Partial mirror symmetry II: Generators and relations

Brent Everitt and John Fountain ⋆

Abstract. We continue our development of the theory of reflection monoids by first deriving a presentation for a general reflection monoid from a result of Easdown, East and Fitzgerald for factorizable inverse monoids. We then derive "Popova" style presentations for reflection monoids built from Boolean hyperplane arrangements and reflection arrangements.

Introduction

In [3] we initiated the formal study of "partial mirror symmetry"–the theory of monoids generated by partial reflections. The principle acheivements of the theory to date, after identifying and formulating the notion itself, are to observe a number of examples of reflection monoids occuring in nature and determine their orders.

In this paper we continue the programme with a general presentation for reflection monoids, which we then interpret for a number of the key examples. Historically, this goes back to Popova [7], who gave a simple presentation for the symmetric inverse monoid $\mathscr{I}_n$ with generators the transpositions $(i, i+1) \in \mathfrak{S}_n$ (the standard Coxeter generators for $\mathfrak{S}_n$ as a Weyl group) and a single idempotent. Just as the symmetric group is the "simplest" family of finite (real) reflection groups, so the symmetric inverse monoid is the simplest family of finite real reflection monoids. In our language, $\mathscr{I}_n$ is the Boolean reflection monoid of type A_{n-1}, just as $\mathfrak{S}_n$ is the Weyl group of type A_{n-1}.

We thus recover Popova's presentation from our general one, as well as a number of others of course. There are other interesting "geometric" interpretations of the Popova presentation: it was recovered in [2] from a presentation for the "braid monoid" on n strands, much as one recovers the Coxeter presentation for $\mathfrak{S}_n$ from a presentation for Artin's braid group.

This paper is organized as follows: we remember reflection monoid terminology from the first paper in the series in §1. The idempotents in our monoids offer many of the difficulties in writing presentations, so they deserve a special section (§2) of their own. Our general presentation is then Theorem 1 of §3, obtained by massaging a presentation for factorizable inverse monoids obtained recently in [1]. The last two sections, §§4-5 interpret the various ingredients of Theorem 1 and perform a few more simplifications for the Boolean and reflection arrangement monoids.

1. Preliminaries on reflection monoids

We summarize the notation and conventions of the first paper in the series: V is a vector space over a field $\mathbb{F}$ and $W \subset GL(V)$ a group generated by reflections. The main theorems of the paper in §3 work at this level of generality, but later we will restrict to the case $\mathbb{F} = \mathbb{R}$ and W finite, in which case $W = W(\Phi)$ is determined by a root system Φ in V. In particular we shall

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Department of Mathematics, University of York, York YO10 5DD, United Kingdom. e-mail: `bje1@york.ac.uk` (Brent Everitt), e-mail: `jbf1@york.ac.uk` (John Fountain).

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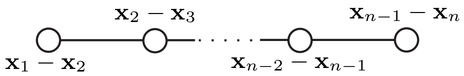
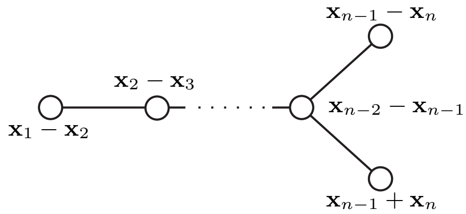
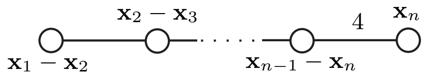
Type	Root system Φ	Coxeter symbol and simple system
$A_{n-1}\,(n \geq 2)$	$\{\mathbf{x}_i - \mathbf{x}_j \ (1 \leq i \neq j \leq n)\}$	
$D_n\,(n \geq 4)$	$\{\pm\mathbf{x}_i \pm \mathbf{x}_j \ (1 \leq i < j \leq n)\}$	
$B_n\,(n \geq 2)$	$\{\pm\mathbf{x}_i \ (1 \leq i \leq n),$ $\pm\mathbf{x}_i \pm \mathbf{x}_j \ (1 \leq i < j \leq n)\}$	

Table 1. Standard root systems $\Phi \subset V$ for the classical Weyl groups [5, §2.10].

be concerned with the finite crystallographic or *Weyl* groups, determined by the Euclidean root systems, with the finite classical systems of types A, D and B given in Table 1. We will always use these versions.

A system $\mathcal{B}$ for W in V is a W-invariant collection of subspaces closed under intersection and containing V. If Ω is any collection of subspaces we write $\langle \Omega \rangle_W$ for the system for W generated by the Ω. The principal example for us is the intersection lattice $L(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$, which we order by reverse inclusion. In particular, the Boolean systems arise from $\mathcal{A}$ the coordinate hyperplanes in V and the arrangement systems from $\mathcal{A}$ the set of reflecting hyperplanes of W.

A partial isomorphism of V is a vector space isomorphism $X \to Y$ between subspaces X, Y of V. A partial reflection is a partial isomorphism obtained by restricting a reflection $s \in GL(V)$ to a subspace. A reflection monoid is a factorizable inverse monoid generated by partial reflections. Alternatively, for a reflection group W and system $\mathcal{B}$ for W, it is the set of partial isomorphisms of the form.

$$M(W, \mathcal{B}) = \{g_X \mid g \in W, X \in \mathcal{B}\},$$

where g_X is the partial isomorphism obtained by restricting the (full) isomorphism g to X. The units are the $g \in W$ and the idempotents the partial identities ε_X that are just the identity map $X \to X$.

If $W = W(\Phi)$ is a Weyl group and $\mathcal{B}$ the Boolean system for W then $M(W, \mathcal{B}) = M(\Phi, \mathcal{B})$ is called a Boolean (reflection) monoid. Similarly, with the arrangement system $\mathcal{H}$ we get $M(\Phi, \mathcal{H})$ the (reflection) arrangement monoids. The third principal example defined in [3, §4.2] is a reflection monoid intimately associated to a connected reductive algebraic monoid $\mathbb{M}$ with 0. We leave a detailed investigation of this to a later date.

2. Idempotents

All the presentations in this section (and the next) will be monoid presentations, (see, eg: [4, §§1.5-1.6]) ie: if S is a set, let S^* be the free monoid on S and if $R \subset S^* \times S^*$, let $\langle\langle R \rangle\rangle$ be the smallest congruence on S^* containing R. Then a monoid M has presentation $\langle S \mid R \rangle$ if $M \cong S^*/\langle\langle R \rangle\rangle$ or, equivalently, if there is a surjective monoid homomorphism $\psi : S^* \to M$ with kernel $\langle\langle R \rangle\rangle$; we say that M has presentation $\langle S \mid R \rangle$ via ψ.

The idempotents in a reflection monoid present their own brand of subtleties, and for this reason it is worth dealing with them separately. Let Ω be a finite set of subspaces of finite dimensional V, and $\mathcal{B} = \langle \Omega \rangle_G$ the system of subspaces for $G \subset GL(V)$ generated by Ω. Recall [3, §2] that if all the $X \in \Omega$ have the same dimension then ordering $\mathcal{B}$ by reverse inclusion gives an atomic poset with atoms $\mathcal{A} = \{Xg \mid X \in \Omega, g \in G\}$.

We will want to keep track of the essentially different ways an element of $\mathcal{B}$ can be expressed as an intersection of atoms. To this end, fix a total ordering $\preceq$ of the atoms, so that an intersection of atoms $X_1\cap\cdots\cap X_k$ is *reduced* if $X_1\prec\cdots\prec X_k$. If $\bigcap X_i$ is any intersection of atoms, then reordering the X_i with respect to $\preceq$ and removing redundancies gives a reduced intersection. Write $\bigcap X_i/\preceq$ for this reduced reordering.

Let E be the semilattice of idempotents in $M(G,\mathcal{B})$ and e_X ($X\in\mathcal{A}$) a collection of symbols parametrised by the atoms. Put $\mathcal{B}'=\mathcal{B}\setminus\{V\}$ and for each $Y\in\mathcal{B}'$ fix a reduced intersection $Y=X_1\cap\cdots\cap X_k$ with the $X_i\in\mathcal{A}$, and let $\widehat{e}_Y$ stand for the expression $e_{X_1}\dots e_{X_k}$. We agree that $\widehat{e}_Y=e_Y$ when $Y\in\mathcal{A}$.

Proposition 1. *E has presentation,*

$$E=\langle e_X\,(X\in\mathcal{A})\,|\,e_X^2=e_X,\,e_Xe_Y=e_Ye_X \textit{ for all } X,Y\in\mathcal{A},\\ \widehat{e}_Y=e_{Y_1}\dots e_{Y_k} \textit{ for all } Y\in\mathcal{B}' \textit{ and } Y=Y_1\cap\cdots\cap Y_k \textit{ reduced}\rangle$$

via $e_X\mapsto\varepsilon_X$.

Proof. We start with the "multiplication table" presentation

$$E=\langle e_Y\,(Y\in\mathcal{B}')\,|\,e_Xe_Y=e_{X\cap Y}\text{ for all }X,Y\in\mathcal{B}'\rangle,$$

from which we can deduce the relations $e_X^2=e_X$ for all $X\in\mathcal{A}$, and $e_Xe_Y=e_Ye_X$ for all $X,Y\in\mathcal{A}$. Similarly, if $Y\in\mathcal{B}'$ and $Y=X_1\cap\cdots\cap X_k$ the reduced intersection chosen for Y with $Y=Y_1\cap\cdots\cap Y_k$ any other reduced intersection, we can deduce $e_Y=e_{X_1}\dots e_{X_k}=e_{Y_1}\dots e_{Y_k}$, ie: the relation $\widehat{e}_Y=e_{Y_1}\dots e_{Y_k}$. Add all these to the relations in the presentation above. Use $e_Y=e_{X_1}\dots e_{X_k}$ to remove generators and replace each occurence of e_Y by $\widehat{e}_Y$, so that

$$E=\langle e_X\,(X\in\mathcal{A})\,|\,e_X^2=e_X,\,e_Xe_Y=e_Ye_X\text{ for all }X,Y\in\mathcal{A},\\ \widehat{e}_Y=e_{Y_1}\dots e_{Y_k}\text{ for all }Y\in\mathcal{B}'\text{ and }Y=Y_1\cap\cdots\cap Y_k\text{ reduced},\\ \widehat{e}_{X\cap Y}=\widehat{e}_X\widehat{e}_Y\text{ for all }X,Y\in\mathcal{B}'\rangle.$$

We can deduce and hence remove the last family using the first three: let $X,Y\in\mathcal{B}$ with $X=X_{11}\cap\cdots\cap X_{1k}$ and $Y=X_{21}\cap\cdots\cap X_{2\ell}$ the reduced intersections chosen for X and Y, so that $\widehat{e}_X=e_{X_{11}}\dots e_{X_{1k}}$ and $\widehat{e}_Y=e_{X_{21}}\dots e_{X_{2\ell}}$. Using the commutativity and idempotency of intersection we can write $X\cap Y=(X_{11}\cap\cdots\cap X_{1k})\cap(X_{21}\cap\cdots\cap X_{2\ell})=X_{i_1}\cap\cdots\cap X_{i_m}$ with the last a reduced intersection and the $i_j\in\{11,\dots,1k,21,\dots,2\ell\}$. These manipulations can be mirrored in $\widehat{e}_X\widehat{e}_Y$ using the first two families of relations so that

$$\widehat{e}_X\widehat{e}_Y=e_{X_{i_1}}\dots e_{X_{i_m}}.$$

On the other hand, $X\cap Y=X_{i_1}\cap\cdots\cap X_{i_m}$ a reduced intersection gives, by the third family of relations, that $\widehat{e}_{X\cap Y}=e_{X_{i_1}}\dots e_{X_{i_m}}$. □

In §§4-5 we will want to be quite specific about the presentation of Proposition 1 for the Boolean and arrangement reflection monoids associated to the classical Weyl groups. This entails a description of the possible reduced intersections for an arbitrary $Y\in\mathcal{B}$.

Let $W=W(\Phi)$ be a Weyl group with Φ a root system as in [3, Table 1] (see also Table 1 of this paper) and $\mathcal{B}=\langle\mathbf{x}_1^\perp,\dots,\mathbf{x}_n^\perp\rangle$ Boolean with atoms the $\mathbf{x}_i^\perp$. Totally ordering $\mathcal{A}$ by $\mathbf{x}_1^\perp\prec\cdots\prec\mathbf{x}_n^\perp$, we have $\mathbf{x}_{i_1}^\perp\cap\cdots\cap\mathbf{x}_{i_k}^\perp$ is reduced if and only if $i_1<\cdots<i_k$. In particular, there is a unique reduced intersection for each element of a Boolean system which reflects the fact that $\mathcal{B}$ is the free semilattice (with identity) on $\mathcal{A}$.

• Should we give a reference for this?

In an arrangement system however, there may be many distinct reduced intersections for a given element. Recall [3, §2] that in this situation we have $\mathcal{B}=L(\mathcal{A})$, the intersection lattice of the arrangement $\mathcal{A}$ of reflecting hyperplanes for W. We parametrize the reduced intersections for $L(\mathcal{A})$ with respect to some $\preceq$ on $\mathcal{A}$, the reflecting hyperplanes of the Weyl group $W(\Phi)$, for Φ classical, recalling the description of $L(\mathcal{A})$ and notation of [3, §2].

First a general definition; let T be a set, and θ a collection of distinct two element subsets $\{i,j\}\subset T$. A subset $T'\subset T$ is *connected by* θ if it is a singleton or for every $x,y\in T'$, there are distinct subsets $\{i_1,j_1\},\dots,\{i_m,j_m\}\in\theta$ such that $x\in\{i_1,j_1\}$, $y\in\{i_m,j_m\}$ and $\{i_k,j_k\}\cap\{i_{k+1},j_{k+1}\}\neq\varnothing$ (and thus they have a single element in common). A subset of T that is maximal with respect to being connected by θ is a connected component.

Starting with the Weyl group $W(A_{n-1})$, let $I=\{1,2,\dots,n\}$ and $\Lambda=\{\Lambda_1,\dots,\Lambda_p\}$ a partition of I. A collection θ of two element subsets of I is a *decomposition* of Λ if the blocks $\Lambda_1,\dots,\Lambda_p$ are the connected components with respect to θ. Let $\mathcal{D}(\Lambda)$ be the set of decompositions of the partition Λ. We refer the reader to [3, §2] for the definition of the subspace $X(\Lambda)$ and note that the proof of the following is now elementary:

Lemma 1. *The map* $\theta\mapsto\cap_{\{i,j\}\in\theta}(\mathbf{x}_i-\mathbf{x}_j)^\perp/\preceq$ *is a bijection from* $\mathcal{D}(\Lambda)$ *to the set of reduced intersections of the subspace* $X(\Lambda)\in L(\mathcal{A})$.

Turning now to $W(B_n)$, we describe the reduced intersections of a subspace of the form $X(\Delta,\Lambda):=X(\Delta,\varnothing,\Lambda)$. A *decomposition* $\theta=(\theta_1,\theta_2)$ of (Δ,Λ) is (a). a decomposition θ_2 of the partition Λ, and (b). a collection θ_1 of distinct subsets of $\Delta^\pm:=\Delta\cup(-\Delta)$ of the form $\{i,-i\}$, $\{i,j\}$ and $\{-i,-j\}$, whose union is $\Delta^\pm$, and is such that if V_Δ is a real space of dimension $|\Delta|$, then the system of homogeneous linear equations $x_i=0,(\{i,-i\}\in\theta_1)$, $x_i-x_j=0,(\{i,j\}\in\theta_1)$ and $x_i+x_j=0,(\{-i,-j\}\in\theta_1)$, has no non-trivial solution.

Writing $\mathcal{D}(\Delta,\Lambda)$ for the set of possible decompositions of (Δ,Λ), we have,

Lemma 2. *The map*

$$\theta\mapsto\left\{\bigcap_{\{i,j\}\in\theta_2}(\mathbf{x}_i-\mathbf{x}_j)^\perp\cap\bigcap_{\{i,j\}\in\theta_1}(\mathbf{x}_i-\mathbf{x}_j)^\perp\cap\bigcap_{\{-i,-j\}\in\theta_1}(\mathbf{x}_i+\mathbf{x}_j)^\perp\cap\bigcap_{\{i,-i\}\in\theta_1}\mathbf{x}_i^\perp\right\}/\preceq,$$

is a bijection from $\mathcal{D}(\Delta,\Lambda)$ *to the set of reduced intersections of* $X(\Delta,\varnothing,\Lambda)$.

Proof. We have $\mathbf{x}=(x_i)\in X(\Delta,\varnothing,\Lambda)$ if and only if $x_i=0$ for $i\in\Delta$ and $x_i=x_j$ for i,j in the same block of the partition Λ. In particular, if an intersection for X has intersectand $\mathbf{x}_i^\perp$ or $(\mathbf{x}_i+\mathbf{x}_j)^\perp$, then $\{i,j\}\subset\Delta$, and the result follows. □

Finally, for $W(D_n)$ we describe the reduced intersections of the subspaces of the form $X(\Delta,\varnothing,\Lambda)$ and $X(\varnothing,\{k\},\Lambda)$, $1\leq k\leq n$. In the first case, a decomposition $\theta=(\theta_1,\theta_2)$ of $(\Delta,\Lambda)=(\Delta,\varnothing,\Lambda)$ is (a). a decomposition θ_2 of the partition Λ, and (b). a collection θ_1 of distinct subsets of $\Delta^\pm:=\Delta\cup(-\Delta)$ of the form $\{i,j\}$ and $\{-i,-j\}$, whose union is $\Delta^\pm$, and such that if V_Δ is a real space of dimension $|\Delta|$, then the system of homogenous linear equations $x_i-x_j=0,(\{i,j\}\in\theta_1)$ and $x_i+x_j=0,(\{-i,-j\}\in\theta_1)$, has no non-trivial solution.

A decomposition θ of $(\Lambda,\{k\}):=(\varnothing,\{k\},\Lambda)$ is a collection of distinct subsets of Λ of the form $\{i,j\}\subset I\setminus\{k\}$ and $\{i,k\}$, with i and k in the same block, and such that the blocks $\Lambda_1,\dots,\Lambda_p$ are the connected components.

Writing $\mathcal{D}(\Delta,\Lambda)$ and $\mathcal{D}(\Lambda,\{n\})$ for the sets of decompositions in the two cases, the proof of the following is similar to Lemma 2.

Lemma 3. *The map*

$$\theta\mapsto\left\{\bigcap_{\{i,j\}\in\theta_2}(\mathbf{x}_i-\mathbf{x}_j)^\perp\cap\bigcap_{\{i,j\}\in\theta_1}(\mathbf{x}_i-\mathbf{x}_j)^\perp\cap\bigcap_{\{-i,-j\}\in\theta_1}(\mathbf{x}_i+\mathbf{x}_j)^\perp\right\}/\preceq,$$

is a bijection from $\mathcal{D}(\Delta,\Lambda)$ *to the set of reduced intersections of* $X(\Delta,\varnothing,\Lambda)$*, and the map,*

$$\theta\mapsto\left\{\bigcap_{\{i,j\}\in\theta}(\mathbf{x}_i-\mathbf{x}_j)^\perp\cap\bigcap_{\{i,k\}\in\theta}(\mathbf{x}_i+\mathbf{x}_n)^\perp\right\}/\preceq,$$

is a bijection from $\mathcal{D}(\Lambda,\{k\})$ *to the set of reduced intersections of* $X(\varnothing,\{k\},\Lambda)$.

3. A presentation for reflection monoids

The main result of this section, Theorem 1 below, is a presentation for the reflection monoid $M(W,\mathcal{B})$ in the case that the system $\mathcal{B}=\langle\Omega\rangle_W$ is generated by a collection Ω of subspaces all having the same dimension, as is the case for instance when Ω is a hyperplane arrangement. The main technical tool is a recent presentation for factorizable inverse monoids, Theorem 2 below.

First we establish the notation and conventions necessary to state the theorem. Let $W\subset GL(V)$ be a reflection group with generating reflections S and $\mathcal{B}=\langle\Omega\rangle_W$ a system of subspaces for W. As usual $\mathcal{B}$ has atoms the subspaces $\mathcal{A}:=\Omega W$. Let Ω_k be a fixed set of orbit representatives for the W-action on the rank k elements of $\mathcal{B}$. In particular Ω_1 is a set of representatives for the W-action on $\mathcal{A}$. Let e_X $(X\in\Omega_1)$ be a collection of symbols parametrised by Ω_1.

For each $g\in W$ we fix a word $\widehat{g}=s_1\dots s_k$ in the generators S representing g, agreeing that $\widehat{s}=s$ when $s\in S$. For $Y\in\mathcal{A}$, we fix a $g\in W$ with $Y=(X)g$ for some $X\in\Omega_1$, and write $\widehat{e}_Y$ for the word $s_k^{-1}\dots s_1^{-1}e_X\,s_1\dots s_k$. For each $Y\in\mathcal{B}$ fix a reduced intersection $Y=X_1\cap\dots\cap X_k$ with respect to some $\preceq$ on $\mathcal{A}$, and let $\widehat{e}_Y$ stand for the expression $\widehat{e}_{X_1}\dots\widehat{e}_{X_k}$. In all cases we take $\widehat{e}_Y=e_Y$ when $Y\in\Omega_1$.

Finally, and possibly somewhat cryptically, let Σ be a set of pairs $(f,(X)g)$ with $X\in\mathcal{B}$, $g\in W$ and f a generator for the isotropy group $W_{(X)g}$, such that the following holds: if Y is any element of $\mathcal{B}$ and t a generator for the isotropy group W_Y, then there is a $(f,(X)g)\in\Sigma$ with $X\leq Y$ and $t=gfg^{-1}$.

Theorem 1. *With the notation above, the reflection monoid $M(W,\mathcal{B})$ has presentation,*

$$\begin{aligned}M(W,\mathcal{B})=\langle s\in S,\,e_X\,(X\in\Omega_1)\,|\,&\text{relations for }W,\ e_X^2=e_X\text{ for }X\in\Omega_1,\\&\widehat{e}_X\widehat{e}_Y=\widehat{e}_Y\widehat{e}_X\text{ for }X\cap Y\in\Omega_2,\\&\widehat{e}_Y=\widehat{e}_{Y_1}\dots\widehat{e}_{Y_m}\text{ for }Y\in\Omega_k,k\geq 2,\\&\qquad\text{and any }Y=Y_1\cap\dots\cap Y_m\text{ reduced},\\&s\,\widehat{e}_X=\widehat{e}_{Xs}s\text{ for }(s,X)\in S\times\mathcal{A},\\&\widehat{e}_{Xg}\widehat{f}=\widehat{e}_{Xg}\text{ for }(f,Xg)\in\Sigma\rangle.\end{aligned}$$

via $\widehat{s}\mapsto s$ and $e_X\mapsto\varepsilon_X$.

Theorem 1 follows by massaging a presentation for factorizable inverse monoids supplied by [1], which we now summarize.

Suppose M is a factorizable inverse monoid with group of units $G=G(M)$, semilattice of idempotents $E=E(M)$, and $\langle S_G\,|\,R_G\rangle$, $\langle S_E\,|\,R_E\rangle$ monoid presentations for G and E. For $g\in G$, fix a word $\widehat{g}$ for g in the generators S_G and similarly a word $\widehat{e}$ in S_E for $e\in E$, with the conventions above applying when $g\in S_G$ and $e\in S_E$. There is anti-action of G on E given by $e\mapsto geg^{-1}\in E$, allowing us to fix a word in S_E for geg^{-1} also. For each $e\in E$ let $G_e=\{g\in G\,|\,eg=e\}$, and $\Sigma_e\subset G_e$ a set of monoid generators for G_e.

Theorem 2 ([1, Theorem 6]**).** *The factorizable inverse monoid M has presentation,*

$$\begin{aligned}M=\langle S_G,\,S_E\,|\,R_G,\,R_E,ge=\widehat{geg^{-1}}\cdot g\text{ for }(g,e)\in S_G\times S_E,\\\widehat{e}\,\widehat{t}=\widehat{e}\text{ for }e\in E,t\in\Sigma_e\rangle.\end{aligned}$$

We now interpret the various ingredients in the case that we have a reflection monoid $M=M(W,\mathcal{B})$, where G is the reflection group W with generating reflections S, and E is generated by the e_X for $X\in\mathcal{A}$ by Proposition 1.

If $s\in S$ and ε_X for $X\in\mathcal{A}$ is a generating idempotent, then $s\varepsilon_Xs^{-1}=\varepsilon_{Xs}$, where $Xs\in\mathcal{A}$, as $\mathcal{A}$ is W-invariant. Thus the relations

$$ge=\widehat{geg^{-1}}\cdot g\text{ for }(g,e)\in S_G\times S_E,$$

in Theorem 2 become $se_X = e_{Xs}s$ for $(s,X) \in S \times \mathcal{A}$. If $e = \varepsilon_Y$, then $G_e = \{g \in W \mid \varepsilon_Y g = \varepsilon_Y\}$, where for any $x \in Y$ we have $x\varepsilon_Y g = x\varepsilon_Y$ iff $xg = x$, ie: G_e is the isotropy group W_Y. Thus we may take for Σ_e any generating set S_Y for the isotropy group W_Y, although in many situations there will be particularly nice ones: if W is a complex reflection group for instance, Steinberg's Theorem [8] allows us to take for S_Y the reflections in the hyperplanes containing Y, and from now on we shall do this.

Thus the relations $\widehat{e}\,\widehat{t} = \widehat{e}$ for $e \in E, t \in \Sigma_e$ become $\widehat{e}_Y\widehat{t} = \widehat{e}_Y$ for $Y \in \mathcal{B}, t \in S_Y$. Summarizing,

Corollary 1. *If $W \subset GL(V)$ is a reflection group and $\mathcal{B} = \langle \Omega \rangle_W$ with atoms $\mathcal{A}$ the Xg for $X \in \Omega$ and $g \in W$, then $M(W,\mathcal{B})$ has presentation,*

$$
\begin{aligned}
M(W,\mathcal{B}) = \langle s \in S,\, e_X\, (X \in \mathcal{A}) \mid \ & \textit{relations for } W,\ e_X^2 = e_X \textit{ for } X \in \mathcal{A},\\
& e_X e_Y = e_Y e_X \textit{ for } X, Y \in \mathcal{A},\\
& \widehat{e}_Y = e_{Y_1} \dots e_{Y_k} \textit{ for } Y \in \mathcal{B}, Y = Y_1 \cap \dots \cap Y_k \textit{ reduced},\\
& se_X = e_{Xs}s \textit{ for } (s,X) \in S \times \mathcal{A},\\
& \widehat{e}_Y \widehat{t} = \widehat{e}_Y \textit{ for } Y \in \mathcal{B}, t \in S_Y \rangle.
\end{aligned}
$$

Deducing Theorem 1 now becomes a matter of removing relations and generators (in that order) from the presentation in Corollary 1. Before we do so, a glance at the presentations in Corollary 1 and Theorem 1 masks the considerable saving in generators and relations of the latter over the former, as we shall see in the next two sections. For example, in the Boolean monoid of type A_{n-1}, Corollary 1 gives n idempotent generators, n relations of the form $e_X^2 = e_X$ and $n(n-1)$ relations of the form $\widehat{e}_X\widehat{e}_Y = \widehat{e}_Y\widehat{e}_X$. There are 2^n subspaces Y in the Boolean system, and if $Y = \mathbf{x}_{i_1}^\perp \cap \dots \cap \mathbf{x}_{i_k}^\perp$ is one of them, then a standard generating set for W_Y has $k-1$ reflecting generators, for a total of $2^{n-1}n(n-1)$ relations of the form $\widehat{e}_Y\widehat{t} = \widehat{e}_Y$. Theorem 1 on the other hand gives a single idempotent generator, a single idempotent relation, a single commuting idempotents relation, and a single relation of the last kind.

First, we deduce some useful intermediate relations:

Lemma 4. *Let $X, Y \in \mathcal{B}$ with $Y = Xg$. Then one can deduce $\widehat{e}_Y = \widehat{g^{-1}}\,\widehat{e}_X\widehat{g}$ from the relations in Corollary 1.*

Proof. We deal first with the case that X and Y are atomic, where we require only the $se_X = e_{Xs}s$ relations. In particular, if $\widehat{g} = s_1 \dots s_k$, then $s_1e_X = e_{Xs_1}s_1$, hence $e_{Xs_1} = s_1^{-1}e_Xs_1$. Proceeding by induction, if

$$e_{Xs_1\dots s_i} = (s_i^{-1}\dots s_1^{-1})e_X(s_1\dots s_i),$$

then $Xs_1 \dots s_i \in \mathcal{A}$, a W-invariant set, thus the relation $s_{i+1}e_{Xs_1\dots s_i} = e_{Xs_1\dots s_{i+1}}s_{i+1}$ of Corollary 1 gives

$$e_{Xs_1\dots s_{i+1}} = (s_{i+1}^{-1}\dots s_1^{-1})e_X(s_1\dots s_{i+1}),$$

and so $e_Y = (s_k^{-1}\dots s_1^{-1})e_X(s_1\dots s_k)$. Suppose now that $\widehat{e}_X = e_{X_1}\dots e_{X_m}$ with the X_i atomic. As $Y = Xg = X_1g \cap \dots \cap X_mg$, with the X_ig also atomic, we get $\widehat{e}_Y = e_{U_1}\dots e_{U_m}$ with $U_1 \cap \dots \cap U_m = X_1g \cap \dots \cap X_mg/\preceq$. The commuting of the idempotents then gives $\widehat{e}_Y = e_{X_1g}\dots e_{X_mg}$, and so

$$\widehat{e}_Y = \prod e_{X_ig} = \prod \widehat{g^{-1}}\, e_{X_i}\, \widehat{g} = \widehat{g^{-1}} \cdot \prod e_{X_i} \cdot \widehat{g} = \widehat{g^{-1}}\,\widehat{e}_X\,\widehat{g}.$$

□

Now to the thinning out of the relations. Let Σ be a set of pairs (f, Xg) with $X \in \mathcal{B}$, $g \in W$ and $f \in S_{Xg}$ a generator for W_{Xg} as in the preamble to the statement of Theorem 1, ie: such that for any $Y \in \mathcal{B}$ and $t \in S_Y$ there is a $(f, Xg) \in \Sigma$ with $X \leq Y$ and $t = gfg^{-1}$.

Lemma 5. *The* $\widehat{e}_Y\widehat{t}=\widehat{e}_Y$ *for* $Y\in\mathcal{B},t\in S_Y$ *are implied by the* $\widehat{e}_{Xg}\widehat{f}=\widehat{e}_{Xg}$ *for* $(f,(X)g)\in\Sigma$*, the idempotent relations and the* $se_X=e_{Xs}s$*.*

Proof. Observe first that the $\widehat{e}_{Xg}\widehat{f}=\widehat{e}_{Xg}$ are indeed a subset of the $\widehat{e}_Y\widehat{t}=\widehat{e}_Y$. As $X\leq Y$ we have $Y=Y\cap X$ giving $\varepsilon_Y=\varepsilon_Y\varepsilon_X$ and so we can deduce $\widehat{e}_Y=\widehat{e}_Y\widehat{e}_X$ from the idempotent relations. Thus,

$$\widehat{e}_Y\widehat{t}=\widehat{e}_Y\widehat{e}_X\widehat{g}\widehat{f}\,\widehat{g^{-1}}=\widehat{e}_Y\widehat{g}\,\widehat{e}_{(X)g}\widehat{f}\,\widehat{g^{-1}}=\widehat{e}_Y\widehat{g}\,\widehat{e}_{(X)g}\widehat{g^{-1}}=\widehat{e}_Y\widehat{e}_X\widehat{g}\,\widehat{g^{-1}}=\widehat{e}_Y,$$

via the assumption and two applications of Lemma 4. □

In the next two lemmas we use the W-action on $\mathcal{B}$ to thin out the idempotent relations. If $X,Y\in\mathcal{B}$, $Y=Xg$ and $Y=Y_1\cap\cdots\cap Y_m$ a reduced intersection, then $Y_1g^{-1}\cap\cdots\cap Y_mg^{-1}/\preceq$ is a reduced intersection for X, from which, together with the commuting of the idempotents, we deduce $\widehat{e}_X=e_{Y_1g^{-1}}\dots e_{Y_kg^{-1}}$.

Lemma 6. *Let* $X,Y\in\mathcal{B}$*, with* $Y=Xg$ *and* $Y=Y_1\cap\cdots\cap Y_k$ *any reduced intersection. Then* $\widehat{e}_Y=e_{Y_1}\dots e_{Y_k}$ *is implied by* $\widehat{e}_X=e_{Y_1g^{-1}}\dots e_{Y_kg^{-1}}$*, the* $e_Xe_Y=e_Ye_X$ *and the* $se_X=e_{Xs}s$*.*

Lemma 7. *(i). Let* $X,Y\in\mathcal{A}$ *and* $g\in W$ *with* $Y=Xg$*. Then the relation* $e_Y^2=e_Y$ *is implied by the relation* $e_X^2=e_X$ *and the* $se_X=e_{Xs}s$*.*

(ii). Let $X_i,Y_i\in\mathcal{A}$*,* $(i=1,2)$ *and* $g\in W$ *with* $Y_i=(X_i)g$*,* $(i=1,2)$*. Then the relation* $e_{Y_1}e_{Y_2}=e_{Y_2}e_{Y_1}$ *is implied by the relation* $e_{X_1}e_{X_2}=e_{X_2}e_{X_1}$ *and the* $se_X=e_{Xs}s$*.*

The proofs of both are an easy application of Lemma 4. Finally, we have the

Proof (of Theorem 1). Lemma 5 allows us to reduce the $\widehat{e}_Y\widehat{t}=\widehat{e}_Y$, for $Y\in\mathcal{B},t\in S_Y$ to the $\widehat{e}_{Xg}\widehat{f}=\widehat{e}_{Xg}$ for $(f,Xg)\in\Sigma$. If Ω_k is a set of orbit representatives for the W-action on the rank k elements of $\mathcal{B}$, then the $\widehat{e}_Y=e_{Y_1}\dots e_{Y_k}$ for $Y\in\mathcal{B}$ relations can be reduced to the cases where $Y\in\Omega_k$ for $k\geq 2$. By Lemma 7(i) we may replace the $e_X^2=e_X$, for $X\in\mathcal{A}W$, by the $e_X^2=e_X$, for $X\in\Omega_1$. The pairs $X\neq Y\in\mathcal{A}W$ correspond to the rank two elements $X\cap Y\in\mathcal{B}$, and as $(X\cap Y)g=(X)g\cap(Y)g$, the $e_Xe_Y=e_Ye_X$, for $X,Y\in\mathcal{A}W$ can be replaced, using Lemma 7(ii), by $e_Xe_Y=e_Ye_X$, for $X\cap Y\in\Omega_2$. So much for the relations. For every $Y\in\mathcal{A}W$ there is an $X\in\Omega_1$ with $e_Y=(s_k\dots s_1)e_X(s_1\dots s_k)$, and so we remove these superfluous generators, replacing each occurence of e_Y in the relations by $\widehat{e}_Y:=(s_k\dots s_1)e_X(s_1\dots s_k)$. □

4. "Popova style" presentations for the Boolean monoids

In this section we recover Popova's presentation [7] for the symmetric inverse monoid by interpreting it as the Boolean monoid of type A and then using Theorem 1. We also do the same for the type B and D Boolean monoids.

Let Φ be a root system as in Table 1, $\mathcal{B}=L(\mathcal{A})$ the intersection lattice of the Boolean arrangement $\mathcal{A}=\{\mathbf{x}_1^\perp,\dots,\mathbf{x}_n^\perp\}$ (with $\mathcal{A}W=\mathcal{A}$) and $X_{i_1\dots i_k}:=\mathbf{x}_{i_1}^\perp\cap\cdots\cap\mathbf{x}_{i_k}^\perp$. We observed in §1 that each $X_{i_1\dots i_k}$ has a unique reduced intersection with respect to the ordering $X_1\prec\cdots\prec X_n$, so the $\widehat{e}_Y=e_{Y_1}\dots e_{Y_k}$ relations are vacuous. The W-action on the rank k elements is transitive, so in particular Ω_1 and Ω_2 each have a single element, say, X_n and $X_{n-1,n}$. Let $e:=e_{X_n}$, and s_i, $(1\leq i\leq n-1)$, the reflection in the hyperplane orthogonal to $\mathbf{x}_i-\mathbf{x}_{i+1}$. We choose $X_i=(X_n)s_{n-1}\dots s_i$ so that $\widehat{e}_{X_i}:=(s_i\dots s_{n-1})\,e\,(s_{n-1}\dots s_i)$.

The result is that the monoids have generators the s_i (the simple reflections for W) and e, with the relations $e_X^2=e_X$ $(X\in\Omega_1)$ just $e^2=e$, and the relations $\widehat{e}_X\widehat{e}_Y=\widehat{e}_Y\widehat{e}_X$ $(X\cap Y\in\Omega_2)$ reducing to the single $es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e$.

Now to the $\widehat{e}_{(X)g}\widehat{f} = \widehat{e}_{(X)g}$ for $(f,(X)g) \in \Sigma$. Any $X_{i_1\dots i_k} \in \mathcal{B}$ has isotropy group a reflection group, generated, according to Steinberg's Theorem, by reflections in the hyperplanes containing $X_{i_1\dots i_k}$. Indeed, we can take

Φ	isotropy group $W_{X_{i_1\dots i_k}}$	Σ
A_{n-1}	$\langle s_{\mathbf{v}} \,\vert\, \mathbf{v} = \mathbf{x}_i - \mathbf{x}_j,\, \{i,j\} \subset \{i_1,\dots,i_k\}\rangle$	$(s_{n-1}, X_{n-1,n})$
B_n	$\langle s_{\mathbf{v}} \,\vert\, \mathbf{v} = \pm\mathbf{x}_i \pm \mathbf{x}_j, \pm\mathbf{x}_i,\, \{i,j\} \subset \{i_1,\dots,i_k\}\rangle$	$\{(s_{n-1}, X_{n-1,n}), (s_n, X_n)\}$
D_n	$\langle s_{\mathbf{v}} \,\vert\, \mathbf{v} = \pm\mathbf{x}_i \pm \mathbf{x}_j,\, \{i,j\} \subset \{i_1,\dots,i_k\}\rangle$	$\{(s_{n-1}, X_{n-1,n}), (s_n, X_{n-1,n})\}$

with s_n the reflection in the hyperplane orthogonal to $\mathbf{x}_n$ in the type B case, and in the hyperplane orthogonal to $\mathbf{x}_{n-1} + \mathbf{x}_n$ in the type D case. Notice that as Y and $t \in S_Y$ vary, so do X and g in the pair $(f,(X)g)$, but $(X)g$ and f remain constant: for example in type A, if t is the reflection in the hyperplane orthogonal to $\mathbf{x}_i - \mathbf{x}_j$ $(j < n)$, then we take $X = X_{ij}$, and $g = (s_j \dots s_{n-1})(s_i \dots s_{n-2})$, giving $(X)g = X_{n-1,n}$ and $f = s_{n-1}$, irrespective of i and j.

The relations are thus $s_{n-1}es_{n-1}es_{n-1} = s_{n-1}es_{n-1}e$ in all three cases, with $es_n = e$ as well in type B and $s_{n-1}es_{n-1}es_n = s_{n-1}es_{n-1}e$ in type D.

We pause to observe that in some sense the geometry of the root systems is reflected in these relations. It is a fundamental fact that if Φ is an irreducible crystallographic root system, then the associated Weyl group $W(\Phi)$ acts on Φ with orbits the roots of a given length. Thus there is a single orbit on Φ in types A, D, E (where all the roots are "long") and two orbits in types B, F, G (where roots are either "short" or long). In particular the transitivity in type A is the reason for the single pair in Σ, while the two orbits in type B result in two pairs.

This completes the presentation given by Theorem 1 for the Boolean monoids, but it turns out that the relations $s\,\widehat{e}_X = \widehat{e}_{(X)s}s$ for $(s,X) \in S \times \mathcal{A}$ can also be significantly reduced in number:

Lemma 8. *The relations* $s_i\,\widehat{e}_{X_j} = \widehat{e}_{(X_j)s_i}s_i$ *for* $1 \leq i \leq n-1$ *and* $1 \leq j \leq n$ *are implied by the relations* $s_i\,e_{X_n} = \widehat{e}_{(X_n)s_i}s_i$ *for* $1 \leq i \leq n-1$, *and the relations for* W.

Proof. By examining the various possibilities for the subspace $(X_j)s_i$, it is to be proved that the relations

$$s_i\,\widehat{e}_{X_j} = \begin{cases} \widehat{e}_{X_j}s_i, & i \neq j-1, j, \\ \widehat{e}_{X_{j-1}}s_i, & i = j-1, \\ \widehat{e}_{X_{j+1}}s_i, & i = j, \end{cases}$$

follow from the given relations, for which there are four cases to consider: (i). $1 \leq i < j-1$: the reflection s_i commutes with $s_j, \dots, s_{n-1}$ and e, giving the result immediately. (ii). $j < i \leq n-1$:

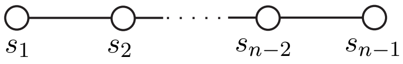
$$\begin{aligned} s_i\,\widehat{e}_{X_j} &= s_i(s_j \dots s_{n-1})e(s_{n-1} \dots s_j) = (s_j \dots s_i s_{i-1} s_i \dots s_{n-1})e(s_{n-1} \dots s_j) \\ &= (s_j \dots s_{i-1} s_i s_{i-1} \dots s_{n-1})e(s_{n-1} \dots s_j) = (s_j \dots s_{n-1})s_{i-1}e(s_{n-1} \dots s_j) \\ &= (s_j \dots s_{n-1})es_{i-1}(s_{n-1} \dots s_j) = (s_j \dots s_{n-1})e(s_{n-1} \dots s_{i-1}s_i s_{i-1} \dots s_j) \\ &= (s_j \dots s_{n-1})e(s_{n-1} \dots s_i s_{i-1} s_i \dots s_j) = (s_j \dots s_{n-1})e(s_{n-1} \dots s_j)s_i \\ &= \widehat{e}_{X_j}s_i, \end{aligned}$$

where we have used the relations $(s_{i-1}s_i)^3 = 1$ in their "braid" form $s_{i-1}s_is_{i-1} = s_is_{i-1}s_i$, and the commuting of s_{i-1} and e. (iii). $i = j-1$: $s_{j-1}\,\widehat{e}_{X_j} = s_{j-1}(s_j \dots s_{n-1})e(s_{n-1} \dots s_j) = (s_{j-1}s_j \dots s_{n-1})e(s_{n-1} \dots s_j)s_{j-1}s_{j-1} = \widehat{e}_{X_{j-1}}s_{j-1}$ (iv). $i = j$: $s_j\,\widehat{e}_{X_j} = s_j(s_j \dots s_{n-1})$ $e(s_{n-1} \dots s_j) = (s_{j+1} \dots s_{n-1})e(s_{n-1} \dots s_j)s_js_j = \widehat{e}_{X_{j+1}}s_j$. □

Putting it all together in the type A case, and observing that $(X_n)s_i = X_n$ when $i \neq n-1$, and $(X_n)s_{n-1} = X_{n-1}$, we get Popova's presentation for the symmetric inverse monoid [7],

Theorem 3. *The Boolean monoid $M(A_{n-1},\mathcal{B})$ has presentation,*

$$M(A_{n-1},\mathcal{B}) = \langle s_1,\dots,s_{n-1},e \,|\, (s_is_j)^{m_{ij}}=1,\ e^2=e,\\ s_ie=es_i\,(i\le n-2),\\ es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e,\\ s_{n-1}es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e\rangle$$



The relation $s_ie_{X_n}=\widehat{e}_{(X_n)s_i}s_i$ is vacuous when $i=n-1$. Recall that the m_{ij} can be read off the Coxeter symbol (which is why we have included it), as the nodes are joined by an edge labelled m_{ij} if $m_{ij}\ge 4$, an unlabelled edge if $m_{ij}=3$, no edge if $m_{ij}=2$ and $m_{ij}=1$ if and only if $i=j$.

Lemma 8 leaves unresolved in types B and D the status of the $s\,\widehat{e}_X=\widehat{e}_{(X)s}s$ relations when $s=s_n$:

Lemma 9. *If $\Phi=B_n$ then the relations $s_n\,\widehat{e}_{X_j}=\widehat{e}_{(X_j)s_n}s_n$ for $1\le j\le n$ are implied by the relations $s_i\,e_{X_n}=\widehat{e}_{(X_n)s_i}s_i$ for $1\le i\le n-1$, the relation $s_n\,\widehat{e}_{X_{n-1}}=\widehat{e}_{X_{n-1}}s_n$, the relation $s_n\,e_{X_n}=e_{X_n}s_n$ and the relations for W.*

Proof. For $1\le j\le n-2$,

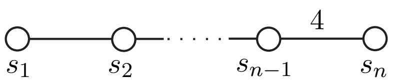
$$\begin{aligned} s_n\,\widehat{e}_{X_j} &= s_n(s_j\dots s_{n-1})e(s_{n-1}\dots s_j)=(s_j\dots s_ns_{n-1})e(s_{n-1}\dots s_j)\\ &=(s_j\dots s_n)\widehat{e}_{X_{n-1}}s_{n-1}(s_{n-1}\dots s_j)=(s_j\dots s_{n-2})\widehat{e}_{X_{n-1}}s_n(s_{n-2}\dots s_j)\\ &=\widehat{e}_{X_j}s_n. \end{aligned}$$

□

Again we have $(X_n)s_i=X_n$ when $i\ne n-1$, and $(X_n)s_{n-1}=X_{n-1}$, resulting in a presentation,

Theorem 4. *The Boolean monoid $M(B_n,\mathcal{B})$ has presentation,*

$$M(B_n,\mathcal{B}) = \langle s_1,\dots,s_n,e \,|\, (s_is_j)^{m_{ij}}=1,\ e^2=e,\\ s_ie=es_i\,(i\le n-2),\ s_ne=es_n,\\ s_ns_{n-1}es_{n-1}=s_{n-1}es_{n-1}s_n,\\ es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e,\ es_n=e,\\ s_{n-1}es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e\rangle.$$



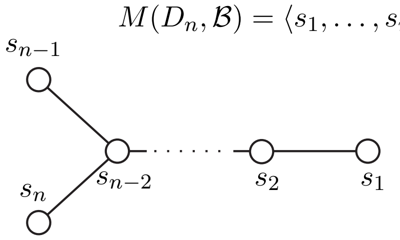
The proof of the following is analogous to Lemma 9:

Lemma 10. *If $\Phi=D_n$ then the relations $s_n\,\widehat{e}_{X_j}=\widehat{e}_{(X_j)s_n}s_n$ for $1\le j\le n$ are implied by the relations $s_i\,e_{X_n}=\widehat{e}_{(X_n)s_i}s_i$ for $1\le i\le n-1$, the relations $s_n\,\widehat{e}_{X_{n-2}}=\widehat{e}_{X_{n-2}}s_n$, $s_n\,\widehat{e}_{X_{n-1}}=e_{X_n}s_n$, $s_n\,e_{X_n}=\widehat{e}_{X_{n-1}}s_n$, and the relations for W.*

Together with $(X_n)s_i=X_n$ when $i\ne n-1,n$, and $(X_n)s_{n-1}=(X_n)s_n=X_{n-1}$, we have,

Theorem 5. *The Boolean monoid $M(D_n,\mathcal{B})$ has presentation,*

$$M(D_n,\mathcal{B}) = \langle s_1,\dots,s_n,e \,|\, (s_is_j)^{m_{ij}}=1,\ e^2=e,\ s_ie=es_i\,(i\le n-2),\\ s_ne=s_{n-1}es_{n-1}s_n,\ es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e,\\ s_ns_{n-1}es_{n-1}=es_n,\ s_{n-1}es_{n-1}es_{n-1}=s_{n-1}es_{n-1}e,\\ s_{n-1}es_{n-1}es_n=s_{n-1}es_{n-1}e,\\ s_ns_{n-2}s_{n-1}es_{n-1}s_{n-2}=s_{n-2}s_{n-1}es_{n-1}s_{n-2}s_n\rangle.$$



5. "Popova style" presentations for the arrangement monoids

We now repeat the process of the previous section, but for the arrangement monoids of types A, B and D. Much is similar, but the non-uniqueness of reduced expressions for subspaces in the arrangement systems does complicate matters a little.

Let $W = W(\Phi)$ with Φ in Table 1 and $\mathcal{H}$ the associated arrangement system. Either using the results of [3, §2.2], or the classical fact that a Weyl group acts transitively on the roots of a given length, we get the W-action is transitive on the reflecting hyperplanes in types A and D and has two orbits, corresponding to the long and short roots, for type B. We take

$$\Omega_1 = (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp \text{ for types } A, D \text{ and } \Omega_1 = \{(\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp, \mathbf{x}_n^\perp\} \text{ for type } B,$$

giving generators the s_i and e for types A and D or the s_i and e_1, e_2 for type B, and relations the usual $(s_i s_j)^{m_{ij}} = 1$ together with $e^2 = e$ or $e_i^2 = e_i, (i = 1, 2)$.

We start with the set Σ, for which the following is one of the nicest properties of the arrangement monoids from a presentation point of view:

Lemma 11. *Let $W \subset GL(V)$ be a reflection group with arrangement system $\mathcal{H}$ and Ω_1 a set of orbit representatives for the W-action on the hyperplanes of $\mathcal{H}$. Then $\Sigma = \{(s, X) \mid X \in \Omega_1\}$, where s is the reflection in the hyperplane X.*

Proof. Let $Y' \in \mathcal{H}$ and t a generating reflection for the isotropy group of Y', which by Steinberg's Theorem is a reflection in a hyperplane Y with $Y \subseteq Y'$. There is thus an $X \in \Omega_1$ and a $g \in W$ with $Y = (X)g$ with the pair (s, X) fulfilling the obligations of the set Σ for the pair (Y', t). □

The relations $\widehat{e}_{(X)g}\widehat{f} = \widehat{e}_{(X)g}$ for $(f, (X)g) \in \Sigma$ are thus $s_{n-1}e = e$ in types A and D, and $s_{n-1}e_1 = e_1, s_n e_2 = e_2$ in type B.

Concentrating now on type A, we have the atoms $\mathcal{A} = \{Y = (\mathbf{x}_i - \mathbf{x}_j)^\perp \mid 1 \leq i < j \leq n\}$, and we write $e_{ij} := \widehat{e}_Y$ for $Y = (\mathbf{x}_i - \mathbf{x}_j)^\perp$, with

$$e_{ij} = \begin{cases} (s_j \dots s_{n-1})(s_i \dots s_{n-2})e\,(s_{n-2} \dots s_i)(s_{n-1} \dots s_j), & \text{for } 1 \leq i < j \leq n-1, \\ (s_i \dots s_{n-2})e\,(s_{n-2} \dots s_i) & \text{for } 1 \leq i < n-2, j = n. \end{cases}$$

Recall from [3, §2.2] that there is a lattice isomorphism $\Lambda \mapsto X(\Lambda)$ from the lattice $\mathcal{P}(n)$ of partitions $\Lambda = \{\Lambda_1, \dots, \Lambda_p\}$ of $I = \{1, \dots, n\}$ to $\mathcal{H}$, and the W-orbits are parametrised by the corresponding partitions $\lambda = (\lambda_1, \dots, \lambda_p)$ of n, where $\lambda_i = |\Lambda_i|$.

The rank two subspaces in $\mathcal{H}$ are the $X(\Lambda)$ for Λ a partition of the form $\Lambda_1 = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5\}, \dots\}$ when $n \geq 4$, or $\Lambda_2 = \{\{i_1, i_2, i_3\}, \{i_4\}, \dots\}$. Indeed, by [6, Proposition 6.72] (see also [3, Proposition 3]), the partitions Λ_1 and Λ_2 are representatives for the W-action on the rank two elements of $\mathcal{P}(n)$, giving,

$$\Omega_2 = \{(\mathbf{x}_{n-3} - \mathbf{x}_{n-2})^\perp \cap (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp, (\mathbf{x}_{n-2} - \mathbf{x}_{n-1})^\perp \cap (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp\}.$$

Thus the $\widehat{e}_X\widehat{e}_Y = \widehat{e}_Y\widehat{e}_X$ ($X \cap Y \in \Omega_2$) family of relations reduces to $e_{n-3,n-2}e = e\,e_{n-3,n-2}$ and $e_{n-2,n-1}e = e\,e_{n-2,n-1}$.

If the *canonical partition of type* $\lambda = (\lambda_1, \dots, \lambda_p)$ is

$$\Lambda = \{\{1, \dots, \lambda_1\}, \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \dots, \{\lambda_1 + \dots + \lambda_{p-1} + 1, \dots, \lambda_1 + \dots + \lambda_p\}\},$$

then we may take the $X(\Lambda)$ below, for Λ canonical, as a set of orbit representatives, for which we fix the reduced intersection,

$$X(\Lambda) = \bigcap_{\lambda_k > 1} \bigcap_{\{i,i+1\} \subset \Lambda_k} (\mathbf{x}_i - \mathbf{x}_{i+1})^\perp, \text{ giving } \widehat{e}_{X(\Lambda)} = \prod_{\lambda_k > 1} \prod_{\{i,i+1\} \subset \Lambda_k} e_{i,i+1}.$$

If θ is a decomposition of Λ as in §2, let

$$e_\theta = \prod_{\{i,j\} \in \theta} e_{ij}.$$

The family of relations $\widehat{e}_Y = \widehat{e}_{Y_1} \dots \widehat{e}_{Y_k}$ for $Y \in \Omega_k, k \geq 2$, and $Y_1 \cap \dots \cap Y_k$ reduced, then becomes, by Lemma 1, the family $\widehat{e}_{X(\Lambda)} = e_\theta$, for all partitions λ of n, Λ the canonical partition of type λ and $\theta \in \mathcal{D}(\Lambda)$.

As with the Boolean monoids, the family of relations $s\,\widehat{e}_X = \widehat{e}_{(X)s}s$ for $(s, X) \in S \times \mathcal{A}$ can also be reduced in number:

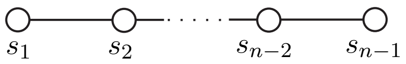
Lemma 12. *If* $Y = (\mathbf{x}_j - \mathbf{x}_k)^\perp$, $(1 \leq j < k \leq n)$*, then the relations* $s_i\widehat{e}_Y = \widehat{e}_{(Y)s_i}s_i$, $(1 \leq i \leq n-1)$ *are implied by the relations* $s_i e_X = \widehat{e}_{(X)s_i}s_i$, $(1 \leq i \leq n-1)$ *for* $X = (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp$*, and the relations* $s_i\widehat{e}_Y = \widehat{e}_{(Y)s_i}s_i$, $(1 \leq i \leq n-2)$ *for* $Y = (\mathbf{x}_i - \mathbf{x}_{i+1})^\perp$.

Proof. A case by case approach similar to Lemma 8. □

The relations are thus $s_i e = e\,s_i$, $(i \neq n-2)$, $s_{n-2}e = s_{n-1}e\,s_{n-1}s_{n-2}$ and $s_i e_{i,i+1} = e_{i,i+1}s_i$ for $(1 \leq i \leq n-2)$. Putting it all together

Theorem 6. *The arrangement monoid* $M(A_{n-1}, \mathcal{H})$ *has presentation,*

$$\begin{aligned} M(A_{n-1}, \mathcal{H}) = \langle s_1, \dots, s_{n-1}, e \,|\, &(s_i s_j)^{m_{ij}} = 1,\ e^2 = e, s_i e = e\,s_i\ (i \neq n-2), \\ &s_{n-1}e = e,\ s_{n-2}e = s_{n-1}e\,s_{n-1}s_{n-2}, \\ &e_{n-3,n-2}e = e\,e_{n-3,n-2}, \\ &e_{n-2,n-1}e = e\,e_{n-2,n-1}, \\ &s_i e_{i,i+1} = e_{i,i+1}s_i,\ (1 \leq i \leq n-2), \\ &\widehat{e}_{X(\Lambda)} = e_\theta, \lambda \textit{ a partition of } n, \theta \in \mathcal{D}(\Lambda)\rangle. \end{aligned}$$



Now to type B, where the atomic elements of $\mathcal{H}$ are the

$$A = \{(\mathbf{x}_i \pm \mathbf{x}_j)^\perp \,|\, 1 \leq i < j \leq n\} \cup \{\mathbf{x}_i^\perp \,|\, 1 \leq i \leq n\},$$

with e_{ij} defined as in type A, except that e_1 replaces e,

$$a_{ij} := \widehat{e}_Y = \begin{cases} (s_j \dots s_{n-1})(s_i \dots s_{n-2})s_n e_1 s_n(s_{n-2} \dots s_i)(s_{n-1} \dots s_j), & 1 \leq i < j \leq n-1, \\ (s_i \dots s_{n-2})s_n e_1 s_n(s_{n-2} \dots s_i), & 1 \leq i < j = n, \\ s_n e_1 s_n & i = n-1, j = n, \end{cases}$$

when $Y = (\mathbf{x}_i + \mathbf{x}_j)^\perp$, and $b_i := \widehat{e}_Y = (s_i \dots s_{n-1})e_2\,(s_{n-1} \dots s_i)$ when $Y = \mathbf{x}_i^\perp$.

The W-orbits on $\mathcal{H}$ are parametrised [6, Proposition 6.75] (see also [3, Proposition 4]) by the pairs (m, λ) of an integer $0 \leq m \leq n$ and a partition λ of $n-m$. The rank two subspaces of $\mathcal{H}$ are thus parametrised by the $(\varnothing, \Gamma, \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5\}, \dots\})$, $(\varnothing, \Gamma, \{\{i_1, i_2, i_3\}, \{i_4\}, \dots\})$, $(\{i_1\}, \Gamma, \{\{i_2, i_3\}, \{i_4\}, \dots\})$ and $(\{i_1, i_2\}, \Gamma, \{\{i_3\}, \dots\})$. By [3, Proposition 4] these four correspond to four W-orbits, where we are free to choose Γ, and the values of the i's, at will. Thus,

$$\begin{aligned} \Omega_2 = \{&(\mathbf{x}_{n-3} - \mathbf{x}_{n-2})^\perp \cap (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp,\ (\mathbf{x}_{n-2} - \mathbf{x}_{n-1})^\perp \cap (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp \\ &(\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp \cap \mathbf{x}_n^\perp,\ \mathbf{x}_{n-1}^\perp \cap \mathbf{x}_n^\perp\} \end{aligned}$$

and so the commuting of the imdepotent relations become $e_{n-3,n-2}e_1 = e_1 e_{n-3,n-2}$, $e_{n-2,n-1}e_1 = e_1 e_{n-2,n-1}$, $e_1 e_2 = e_2 e_1$ and $b_{n-1}e_2 = e_2 b_{n-1}$.

In general we have orbit representatives the $X(\Delta, \Lambda) = X(\Delta, \varnothing, \Lambda)$ of [3, §2.2], where Λ is the canonical partition of $\{1, \dots, n-m\}$ and $\Delta = \{n-m+1, \dots, n\}$. Let

$$X(\Delta, \Lambda) = \bigcap_{\lambda_k > 1} \bigcap_{\{i,i+1\} \subset \Lambda_k} (\mathbf{x}_i - \mathbf{x}_{i+1})^\perp \cap \bigcap_{i \in \Delta} \mathbf{x}_i^\perp, \text{ giving } \widehat{e}_{X(\Delta,\Lambda)} = \prod_{\lambda_k > 1} \prod_{\{i,i+1\} \subset \Lambda_k} e_{i,i+1} \prod_{i \in \Delta} b_i.$$

If $\mathcal{D} = \mathcal{D}(\Delta, \Lambda)$ is the set of decompositions of (Δ, Λ) as in §2, then for $\theta = (\theta_1, \theta_2) \in \mathcal{D}$, let

$$e_\theta = \prod_{\{i,j\}\in\theta_2} e_{ij} \prod_{\{i,j\}\in\theta_1} e_{ij} \prod_{\{-i,-j\}\in\theta_1} a_{ij} \prod_{\{i,-i\}\in\theta_1} b_i.$$

The relations $\widehat{e}_Y = \widehat{e}_{Y_1} \dots \widehat{e}_{Y_k}$ for $Y \in \Omega_k, k \geq 2$, and $Y_1 \cap \dots \cap Y_k$ reduced, then become $\widehat{e}_{X(\Delta,\Lambda)} = e_\theta$, for all pairs (m, λ) consisting of an integer $1 \leq m \leq n$ and λ a partition of $n - m$, Λ the canonical partition of type λ, $\Delta = \{n - m + 1, \dots, n\}$ and $\theta \in \mathcal{D}(\Delta, \Lambda)$.

It remains to consider the $s\,\widehat{e}_Y = \widehat{e}_{(Y)s} s$, where Lemma 12 applies equally to type B, while for $Y = \mathbf{x}_i^\perp$ we can use Lemma 9 from the Boolean case. This leaves unresolved the cases where $s = s_n$ or $Y = (\mathbf{x}_i + \mathbf{x}_j)^\perp$, for which the proof of the following is much the same as for Lemma 12:

Lemma 13. *Let $\Phi = B_n$. Then (i). the relations $s_n\,\widehat{e}_Y = \widehat{e}_{(Y)s_n} s_n$ for $Y = (\mathbf{x}_i - \mathbf{x}_n)^\perp$ can be deduced from the relations for W, and*

(ii). the relations $s_i\,\widehat{e}_Y = \widehat{e}_{(Y)s_i} s_i$ for $Y = (\mathbf{x}_i + \mathbf{x}_j)^\perp$ are implied by the relations $s_i\,\widehat{e}_Y = \widehat{e}_{(Y)s_i} s_i$ for $Y = (\mathbf{x}_i - \mathbf{x}_j)^\perp$, by the $s_i a_{i,i+1} = a_{i,i+1} s_i$ $(1 \leq i \leq n-2)$; by $s_n a_{ij} = a_{ij} s_n$ $(1 \leq i < j \leq n-1)$, and by $s_{n-1} a_{n-1,n} = a_{n-1,n} s_{n-1}$.

The first part of the Lemma leaves the $Y = (\mathbf{x}_i - \mathbf{x}_j)^\perp$ for $1 \leq i < j \leq n-1$, ie: the relations $s_n e_{ij} = e_{ij} s_n$ $(1 \leq i < j \leq n-1)$.

Theorem 7. *The arrangement monoid $M(B_n, \mathcal{H})$ has presentation,*

$$\begin{aligned} M(B_n, \mathcal{H}) = \langle s_1, \dots, s_n, e_1, e_2 \mid\ & (s_i s_j)^{m_{ij}} = 1,\ e_i^2 = e_i, s_{n-1} e_1 = e_1, s_n e_2 = e_2, \\ & s_i e_1 = e_1 s_i\ (i \neq n-2, n), s_i e_2 = e_2 s_i\ (i \leq n-2), \\ & e_1 e_2 = e_2 e_1, s_n e_2 = e_2 s_n, s_{n-2} e_1 = s_{n-1} e_1 s_{n-1} s_{n-2}, \\ & e_{n-3,n-2} e_1 = e_1 e_{n-3,n-2},\ e_{n-2,n-1} e_1 = e_1 e_{n-2,n-1}, \\ s_n s_{n-1} e_2 s_{n-1} = s_{n-1} e_2 s_{n-1} s_n,\ & b_{n-1} e_2 = e_2 b_{n-1},\ s_{n-1} a_{n-1,n} = a_{n-1,n} s_{n-1}, \\ s_i e_{i,i+1} = e_{i,i+1} s_i,\ (1 \leq i \leq n-2),\ & s_n e_{ij} = e_{ij} s_n,\ (1 \leq i < j \leq n-1), \\ s_i a_{i,i+1} = a_{i,i+1} s_i,\ (1 \leq i \leq n-2),\ & s_n a_{ij} = a_{ij} s_n,\ (1 \leq i < j \leq n-1), \\ \widehat{e}_{X(\Delta,\Lambda)} = e_\theta,\ & m \text{ an integer}, \lambda \text{ a partition of } n - m, \theta \in \mathcal{D}(\Delta, \Lambda) \rangle. \end{aligned}$$

This brings us finally to type D, where the atomic elements are the $A = \{(\mathbf{x}_i \pm \mathbf{x}_j)^\perp \mid 1 \leq i < j \leq n\}$, with e_{ij} as in type A, and

$$a_{ij} = \widehat{e}_Y = \begin{cases} (s_j \dots s_{n-1})(s_i \dots s_{n-3}) g^{-1} e g (s_{n-3} \dots s_i)(s_{n-1} \dots s_j), & 1 \leq i < j \leq n-1, \\ (s_i \dots s_{n-3}) g^{-1} e g (s_{n-3} \dots s_i), & 1 \leq i < j = n, \\ s_{n-2} g^{-1} e g s_{n-2} & i = n-1, j = n, \end{cases}$$

when $Y = (\mathbf{x}_i + \mathbf{x}_j)^\perp$, and where $g = s_{n-2} s_{n-1} s_n$.

The W-orbits on $\mathcal{H}$ are parametrised [3, Proposition 5] by the pairs (m, λ) of an integer $0 \leq m \leq n$ with $m \neq 1$, and $\lambda = (\lambda_1, \dots, \lambda_p)$ a partition of $n - m$, except if $m = 0$ and all the λ_i are even, in which case there are two orbits corresponding to this pair. The rank two subspaces of $\mathcal{H}$ are thus the $X(\varnothing, \Gamma, \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5\}, \dots\})$, $X(\varnothing, \Gamma, \{\{i_1, i_2, i_3\}, \{i_4\}, \dots\})$ and $X(\{i_1, i_2\}, \Gamma, \{\{i_3\}, \dots\})$. By [3, Proposition 5] these correspond to *four* $W(D_n)$-orbits, where we are free to choose i's at will, as well as Γ in the second two cases, and $\Gamma = \varnothing$ or $\{n\}$ in the first. Thus,

$$\begin{aligned} \Omega_2 = \{ & (\mathbf{x}_{n-3} - \mathbf{x}_{n-2})^\perp \cap (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp,\ (\mathbf{x}_{n-3} - \mathbf{x}_{n-2})^\perp \cap (\mathbf{x}_{n-1} + \mathbf{x}_n)^\perp, \\ & (\mathbf{x}_{n-2} - \mathbf{x}_{n-1})^\perp \cap (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp, (\mathbf{x}_{n-1} - \mathbf{x}_n)^\perp \cap (\mathbf{x}_{n-1} + \mathbf{x}_n)^\perp \}, \end{aligned}$$

and so the commuting of the idempotents relations become $e_{n-3,n-2}e = e\,e_{n-3,n-2}$,

$$e_{n-3,n-2}s_{n-2}g^{-1}egs_{n-2} = s_{n-2}g^{-1}egs_{n-2}e_{n-3,n-2},$$

$e_{n-2,n-1}e = e\,e_{n-2,n-1}$, and $s_{n-2}g^{-1}egs_{n-2}e = es_{n-2}g^{-1}egs_{n-2}$.

In general we get orbit representatives the $X(\Delta, \Lambda) = X(\Delta, \varnothing, \Lambda)$ of [3, §2.2] where Λ is the canonical partition of $\{1, \dots, n-m\}$ and $\Delta = \{n-m+1, \dots, n\}$, except for $\Delta = \varnothing$ and the $|\Lambda_i|$ all even, where we have representatives $X(\Lambda, \varnothing) = X(\varnothing, \varnothing, \Lambda)$ and $X(\Lambda, \{n\}) = X(\varnothing, \{n\}, \Lambda)$.

All of which results in the expressions,

$$\widehat{e}_{X(\Delta,\Lambda)} = \prod_{\lambda_k>1} \prod_{\{i,i+1\}\subset\Lambda_k} e_{i,i+1} \prod_{\{i,i+1\}\in\Delta} e_{i,i+1} \prod_{\{i,i+1\}\in\Delta} a_{i,i+1},$$

(which is also valid for $X(\Lambda, \varnothing)$) and $\widehat{e}_{X(\Lambda,\{n\})} = \widehat{e}_{X(\Lambda,\varnothing)}a_{n-1,n}$. Lemma 3 allows us to read off, for $\theta \in \mathcal{D}(\Delta, \Lambda)$,

$$e_\theta = \prod_{\{i,j\}\in\theta_2} e_{ij} \prod_{\{i,j\}\in\theta_1} e_{ij} \prod_{\{-i,-j\}\in\theta_1} a_{ij}, \text{ and } e_\theta = \prod_{\{i,j\}\in\theta} e_{ij} \prod_{\{i,n\}\in\theta} a_{in},$$

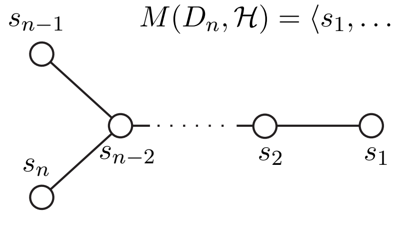
for $\theta \in \mathcal{D}(\Lambda, \{n\})$.

Finally the $s\,\widehat{e}_Y = \widehat{e}_{(Y)s}s$ relations, where Lemma 12 also applies to type D, leaving unresolved the cases where $s = s_n$ or $Y = (\mathbf{x}_i + \mathbf{x}_j)^\perp$:

Lemma 14. *Let $\Phi = D_n$. Then (i). the relations $s_n\,\widehat{e}_Y = \widehat{e}_{(Y)s_n}s_n$ for $Y = (\mathbf{x}_j - \mathbf{x}_k)^\perp$ are implied by the relations for W, $s_ne_{j,n-1} = a_{jn}s_n$ $(1 \leq j \leq n-2)$, $s_ne_{jn} = a_{j,n-1}s_n$ $(1 \leq j \leq n-2)$ and $s_ne = e\,s_n$;*

(ii). the relations $s_i\,\widehat{e}_Y = \widehat{e}_{(Y)s_i}s_i$ for $Y = (\mathbf{x}_j + \mathbf{x}_k)^\perp$ are implied by the relations $s_i\,\widehat{e}_Y = \widehat{e}_{(Y)s_i}s_i$ for $Y = (\mathbf{x}_j - \mathbf{x}_k)^\perp$, by $s_{n-2}a_{n-1,n} = a_{n-2,n}s_{n-2}$, by the $s_ia_{i,i+1} = a_{i,i+1}s_i$ $(1 \leq i \leq n-1)$, by the $s_na_{jn} = e_{j,n-1}s_n$ $(1 \leq j \leq n-2)$, by the $s_{n-1}a_{jk} = a_{jk}s_{n-1}$ $(1 \leq j < k \leq n-2)$, and by the $s_na_{jk} = a_{jk}s_n$ $(1 \leq j < k \leq n-1)$.

Theorem 8. *The arrangement monoid $M(D_n, \mathcal{H})$ has presentation,*



$$\begin{aligned} M(D_n, \mathcal{H}) = \langle s_1, \dots, s_n, e \,|\, & (s_is_j)^{m_{ij}} = 1,\ e^2 = e, s_{n-1}e = e, \\ & s_ie = es_i\ (i \neq n-2, n),\ s_{n-2}e = s_{n-1}es_{n-1}s_{n-2}, \\ & e_{n-3,n-2}e = e\,e_{n-3,n-2},\ e_{n-2,n-1}e = e\,e_{n-2,n-1} \\ & s_{n-2}g^{-1}egs_{n-2}e = es_{n-2}g^{-1}egs_{n-2}, \\ & e_{n-3,n-2}s_{n-2}g^{-1}egs_{n-2} = s_{n-2}g^{-1}egs_{n-2}e_{n-3,n-2}, \\ & s_{n-2}a_{n-1,n} = a_{n-2,n}s_{n-2},\ s_ie_{i,i+1} = e_{i,i+1}s_i,\ (1 \leq i \leq n-2), \\ & s_ne_{j,n-1} = a_{jn}s_n,\ (1 \leq j \leq n-2),\ s_ne_{jn} = a_{j,n-1}s_n,\ (1 \leq j \leq n-2), \\ & s_ia_{i,i+1} = a_{i,i+1}s_i,\ (1 \leq i \leq n-1),\ s_na_{jn} = e_{j,n-1}s_n,\ (1 \leq j \leq n-2), \\ & s_{n-1}a_{jk} = a_{jk}s_{n-1},\ (1 \leq j < k \leq n-2),\ s_na_{jk} = a_{jk}s_n,\ (1 \leq j < k \leq n-1), \\ & \widehat{e}_{X(\Lambda,\{n\})} = e_\theta,\ \theta \in \mathcal{D}(\Lambda, \{n\}),\ \widehat{e}_{X(\Delta,\Lambda)} = e_\theta, m \neq 1, \lambda \text{ a partition of } n-m, \\ & \theta \in \mathcal{D}(\Delta, \Lambda)\rangle. \end{aligned}$$

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