

A FAMILY OF CONFORMALLY ASYMMETRIC RIEMANN SURFACES

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(Received 17 January, 1996)

Abstract. We give explicit examples of asymmetric Riemann surfaces (that is, Riemann surfaces with trivial conformal automorphism group) for all genera $g \geq 3$. The technique uses Schreier coset diagrams to construct torsion-free subgroups in groups of signature $(0; 2, 3, r)$ for certain values of r .

1. Introduction. Suppose Γ is a co-compact Fuchsian group acting properly discontinuously on the hyperbolic plane \mathbb{H}^2 . The group Γ then has a presentation

$$\langle a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_s \mid x_1^{m_1} = \dots = x_s^{m_s} = \prod_{i=1}^s x_i \prod_{j=1}^g [a_j, b_j] = 1 \rangle,$$

which is encapsulated by its signature $(g; m_1, m_2, \dots, m_s)$, where g is the genus of the surface \mathbb{H}^2/Γ and m_1, m_2, \dots, m_s are the orders of the maximal finite subgroups in Γ . In the case where $g = 0$ and $s = 3$, Γ is called a *triangle group*. The hyperbolic area of a fundamental region for the action of Γ on \mathbb{H}^2 is proportional to

$$\mu(\Gamma) = 2g - 2 + \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right),$$

where, necessarily, $\mu(\Gamma) > 0$.

If Γ is torsion-free, so that it acts freely on \mathbb{H}^2 and has signature $(g; -)$, then $S = \mathbb{H}^2/\Gamma$ is a Riemann surface of genus $g \geq 2$. Its group of conformal automorphisms $\text{Aut}(S)$ is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normaliser of Γ in $PSL(2, \mathbb{R})$, the orientation preserving conformal isometry group of the hyperbolic plane. In this paper we construct Riemann surfaces S for all genera $g \geq 3$ where $\text{Aut}(S)$ is trivial, by searching for appropriate Γ amongst the subgroups of certain triangle groups. It is worth recalling that any genus 2 surface, being hyperelliptic, admits a conformal involution. In [2], an infinite family of such surfaces is constructed, although the set of genera not accounted for is also infinite.

2. Coset diagrams for triangle groups. Consider the triangle group $\Delta = \Delta(p, q, r)$ having signature $(0; p, q, r)$ and presentation

$$\langle x_1, x_2, x_3 \mid x_1^p = x_2^q = x_3^r = x_1 x_2 x_3 = 1 \rangle \cong \langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle.$$

If Γ is a subgroup of finite index n in Δ then the action of the triangle group on the right cosets of Γ induces a homomorphism $\theta: \Delta \rightarrow S_n = \text{Sym}\{1, 2, \dots, n\}$. By theorem 1 of [3], Γ will be torsion-free when $\theta(x)$, $\theta(y)$ and $\theta(xy)$ are *regular* permutations—that is, they are composed entirely of p -cycles, q -cycles and r -cycles respectively. At this stage, we just insist that $\theta(y)$ is regular. We then depict this representation using a (slightly simplified)

Schreier coset diagram D: for each q -cycle in $\theta(y)$ draw a shaded q -gon, labelling its vertices in an anticlockwise direction with the points from the cycle; for each point $i \in \{1, 2, \dots, n\}$ and its image j under $\theta(x)$, draw a directed arc running from i to j (except for those i fixed by $\theta(x)$ which are left unconnected by arcs).

On the other hand, suppose D is some diagram composed of n vertices grouped into shaded q -gons, and such that each vertex is incident with either no directed arcs or precisely two—one incoming and the other outgoing. The arcs and q -gons induce permutations $\alpha, \beta \in S_n$ according to the above scheme. These permutations are unique up to relabelling of the vertices and satisfy $\alpha^p = \beta^q = (\alpha\beta)^r = 1$, where p and r are the orders of the permutations α and $\alpha\beta$. Hence we define a homomorphism $\theta: \Delta = \Delta(p, q, r) \rightarrow S_n$ by $\theta(x) = \alpha$ and $\theta(y) = \beta$, and refer to D as a diagram for Δ . If $k \in \{1, 2, \dots, n\}$, and G_k is the stabiliser in $G = \langle \theta(x), \theta(y) \rangle$ of k , then $\theta^{-1}(G_k)$ is a subgroup Γ_k of index n in Δ . For different vertex labellings and different k the subgroups Γ_k are conjugate in Δ , so without loss of generality we simply refer to the subgroup Γ arising from D . By construction, the permutation $\theta(y)$ is regular, and if $\theta(x)$ and $\theta(xy)$ are also regular, then Γ is torsion-free (again by [3]).

The relations $x^p = y^q = (xy)^r = 1$ induce a natural embedding of a diagram D into an orientable surface by defining the faces of the embedding to be the cycles of $\theta(x)$, $\theta(y)$ and $\theta(xy)$, ignoring repetitions. By Theorem 2 of [4], the genus of this embedding coincides with the genus of $S = \mathbb{H}^2/\Gamma$, where Γ is the group arising from D . We can thus compute the genus of this surface either by using the Riemann-Hurwitz formula $n\mu(\Delta) = \mu(\Gamma)$ (where n is the number of vertices of D and hence the index of Γ in Δ) or by taking an Euler count on the vertices, q -gonal sides, arcs and faces of D . Note that taking a face of the third kind—that is, one that is unshaded and not the interior of an x -cycle—and counting the number of q -gonal sides adjacent to it gives the length of a cycle of $\theta(xy)$. Hence, for the group Γ arising from D to be torsion-free, this count must be r for each such face of the embedding.

A family of examples, D'_m , $m \geq 4$, on $12m$ vertices and where the embedding is planar is illustrated in Figure 1—in fact the diagrams we shall use for our construction (where for transpositions in $\theta_m(x)$ we have replaced the two arcs running between a pair of vertices by a single undirected edge). Counting the number of triangular sides adjacent to the two unshaded faces we see that $\theta_m(xy)$ is composed of two cycles of length $6m$, $m \geq 4$. Clearly $\theta_m(x)$ and $\theta_m(y)$ have orders two and three respectively, so that the group Γ_m arising from D_m has index $12m$ in the triangle group $\Delta(2, 3, 6m)$, $m \geq 4$. Note that Γ_m is not torsion-free in this case, having signature $(0; 2^{4m})$.

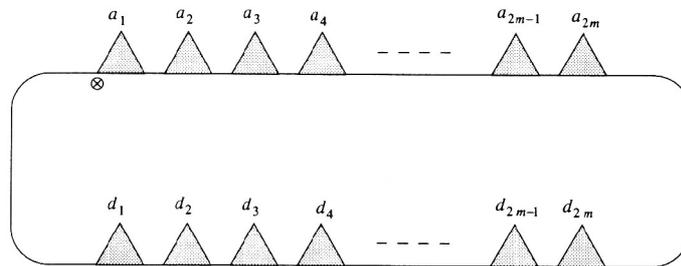


Figure 1.

An *automorphism* of a coset diagram D for some Δ is a permutation $\sigma \in S_n$ of its vertices such that $\sigma\theta(x) = \theta(x)\sigma$ and $\sigma\theta(y) = \theta(y)\sigma$. The automorphisms form a group under composition, $\text{Aut}(D)$, which is then the centraliser in S_n of $G = \langle \theta(x), \theta(y) \rangle$. Observe that σ is uniquely determined by its effect on a vertex of D , so that in particular, if σ fixes a vertex it must be the identity permutation. By Theorem 2.5 of [6] and Theorem 1 of [4], $\text{Aut}(D)$ is isomorphic to $N_\Delta(\Gamma)/\Gamma$, where $N_\Delta(\Gamma)$ is the normaliser in Δ of the group Γ arising from D . Hence, $\text{Aut}(D)$ acts as a group of conformal automorphisms of the Riemann surface $S = \mathbb{H}^2/\Gamma$, but of course in general, it will be a subgroup of the full automorphism group $\text{Aut}(S)$. We are interested in when the two coincide.

Suppose that Δ is *non-arithmetic* and *maximal*, in the sense that it is not properly contained in any triangle group. If Γ is a subgroup of finite index in Δ and $g \in N(\Gamma)$, the normaliser in $PSL(2, \mathbb{R})$ of Γ , then g is contained in the commensurator of Γ , $\text{Comm}(\Gamma) = \{g \in PGL(2, \mathbb{R}) \mid \Gamma, g^{-1}\Gamma g \text{ commensurable}\}$ —where two Fuchsian groups are commensurable if their intersection has finite index in both. Since $\text{Comm}(\Gamma)$ is contained in $\text{Comm}(\Delta)$, we have $g \in \text{Comm}(\Delta)$. By a theorem of Margulis [1], and since Δ is non-arithmetic, $\text{Comm}(\Delta)$ contains Δ as a subgroup of finite index. An elementary area calculation using the Riemann-Hurwitz formula gives that $\text{Comm}(\Delta)$ is also a triangle group, and hence by maximality $\text{Comm}(\Delta) = \Delta$. Thus, $g \in \Delta$ and so $N(\Gamma)/\Gamma = N_\Delta(\Gamma)/\Gamma$, that is, $\text{Aut}(S) \cong \text{Aut}(D)$.

Finally, we describe a method for combining diagrams to obtain new ones. Suppose D_1 and D_2 are diagrams for the same triangle group $\Delta(p, q, r)$, where the groups Γ_1 and Γ_2 arising from them have index n_1 and n_2 in Δ respectively. A t -handle $[u, v]_t$ is a pair of vertices u and v such that $\theta_i(x):u \rightarrow u$, $\theta_i(x):v \rightarrow v$ and $\theta_i(xy):u \rightarrow v$, $i = 1$ or 2 . Suppose that between them, the diagrams D_1 and D_2 contain the p t -handles $[u_1, v_1]_t, [u_2, v_2]_t, \dots, [u_p, v_p]_t$. The *composition* of D_1 and D_2 is then the diagram D obtained by connecting D_1 and D_2 with the two p -cycles (u_1, u_2, \dots, u_p) and (v_1, v_p, \dots, v_2) : that is, drawing directed arcs from vertex u_1 to u_2 , u_2 to u_3, \dots, u_p to u_1 , v_1 to v_p , v_p to v_{p-1}, \dots , and v_2 to v_1 .

Clearly, the permutations α and β arising from D are elements of $S_{n_1+n_2}$, with β composed entirely of q -cycles and α having order p . A cycle of $\alpha\beta$ that does not pass through any of the vertices u_i and v_i is unaffected by composition (up to the obvious relabelling of the vertices of one of the diagrams). If $(u_i, w_{i,1}, w_{i,2}, \dots, w_{i,t-1}, v_i, w_{i,t+1}, \dots)$ and $(u_{i+1}, w_{i+1,1}, w_{i+1,2}, \dots, w_{i+1,t-1}, v_{i+1}, w_{i+1,t+1}, \dots)$ are the cycles in $\alpha\beta$ that pass through the handles $[u_i, v_i]$ and $[u_{i+1}, v_{i+1}]$ respectively *before* composition, then after, the first becomes $(u_i, w_{i+1,1}, w_{i+1,2}, \dots, w_{i+1,t-1}, v_i, w_{i,t+1}, \dots)$, again, up to relabelling. Thus, for all i , the lengths of these cycles is unchanged, and so $\alpha\beta$ has order r . We define θ in the usual way, giving that D is also a diagram for $\Delta(p, q, r)$ with the resulting group Γ having index $n_1 + n_2$ in Δ .

3. The construction. Consider the diagrams D_m for $\Delta(2, 3, 6m)$, $m \geq 4$, illustrated in Figure 1. By [5], the triangle group $\Delta(2, 3, r)$ is arithmetic exactly when $r = 7, 8, 9, 10, 11, 12, 14$ and 18 , and a simple area calculation gives that $\Delta(2, 3, r)$ is maximal for all r . We observed in the previous section that $\theta_m(xy)$ is composed of two cycles of length $6m$ while $\theta_m(y)$ is obviously composed of $4m$ 3-cycles. Thus, both $\theta_m(y)$ and $\theta_m(xy)$ are regular permutations.

The vertices a_1, a_2, \dots, a_{2m} and d_1, d_2, \dots, d_{2m} can be grouped into the 2-handles

$[a_1, a_2]_2, [a_3, a_4]_2, \dots, [a_{2m-1}, a_{2m}]_2, [d_1, d_2]_2, [d_3, d_4]_2, \dots, [d_{2m-1}, d_{2m}]_2$. Since $\theta_m(x)$ has order two in this case, a single composition requires just two of these 2-handles. Perform $2m$ such compositions on D_m by joining the 2-handle $[a_i, a_{i+1}]_2$ to the 2-handle $[d_{i+2}, d_{i+3}]_2$, where subscripts are taken modulo $2m$. Alternatively, we achieve the same result by considering the handles $[a_1, a_{2m-2}]_{4m-6}, [a_2, a_{2m-3}]_{4m-10}, \dots, [a_{m-1}, a_m]_2$ and joining them to the handles $[d_3, d_{2m}]_{4m-6}, [d_4, d_{2m-1}]_{4m-10}, \dots, [d_{m+1}, d_{m+2}]_2$, respectively; (this observation will become useful shortly). Call the resulting diagrams D'_m . Notice that after these compositions, the resulting $\theta'(x)$ is composed entirely of transpositions, and so is a regular permutation. Thus, the groups $\Gamma'_m, m \geq 4$, arising from the D'_m are torsion-free in $\Delta(2, 3, 6m)$. These diagrams have $12m$ vertices, $12m$ triangular sides, $6m$ edges, $4m$ shaded faces and 2 unshaded faces. An Euler count thus gives $2 - 2g = 12m - 12m - 6m + 4m + 2 = -2m + 2 \Rightarrow g = m, m \geq 4$.

Now, chasing a path around the diagram through the handles $[a_1, a_{2m-2}]_{4m-6}$ and $[d_3, d_{2m}]_{4m-6}$, we see that the element $\theta'((xy)^{4m-6}(xy^2)^{4m-6})$ fixes the vertices a_{2m-2} and d_{2m} . Moreover, a careful but routine examination of the cycles in $\theta'(xy)$ and $\theta'(xy^2)$ reveals that these are the only two vertices fixed by this element (essentially since $4m - 6$ is the largest t for which D_m contains a t -handle, the handles $[a_1, a_{2m-2}]_{4m-6}$ and $[d_3, d_{2m}]_{4m-6}$ being the only two such examples.) Thus, since any $\sigma \in \text{Aut}(D'_m)$ centralises $\theta'((xy)^{4m-6}(xy^2)^{4m-6})$, it must either stabilise both a_{2m-2} and d_{2m} , or transpose them. Notice that the vertex in D'_m marked \otimes and the vertex d_1 are the images of a_{2m-2} and d_{2m} respectively under $\theta'(y^2xy^2xyx)$. Hence if (a_{2m-2}, d_{2m}) is a transposition in σ , then so is (\otimes, d_1) . But (a_{2m-2}, d_{2m}) is a transposition in $\theta'(x)$ while (\otimes, d_1) is not, and so σ does not commute with $\theta'(x)$, giving that σ must fix both a_{2m-2} and d_{2m} , and is the identity. Thus, $\text{Aut}(D'_m) \cong \text{Aut}(\mathbb{H}^2/\Gamma'_m)$ is trivial.

Finally (and somewhat unfortunately) the case $g = 3$ is handled separately using the diagram D in Figure 2. Chasing the action of $\theta(xy)$, it is easy to see that D is a diagram for $\Delta(2, 6, 9)$ where $\theta(y)$ and $\theta(xy)$ are regular permutations. Also, $\Delta(2, 6, 9)$ is non-arithmetic ([5]) and maximal. Composing the 2-handles $[a_1, a_2]_2$ and $[a_3, a_4]_2$ as well as the pair $[a_5, a_6]_2$ and $[a_7, a_8]_2$ yields D' , in which the resulting $\theta'(x)$ is also regular. An Euler count gives that \mathbb{H}^2/Γ' has genus 3, and a simple calculation using the group theory package MAGMA evaluates the order of the centraliser in S_{18} of $\langle \theta'(x), \theta'(y) \rangle$ (being trivial).

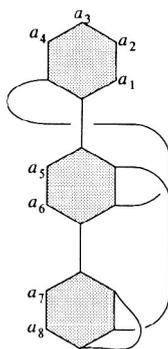


Figure 2.

4. Acknowledgements. The author is grateful to David Singerman and Colin Machlachlan for helpful comments during the preparation of this paper, and to the Department of Mathematical Sciences at the University of Aberdeen for its hospitality.

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