots \oplus k \oplus k \)

If \( V \) a \( k \)-space, \( \text{GL}(V) = \text{gp. invertible linear maps of } V \)

Then a \( k \)-reflection \( g : = \text{subgp. of } \text{GL}(V) \text{ generated} \)

by \( k \)-reflections.

In these lectures, we will restrict to \( \text{finite, real reflectiongps.} \)

Throughout, \( V \) a real space with basis \( x_1, \ldots, x_n \) and inner

product \( (x_i, x_j) = \delta_{ij} \) (Euclidean).

\( v \in V, s_v := \text{linear map } V \rightarrow V \)

(\( x \) a hyperplane) fixing \( v \) pt.-wise and sending \( v \mapsto -v \)

(Ex: \( s_v : u \mapsto u - 2 \frac{(u, v)}{(v, v)} v \)).

\( W \in \text{GL}(V) \text{ finite is a reflection gp } \iff W = \{ s_{x_1}, \ldots, s_{x_n} \}. \)
\( E_8 \) : \( V = \mathbb{R}^2 \)

\( W = \text{symmetries of square} \)

\( \Phi = \{ v \in V \mid v \neq 0 \} \)

\[ \begin{aligned}
&\text{Finite set of non-zero vectors} \\
&v \in \Phi \implies 1_v \cap \Phi = \pm v \\
&v \in \Phi \implies \overline{v} = \overline{v}
\end{aligned} \]

- A \( \Phi \subset V \) satisfying (\( \ast \)) a root system (\( \Phi \) \( \Phi \) a root)

\[ W(\Phi) = \langle sv \mid v \in \Phi \rangle \subset GL(V) \text{ a finite ref. grp.} \]

root systems \( \leftrightarrow \) finite ref. grps.

(combinatorics) \quad (g.p. Kazuo)

\( \Phi \) reducible iff \( V = V_1 \oplus V_2 \) with \( \Phi = \Phi_1 \cup \Phi_2 \);

irreducible otherwise.

\[ \Phi = \Phi_1 \cup \Phi_2 \]

(Ex: \( \Phi \) reducible \( \implies W(\Phi) \cong W(\Phi_1) \times W(\Phi_2) \)).

- \( \Phi \) is real, root systems have been classified:

\[ A_{n-1}(n \geq 2), \quad B_n(n \geq 2), \quad D_n(n \geq 4), \quad I_2(m)(m \geq 3) \]

\[ H_3, H_4, F_4, E_6, E_7, E_8. \]
- \textbf{Eg:} \( B_n = \{ x^i, x^i \} \cup \{ x_i \} \quad (B_2 = \mathbb{X}) \)

\[ W(B_n) \cong \text{symmetries of } n\text{-cube }/n\text{-octahedron} \]

- \textbf{Eg:} \( A_{n-1} = \{ x^i - x^j \} \quad \delta_{x^i} : x^i - x^j \to x^j - x^i \)

\[ \to (i,j) \in G_n \quad \text{under permutation} \]

\[ W(A_{n-1}) \cong G_n \subset GL(V) \quad \text{action} \quad x_i \to x_i^\pi \]

\[ = \text{symmetries of } (n-1)\text{-simplex } \subset \mathbb{R}^n \]

\[ = \{ \sum x_i^\pi \mid \sum x_i = 1 \text{ and } x_i > 0 \} \]

- \textbf{Eg:} \( W(C_n) \cong \text{symmetries of } n\text{-dim. cross polytope} \)

- \textbf{Eg:} \( W(H_3) \cong \text{icosahedron/dodecahedron} \)

\[ W(H_4) \cong \text{120/600-cell} \]

\[ W(F_4) \cong \text{24-cell} \]

\[ \text{4-dim. Platonic solids} \]
Lecture 3: Hyperplane arrangements

Throughout: V Euclidean with orthonormal basis $e_1, \ldots, e_n$.

- A real arrangement $\mathcal{A}$ is a set of linear hyperplanes in $V$.

Describing hyperplanes: (i). $v \in V$ and hyperplane $H_v = \{ u \in V | (u, v) = 0 \}$, or (ii). coordinate maps $x_i : V \to \mathbb{R}$ with $x_i(\sum a_i x_i) = x_i$. If $v = \sum a_i x_i$ then $v^\perp$ the kernel of map $a_1 x_1 + \cdots + a_n x_n : V \to \mathbb{R}$. (c.f. Cartesian equation)

- Eg: The Borel arrangement $\mathcal{A} = \{ e_1, \ldots, e_n \}$ or 
  \{ hyperplanes with equation $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ \}.

- Eg: Braid arrangement $\mathcal{A} = \{ $ hyperplanes $x_i - x_j = 0 \ for \ all \ i \neq j \}$.

Recall (Lecture 2) the root system $\mathcal{A}_{n-1} = \{ x_i - \pm \epsilon_i \ | \ i \neq j \} \subset V$

with reflection gp. $W(\mathcal{A}_{n-1}) \simeq \mathfrak{S}_n$

Thus, braid arrangement = the reflecting hyperplanes of the set gp.

In general, a reflection arrangement = \{ reflecting hyperplanes of some reflection gp \}.

Combinatorics (of hyperplane arrangements).
Intersection lattice $L(\mathcal{A}) := \text{all possible intersections of elements of } \mathcal{A}$

- a poset, partially ordered by reverse inclusion.

Eq.: Boolean arrangement (n=3): $\mathcal{A} = \{ x_1 = 0, x_2 = 0, x_3 = 0 \}$

$\equiv (\text{as a poset}) \text{ poset of subsets of } \{1,2,3\} \text{, i.e. a Boolean algebra}$

Eq.: braid arrangement: let $I = \{i_1, ..., i_n\}$, A partition of $I$ blocks 

is a $\Lambda = \{\Lambda_1, ..., \Lambda_p \}$ with $\Lambda_i \subset I$, $\Lambda_i \cap \Lambda_j = \emptyset$ and $I = \bigcup \Lambda_i$.

If $\Lambda' = \{\Lambda'_1, ..., \Lambda'_{p'} \}$ then define $\Lambda \leq \Lambda' \iff \text{for each } \Lambda_i \text{ there is a } \Lambda'_i \text{ with } \Lambda_i \subset \Lambda'_i$. Write $\Pi(n)$ for set of partitions of $I$; the set $\Pi(n)$ with partial order $\leq$ is a poset called the partition lattice.

$n=3$:

$\Pi(n)$ has a bijection

Ex: there is a map $\Lambda \mapsto x(\Lambda)$ from $\Pi(n) \rightarrow L(\mathcal{A})$ such that $\Lambda \leq \Lambda' \iff x(\Lambda) \equiv x(\Lambda')$.

$\implies$ an isomorphism of posets.

Defn: for any $L(\mathcal{A}) \rightarrow \mathbb{Z}$, Möbius function $\mu: L(\mathcal{A}) \rightarrow \mathbb{Z}$ is

$$\mu(x) = \begin{cases} 1, & \text{if } x = V \\ - \sum_{y < x} \mu(y), & \text{if } x \neq V \end{cases}$$
and the Poincaré polynomial is

\[ \pi(A, t) := \sum_{x \in \mathcal{L}(A)} \mu(x) (-t)^{\text{codim } X} \]

**Beautiful Result 1 (Zaslavsky):** \( \pi(A, 1) = \text{N}^w \)-regions, \( w \):

\[ \text{N}^w \text{ of connected components in } V - \bigcup_{x \in A} x \]

**Topology (of hyperplane arrangements):**

A \( \subseteq \mathbb{V} \) hyperplane arrangement means

\[ A_c = \{ X \oplus \mathbb{C} \mid x \in A \} \]

**Exercise (amusing):** \( V \setminus X \) disconnected

\[ V_c \setminus (X \oplus \mathbb{C}) \text{ connected} ! \]

Form space \( M_A := V_c - \bigcup_{x \in A} x \oplus \mathbb{C} \), the complement of the hyperplanes in \( V_c \).

The Poincaré polynomial

of \( M_A \) defined to be \( \text{Poin}(M_A, t) := \sum_{k \geq 0} \text{rk } H^k(M_A, \mathbb{C}) t^k \).

**Beautiful Result 2 (Arnold):** \( A = \text{braid arrangement} \)

\[ \Rightarrow \text{Poin}(M_A, t) = (1 + t)(1 + 2t) \cdots (1 + (n-1)t) \]

**Beautiful Result 3 (Orlik-Solomon):** for any \( A \),

\[ \text{Poin}(M_A, t) = \pi(A, t) \]
Lecture 4: Inverse monoids

- \( X = \mathbb{E} \), \( \cdots \). Recall symmetric inverse monoid \( \text{In} \) has elements partial bijections \( Y \to \mathbb{Z} \) (\( Y \subseteq X \)) under composition of partial maps.
- \( \text{dom} x = Y, \text{im} x = \mathbb{Z} \).
- \( \text{In} \text{ monoid: } \begin{cases} \text{compositions of partial maps are associative} \\ \text{id: } X \to X \text{ with } \text{id} \cdot x = x = x \cdot \text{id} \text{ for all } x \end{cases} \)
- \( Y \subseteq X \), \( \text{id}_Y : Y \to Y \text{ partial identity} \); if \( x \in \text{In} \text{ and } Y = \text{dom} x \)
  \[ \begin{aligned} \text{then } \text{id}_Y \cdot x &= x \text{ (but, } \text{id}_Y \cdot \beta \neq \beta \text{ in general).} \end{aligned} \]

\( X \) \[ \begin{aligned} \begin{array}{ccc} \circ & \alpha \to \circ \to & \circ \\
Y & \text{ and } & \mathbb{Z} \end{array} \end{aligned} \]

For \( x \in \text{In} \), there is \( x^{-1} \in \text{In} \) defined by

- \( x^{-1}(z) = y \iff z \in \mathbb{Z} \text{ and } x(y) = z \)
- \( x^{-1} \): partial bijection \( \text{with } \text{dom} x^{-1} = \text{im} x, \text{im} x^{-1} = \text{dom} x \)
- have \( x x^{-1} = \text{id}_Y \), \( x^{-1} x = \text{id}_X \) (note \( ax^{-1} \neq x^{-1} a \))
- \[ \begin{aligned} \Rightarrow x x^{-1} &= x \text{ and } x^{-1} x &= x^{-1} \end{aligned} \]

In inverse monoid: \( \begin{cases} \text{a monoid } M \text{ s.t. for all } a \in M \text{ there is a} \\ \text{unique } a^1 \in M \text{ with } a a^1 a = a \text{ and} \\ a^1 a a^1 = a \end{cases} \)

\( \text{Caution: in an inverse monoid we have } (a^{-1})^{-1} = a \text{ and } (ab)^{-1} = \)

- \( b^{-1} a^{-1} \); we do not have \( a a^{-1} = a^{-1} a \text{ or } ab = ac \Rightarrow b = c \text{ (as we do in groups).} \)

- In \( \text{In} \) we have the bijections \( \alpha : X \to X, \bar{\alpha} : \mathbb{E} \ni x \) is a subgroup of \( \text{In} \).
consisting of precisely those $a \in \mathcal{I}_N$ with $a^{-1} \cdot \text{id} = \text{id} = \text{id} \cdot a$.

For $M$ inverse monoid, the units $E = \mathcal{E}(M)$ can form a set $\mathcal{N}$ with

$$aa^{-1} = \text{id} = a^{-1}a.$$  

1. $Y \subseteq X \Rightarrow \text{id}_Y \cdot \text{id}_Y = (\text{id}_Y^2) = \text{id}_Y$. For $M$ (inverse) monoid

the idempotents $E = \mathcal{E}(M)$ are those $e \in M$ s.t. $e^2 = e$.

(In $\mathcal{I}_N$ this includes the empty map $\emptyset : \emptyset \to \emptyset$)

The units act on the idempotents: in $\mathcal{I}_N$, if $x \in \mathcal{E}_N$ then $x^* \cdot \text{id}_Y \cdot x = \text{id}_Y$, and in general, for $g \in \mathcal{E}$, $e \in E$ have $g \cdot e \cdot g \in E$.

2. $\alpha \in \mathcal{I}_N$ with $Y = \text{dom} \alpha$ \Rightarrow there is $\hat{\alpha} \in \mathcal{E}_N$ with $\hat{\alpha} \cdot \text{id}_Y = \alpha$.

$$\begin{bmatrix} \circ & \alpha \to \circ \\
Y & \circ \end{bmatrix} \quad (\hat{\alpha} = \alpha \text{ on } Y \text{ and is any bijection } X \setminus Y \to X \setminus \hat{\alpha}(Y))$$

Then, every $\alpha \in \mathcal{I}_N$ a restriction of a unit (warning: $\hat{\alpha}$ not unique).

Put another way: $\alpha = \text{id}_Y \cdot \hat{\alpha} \in \mathcal{E}(\mathcal{I}_N) \& (\mathcal{I}_N)$.

In general, a monoid is factorable $\iff M = \mathcal{E} \mathcal{G}$.  

3. Partially order $\mathcal{E}(\mathcal{I}_N)$ by $\text{id}_Y \leq \text{id}_Z \iff Y \subseteq Z \iff \text{id}_Y \cdot \text{id}_Z = \text{id}_Z$  

(for general $M$, partially order $\mathcal{E}(M)$ by $e \leq f \iff ef = f$)
\[ E(I_3) : \]

\[ \text{id}_{i_1}, \text{id}_{i_2}, \text{id}_{i_3}, \text{id}_{i_{1,2}}, \text{id}_{i_{1,3}}, \text{id}_{i_{2,3}}, \text{id}_{x} \]  

(Boolean algebra)

General principle: much of the structure of \( M \) determined by
the group \( G(M) \), the point \( E(M) \) and the action of \( G \) on \( E \).

- All of the above holds for: \( K \) field, \( V \) vector space over \( K \)

\[ ML(V) = \text{vector space isoms. } Y \to \mathbb{Z} \] \( (Y,Z \text{ subspaces of } V) \)

under composition of partial maps.
Lecture 5: Reflection monoids

- Recall (lecture 4): \( V \) vector space, \( Y, Z \subset V \) subspaces, then an isomorphism \( Y \to Z \) a partial (linear) isomorphism \( \Rightarrow ML(V) \)

Linear monoid of partial isoms under composition of partial maps.

- Notation: \( g \in GL(V) \) (\( \bar{a}: V \to V \) full isom.) and \( Y \subset V \) subspace, write \( g_Y \) for partial isom. \( Y \to Y \bar{g} \). In particular, \( \sigma \in GL(V) \)

Recall \( g_Y, h_Z \in ML(V) \)

\[ g_Y h_Z = (gh)_Y \circ h_{\bar{g}^{-1}}. \]

- \( WCGL(V) \) a group. A set \( S \) of subspaces of \( V \) is a system in \( V \)

Let \( W \Leftrightarrow (1). V \in S, (2). SW = S (\bar{a}: X \in S, g \in W \Rightarrow X \bar{g} \in S) \)

(3). \( X, Y \in S \Rightarrow X \cap Y \in S. \)

- \( E_g : V \) Euclidean, orthonormal basis \( \{x_1, ..., x_n\} \) recall root system \( \Delta = \{ \pm \frac{1}{2}, 1 \} \) and reflection group \( W(\Delta) \)

\[ G \cong GL(V) \text{ under permutation action } (x_i \mapsto x_{i \pi}) \]

(called \( g \in GL(V) \) under this isom.)

Let \( I = \{0, 1, 2\} \) and for \( J \subseteq I, \delta(J) = \bigoplus_{j \in J} \mathbb{R} x_j \subset V \) and \( S_I = \{ x(J) \mid J \subseteq I \} \), \( x(\emptyset) = 0 \).
Then (1). \( V = X(I), \) (2). \( X(I)g(\sigma) = X(I\sigma) \) and (3). \( X(I_1) \cap X(I_2) = X(I_1 \cap I_2) \). \( \Rightarrow \) \( S \) is system in \( V \) for \( W(An) \).

- \( E_g: \) Some \( W \) as above; \( A = \) braid arrangement (lecture 3)

\[ \{ \text{hyperplanes } x_i - x_j = 0 \} = \text{reflected hyperplanes of } W(An). \]

Let \( S_1 = L(A) \) intersect lattice. Recall there is an isomorphism of posets \( T(A) \rightarrow S_1 \), written \( A = \{ \Lambda_1, \ldots, \Lambda_n \} \rightarrow X(\Lambda) \in S_1 \). It turns out that \( X(\Lambda \cdot \sigma) = X(\Lambda \Pi) \) with \( \Pi = \{ \Lambda_{1\Pi}, \ldots, \Lambda_{n\Pi} \} \)

\( \Rightarrow S_2 \) a system for \( W(An) \).

- \( E_g: \) In general, if \( W \subset GL(V) \) a reflection gp., \( A = \) reflecting hyperplanes of \( W \) and \( S = L(A) \), then let \( X \in S, g \in W \) and \( s = \text{reflection in } X \)

\[ \Rightarrow gsg = s' = \text{reflection in } Xg = Xg \in A \Rightarrow AW = A \Rightarrow SW = S. \]

Thus, \( S \) a system in \( V \) for \( W \).

- \( W \subset GL(V) \), \( S \) system in \( V \) for \( W \). Let

\[ M(W, S) = \{ gy \mid g \in W, y \in S \} \subset MGL(V). \]

**Theorem:** TFAE: (1). \( MGL(V) \) factorizable inverse monoid generated by partial reflections, (2). There is a reflection group
\( WCGL(v) \) and a system \( S \) for \( W \in V \) with \( M = M(W, S) \).

A reflection monoid is an \( M \) satisfying \((1)\) or \((2)\).

- **Eg:** \( W = W(\Gamma_{n-1}) \cong S_n \) (act. 1), writing \( M(\Gamma_{n-1}, S_1) \) for \( M(W, S_1) \), we have \( M(\Gamma_{n-1}, S_1) \cong I_n \).

- **Theorem:** \( WCGL(v) \) finite and system \( S \) finite. Then

\[
1M(W, S) = \sum_{x \in S} [W : W_x]
\]

\( (W_x : \text{isotropy grp. } x = \{ g \in W \mid xg = x \text{ for all } y \in X \}) \).

- **Eg:** \( W = W(\Gamma_{n-1}) \cong S_n \), \( S_2 \) system a base. If \( x = x(n) \) for \( n = \delta_1, \ldots, \delta_p \) then \( W_x \cong S_{\delta_1} \times \cdots \times S_{\delta_p} \) and

\[
1M(\Gamma_{n-1}, S_2) = \sum_{\Lambda} [G_\Lambda : G_{\Lambda_1} \times \cdots \times G_{\Lambda_p}]
\]

- **In general:** if \( W = W(\Phi) \) then a \( \Phi \subseteq \Phi \) satisfying the axioms for a root system (see Lecture 2) is a sub-root system. A reflection subgroup is a \( W(\Phi) \) for \( \Phi \) a sub-root system. Then,

**Theorem:** \( WCGL(v) \) finite and \( S_2 = L(\Phi) \) the intersection lattice of the reflecting hyperplanes. Then \( M(W, S_2) \) has order the sum of the indices of the reflection subgroups.