Lectures on Buildings

Summer 2011

Lecture 1 Motivating Example (flag complex of a vector space)

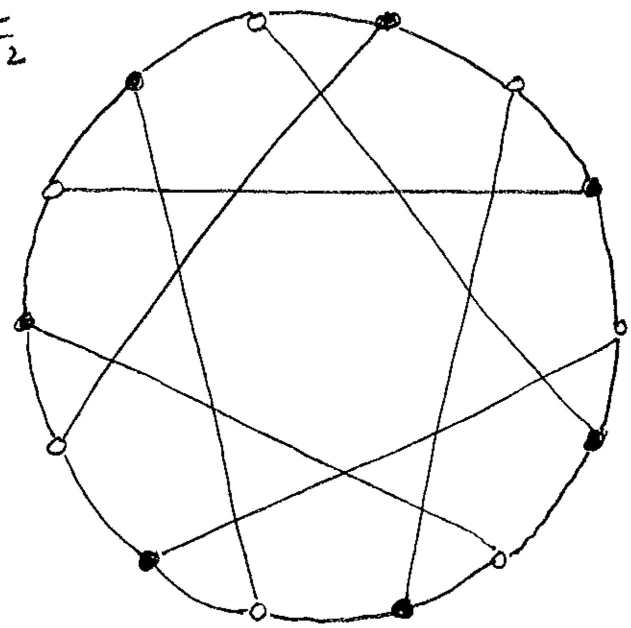
- k field, V 3-dim vector space $/ k$.

$\Delta =$ 1-dim. simplicial complex (i.e.: graph) with vertices = proper non-trivial subspaces of V (hence either 1, 2-dimensional)

edges $\underset{V_i}{\circ}$ —— $\underset{V_j}{\bullet}$ $\iff V_i \subset V_j$

(V_i: 1-dim, V_j: 2-dim)

- Eg: $k = \mathbb{F}_2$



call the edges "chambers".

Identify Δ with its chambers.

- finding a shortest route from one chamber to another:

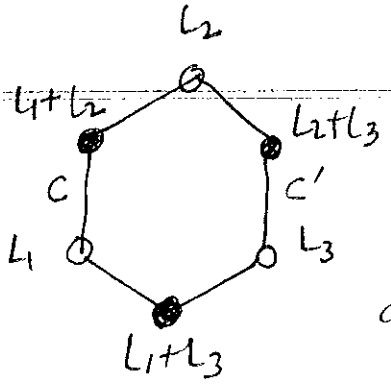
$C = V_1 \subset V_2 \rightsquigarrow C' = V_1' \subset V_2'$

(assume $V_1 \neq V_1'$, $V_2 \neq V_2'$)

change notation: L_1, L_2, L_3 lines with $L_1 = V_1$, $L_3 = V_1'$ and $L_2 = V_2 \cap V_2'$

Ex: $V_2 = L_1 + L_2$, $V_2' = L_2 + L_3$

Thus:



local picture $\supseteq C, C'$.

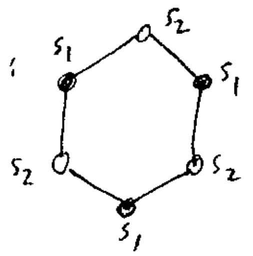
(note: doesn't see k)

call chambers $V_1 \subset V_2$ and $V_1' \subset V_2'$ s_i-incident

$\Leftrightarrow$ they only differ in i-th position

i.e.: $V_1 \subset V_2 \supset V_1'$ s_1-incident $\quad$ $V_2 \supset V_1 \subset V_2'$ s_2-incident

label vertices by s_1, s_2:

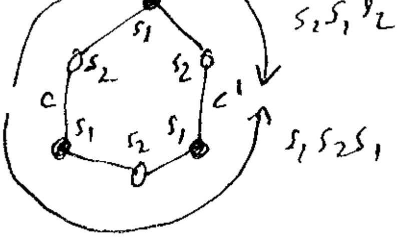


shortest route(s) in local picture $C \underset{s_2 s_1 s_2}{\xrightarrow{s_1 s_2 s_1}} C'$ (read routes from l. to r.)

define $\delta(C, C') :=$ set of words in s_1, s_2 obtained by performing this process in all possible ways

- want $\delta(C,C')$ to be an $\in$ a group: if $s_1 := (1,2)$, $s_2 := (2,3) \in \mathfrak{S}_3$

then the words in $\delta(C,C')$ all represent same $\pi \in \mathfrak{S}_3$*



$\left.\begin{matrix} s_2 s_1 s_2 \\ s_1 s_2 s_1 \end{matrix}\right\} = (1,3) \in \mathfrak{S}_3$

*(we will see why in lecture 4).

- these choices are (in some sense) canonical:

$C = 0 \subset L_1 \subset L_1 + L_2 \subset V = V_0 \subset \cdots \subset V_3$

$C' = 0 \subset L_3 \subset L_2 + L_3 \subset V = V_0' \subset \cdots \subset V_3'$

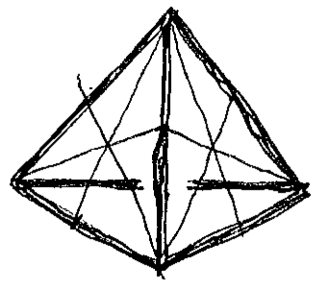
filtrate V_i'/V_{i-1}' by $\cap$ with $V_0 \subset \cdots \subset V_3$

(i.e.: $(V_i' \cap V_0)/V_{i-1}' \subset \cdots \subset (V_i' \cap V_3)/V_{i-1}'$

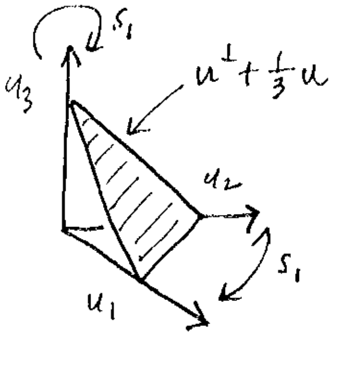
i	V_i/V_{i-1}	filtration	"jump" index j
1	L_3	$0 \subset 0 \subset 0 \subset L_3$	3
2	L_2+L_3/L_3	$0+L_3 \subset 0+L_3 \subset L_2+L_3 \subset L_2+L_3$	2
3	V/L_2+L_3	$0+(L_2+L_3) \subset V+(L_2+L_3) \subset V+(L_2+L_3) \subset V+(L_2+L_3)$	1

defining $\pi(i)=j$ gives $(1,3) \in \mathfrak{S}_3$.

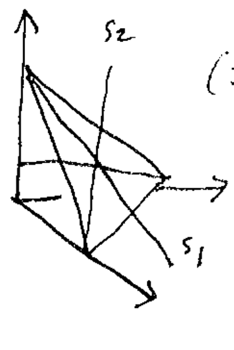
- summary: building is a set of chambers with "s_i-incidence" ($s_i \in$ some set S) and a "W-valued metric" for W some group containing S.

- Ex: repeat with $\dim V = 4$; replace by 
  (= ∂ of tetrahedron, barycentrically subdivided)
  (c.f.  $\approx$  = ∂ of triangle barycentrically subdivided).

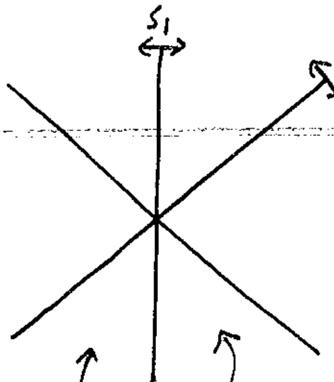
- $W=\mathfrak{S}_3$ as reflection group: V now Euclidean with orthonormal basis $\{u_1,u_2,u_3\}$; $\mathfrak{S}_3 \circlearrowright V$ via $\pi \cdot u_i := u_{\pi(i)}$ (i.e.: permuting coordinates); $u=\sum u_i$ fixed by all of $\mathfrak{S}_3$
  $\rightsquigarrow$ pass to action on $u^\perp = \{\sum \lambda_i u_i \mid \sum \lambda_i = 0\}$



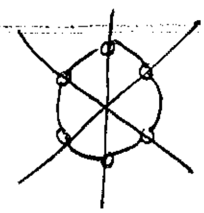
$s_1=(1,2)$
$s_2=(2,3)$



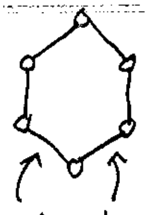
(= triangle with ∂ barycentrically subdivided) transfer $\rightsquigarrow$ to $u^{\perp}$

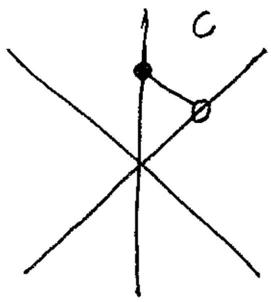


intersect $\rightsquigarrow$ with S^1



$\approx$





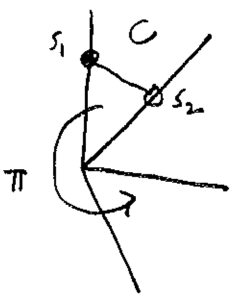
$\mathfrak{S}_3$ transitive on chambers and

$\pi C = C \Rightarrow \pi = 1$.

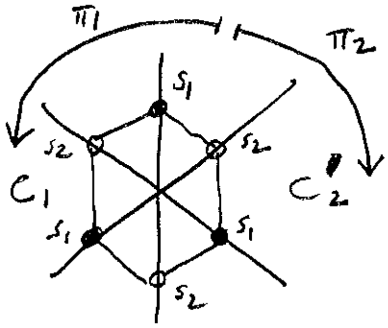
$\Rightarrow \mathfrak{S}_3 \overset{1-1}{\longleftrightarrow}$ chambers

via $\pi \leftrightarrow \pi C$

get s_1, s_2 labels by transfering:



gives $\rightsquigarrow$



in this context $\delta(C_1, C_2) = \pi_1^{-1}\pi_2$ (Eg: $\pi_1 = s_1 s_2$, $\pi_2 = s_2$, $\delta(C_1, C_2) = s_2 s_1 s_2$)

<u>summary</u>: building a set of chambers and "W-valued metric" for W a reflection group, and δ arises from the "geometry" of W.

Lecture 2 Reflection groups and Coxeter groups

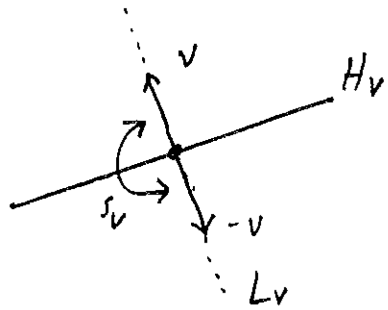
- V = finite dim. $\mathbb{R}$-vector space

<u>reflection</u> = linear map $s: V \to V$ s.t. there is linear hyperplane H_s and

$V = H_s \oplus L_s$ with $s|_{H_s} = 1$ and $s|_{L_s} = -1$

reflection group W = gp. generated by finitely many reflections.

- **Eg**: V Euclidean with inner product; <u>orthogonal</u> reflection has $L_s = H_s^{\perp}$; alternatively, $H_s = H_v = v^{\perp}$ for some $v (\neq 0)$ in V and $s = s_v$ fixes H_v pointwise and $v \overset{s_v}{\longmapsto} -v$. (uniquely determined by v or H_v $(= v^{\perp})$).

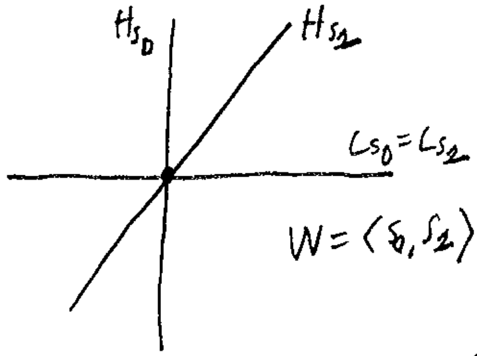


- **Ex**: let $\mathcal{H} = \{H_{v_1}, \dots, H_{v_m}\}$ and $W = \langle \text{orthog. refs. } s_{v_i} \rangle$. Then W finite $\Leftarrow$ $s_{v_i}\mathcal{H} = \mathcal{H}$ for all i. (or $W\mathcal{H} = \mathcal{H}$)

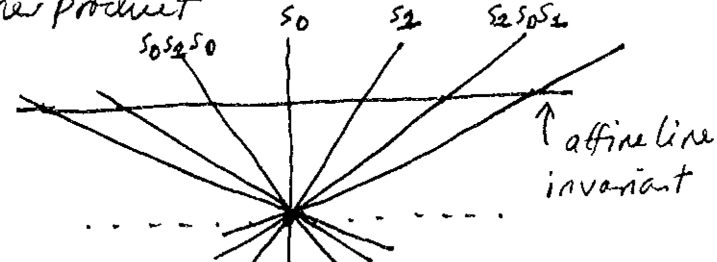
- **Eg**: $\{v_1, \dots, v_{n+1}\}$ orthon. basis for V and $\mathcal{H} = \{ (v_i - v_j)^{\perp} : 1 \leq i \neq j \leq n+1 \}$ then $W\mathcal{H} = \mathcal{H}$. Indeed $W \cong \mathfrak{S}_{n+1}$ via $s_{v_i - v_j} \mapsto (i, j)$

- The W above leave the sphere $\{v : \|v\| = 1\}$ invariant and so are often called <u>spherical</u> reflection gps.

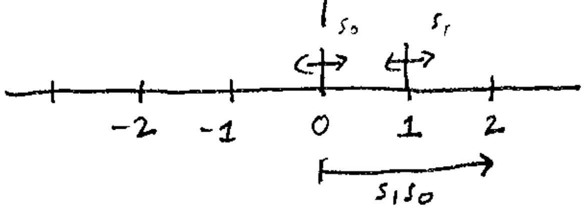
- **Eg**: V 2-dimensional, but no inner product



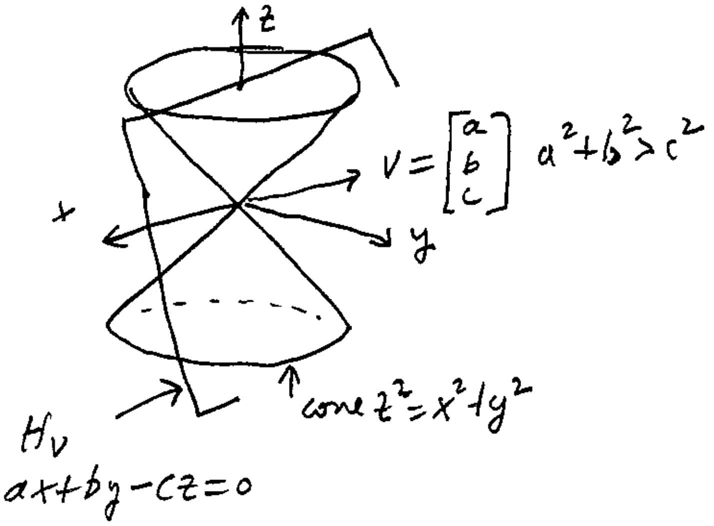
$W = \langle s_0, s_2 \rangle$



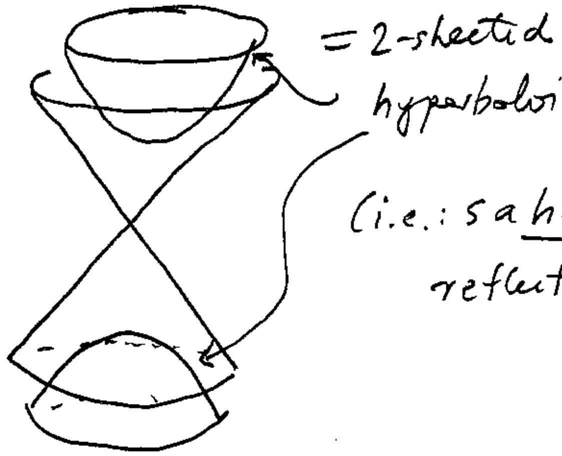
W <u>affine</u> reflection group $\cong$



• **Eg**: $\dim V = 3$ (no inner product)



Ex: s reflection with $H_s = H_v$ and $L_s = \mathbb{R}v$ leaves invariant set $x^2+y^2-z^2=-1$



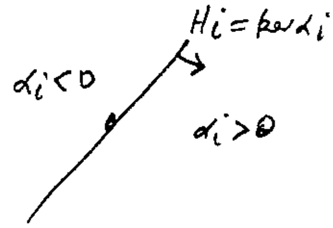
(i.e.: s a hyperbolic reflection)

• return to finite case: V Euclidean, $\mathcal{H} = \{H_i\}_{i\in I}$ finite, $W = \langle s_i \rangle$ (orthog. refs.)

with $W\mathcal{H} = \mathcal{H}$ ($\Rightarrow$ W finite and $\mathcal{H}$ all reflecting hyperplanes of W)

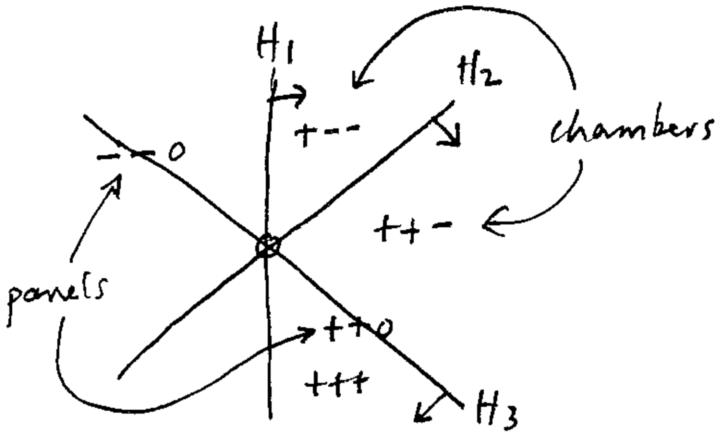
Choose $\alpha_i (\neq 0) \in V^*$ with $H_i = \ker \alpha_i$

For each $i \in I$ choose $\varepsilon_i \in \{\pm 1\}$



chamber := non-empty set of form $\{v \mid \alpha_i(v) = \lambda_v \varepsilon_i \ (\lambda_v > 0)\}$

panel := non-empty set of form $\{v \mid \alpha_{i_0}(v) = 0$ for some i_0, $\alpha_i(v) = \lambda_v \varepsilon_i \ (\lambda_v > 0)$ for $i \neq i_0\}$



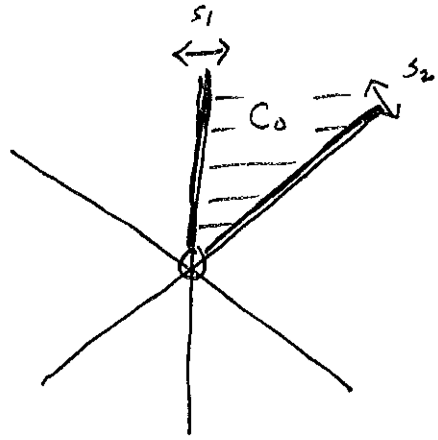
(note: +−0 is empty and so not a panel)

W acts regularly on chambers

i.e.: $\begin{cases} W \text{ acts transitively} \\ \text{and } wC = C \Rightarrow w = 1 \end{cases}$

fix a chamber $C_0 \Rightarrow W \overset{1-1}{\longleftrightarrow}$ chambers via $w \longleftrightarrow wC_0$

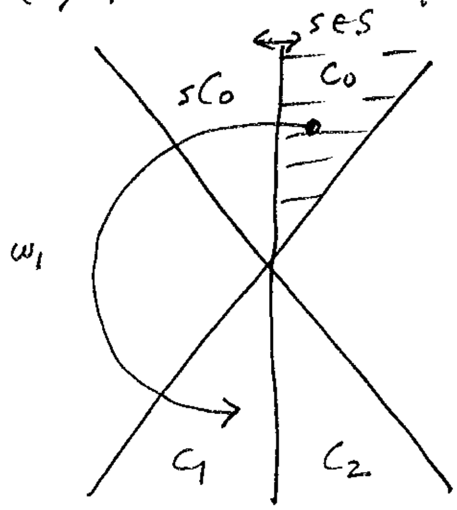
A a <u>panel</u> of $C \overset{def}{\iff} A \subset \overline{C}$



$S = \{s_1, \dots, s_n\}$ s.t. H_i is spanned by a panel of C_0

C_1, C_2 (chambers) are <u>adjacent</u>

$\iff$ have a common panel.

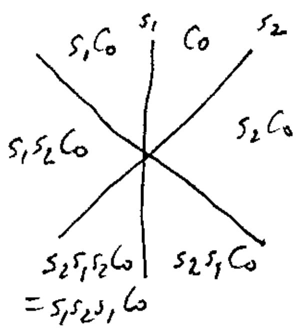


If $C_1 = w_1 C_0$, $C_2 = w_2 C_0$ adjacent then $w_1^{-1} C_1$, $w_1^{-1} C_2$ adjacent and the common panel of $C_1 C_2$ sent to a panel of C_0 corresponding to $s \in S$, with $w_1^{-1} C_2 = s C_0$.

i.e.: $C_2 = w_1 s C_0$, thus

(*) chambers adjacent to chamber $w_1 C_0$ are the $w_1 s C_0$ $(s \in S)$.

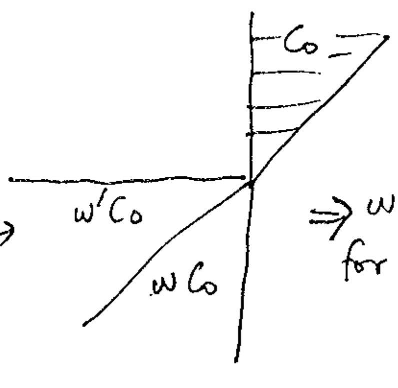
<u>fact 1</u>: W generated by the $s \in S$



[<u>proof</u> (sketch): induction on the

"distace" of wC_0 from C_0

"closer" to C_0 $\longrightarrow$



$\Rightarrow w = w's$ for some s]

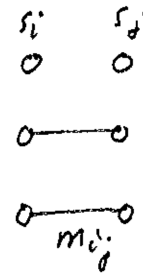
for $s_1, s_2 \in S$ the element $s_1 s_2$ a rotation of finite order m_{ij} say.

<u>fact 2</u>: $W \cong \langle s \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$

where $m_{ij} \in \mathbb{Z}^{\geq 1}$ s.t. $m_{ij} = m_{ji}$ and $m_{ij} = 1 \Leftrightarrow i = j$ (in particular, $s_i^2 = 1$)

An (abstract) group with such a presentation where the $m_{ij} \in \mathbb{Z}^{\geq 1} \cup \infty$ (S finite) is a <u>Coxeter group</u>. (write (W,S))

<u>symbol for W</u>: nodes the $s \in S$ and

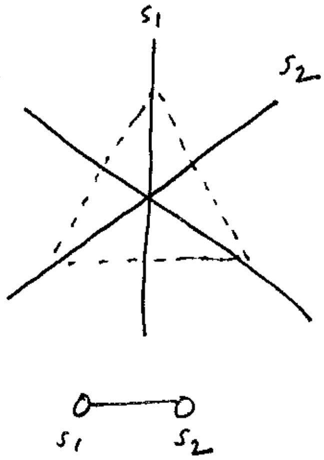


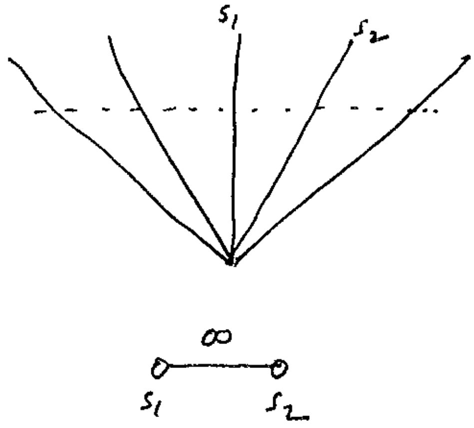
$m_{ij} = 2$

$m_{ij} = 3$

$m_{ij} > 3$

<u>Eg</u>:





Lecture 3 Chamber systems + Coxeter complexes

- set Δ is a <u>chamber system</u> over (finite) $I \Leftrightarrow$ each $i \in I$ determines equivalence relation on Δ $\sim_i$ called <u>i-adjacency</u>. ($c, c' \in \Delta$ chambers, $c \sim_i c'$ i-adjacent)

<u>Gallery</u> = sequence chambers $c_0 \sim_{i_1} c_1 \sim_{i_2} \cdots \sim_{i_k} c_k$ with $c_{j-1} \neq c_j$

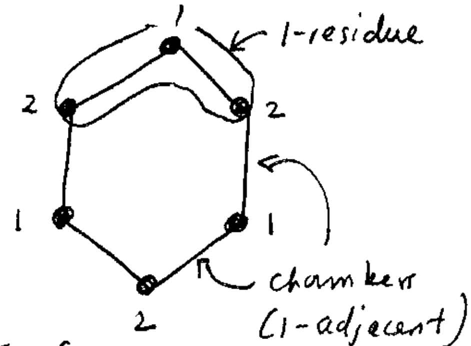
has <u>type</u> $i_1 i_2 \dots i_k$. Write $c_0 \xrightarrow[f]{} c_k$ ($f = i_1 i_2 \dots i_k$)

J-gallery for $J \subseteq I$ a gallery type $i_1 \cdots i_k$ with $i_j \in J$.

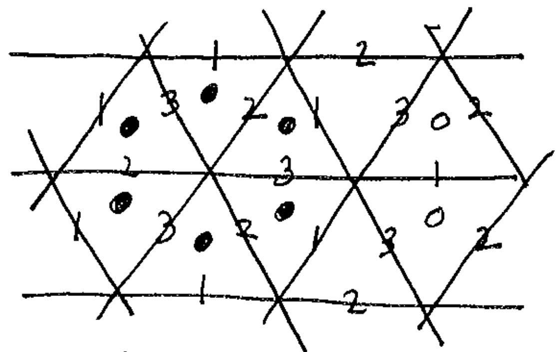
$\Delta' \subseteq \Delta$ is J-connected $\Leftrightarrow$ any two chambers in Δ' can be joined by a J-gallery.

J-residues of Δ = J-connected components.

Egs:



$I = \{1,2\}$



● = $\{2,3\}$-residue

o = $\{1\}$-residue

J-residue has rank $|J|$

chambers = rank 0 residues

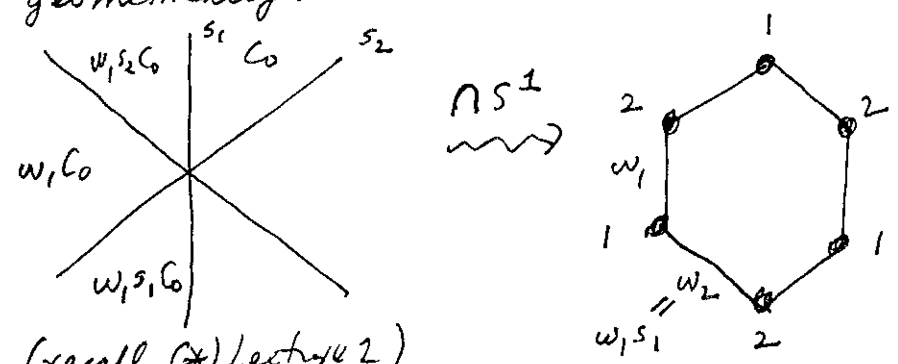
panel := rank 1 residue.

- Eg: Coxeter complex. (W,S) Coxeter group with $S = \{s_1, \dots, s_n\}$

chambers = $w \in W$ ($w_1 \sim_i w_2 \overset{\text{def}}{\Longleftrightarrow} w_2 = w_1 s_i$), $I = \{1, \dots, n\}$, write Δ_W

$(W,S) = \underset{s_1}{\circ}\!\!-\!\!\underset{s_2}{\circ}$

geometrically:



(recall (*) Lecture 2)

an s_i-panel has form $w \sim_i w s_i$ (or $w s_i \sim_i w$)

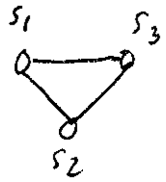
i.e.: every panel $\supseteq$ exactly two chambers.

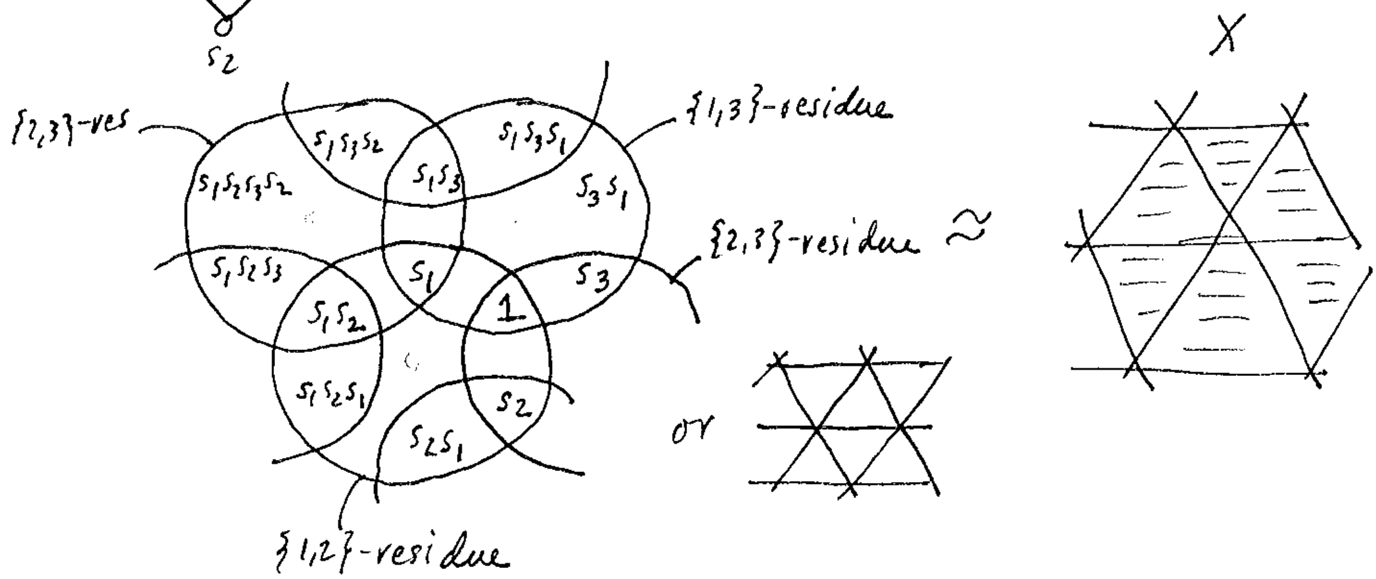
[• aside: (abstract) simplicial complex with vertex set V = collection X of finite subsets of V s.t.
$$\begin{cases} \sigma \in X,\ \underset{(\emptyset \neq)}{\tau} \subseteq \sigma \Rightarrow \tau \in X \\ \{v\} \in X \text{ for } v \in V \end{cases}$$

(call σ a $(|\sigma|-1)$-simplex)

Eg: Δ = chamber system / I, V = set of rank $|I|-1$ residues

X = set of $\sigma \subseteq V$ s.t. $\underset{k\text{-simplex}}{\sigma = \{R_0, \ldots, R_k\}} \Leftrightarrow \bigcap R_i \neq \emptyset$.

(W,S) =  and Δ = Coxeter complex



note: (1). X = nerve of covering of Δ by rank $|I|-1$ residues.

(2). $\bigcap R_i \neq \emptyset \Rightarrow \bigcap R_i$ residue too.
(R_i over J_i) (over $\bigcap J_i$)

moral: chambers in $\Delta \sim$ top. dim. simplicies of X; residues give lower dim. simplicies] — rank k residues $\longleftrightarrow$ codim k simplicies

- Δ_W = Coxeter cx. of (W,S), $f = i_1 i_2 \cdots i_k$

  $S = \{s_1, \dots, s_n\}$
  $I = \{1, \dots, n\}$
  let $s_f := s_{i_1} s_{i_2} \cdots s_{i_k} \in W$

gallery $C \xrightarrow{f} C'$ $\Leftrightarrow$ $C \sim w$, $C' \sim w'$ and $w' = w s_f$ in W.

<u>gallery</u> $C \xrightarrow{f} C'$ <u>minimal</u> $\overset{\text{def}}{\Longleftrightarrow}$ there is no gallery from C to C' passing thro' fewer chambers.

<u>word</u> $s_f = s_{i_1} \cdots s_{i_k}$ <u>reduced</u> $\overset{\text{def}}{\Longleftrightarrow}$ there is no word representing s_f involving fewer $s \in S$ (counted with multiplicity)

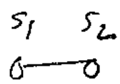
thus: $C \xrightarrow{f} C'$ minimal $\Leftrightarrow$ s_f reduced.

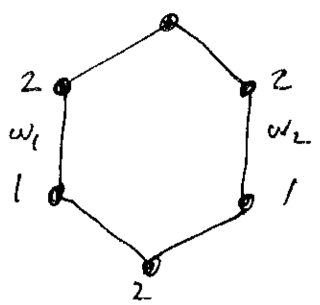
- define a "W-valued metric" on Δ_W: $\delta_W : \Delta_W \times \Delta_W \to W$

by $\delta_W(w_1, w_2) = w_1^{-1} w_2$.

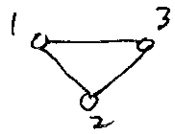
Then $\delta_W(w_1, w_2) = s_f \Leftrightarrow w_1 s_f = w_2 \Leftrightarrow$ gallery $w_1 \xrightarrow{f} w_2$

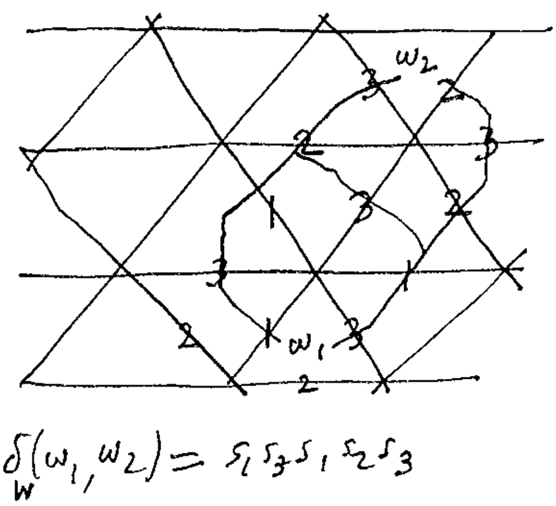
- <u>Eg's</u>:

$(W,S) =$ 

$\Delta_W =$ 

$\delta_W(w_1, w_2) = s_2 s_1 s_2$
$= s_1 s_2 s_1$

$(W,S) =$ 

$\Delta_W =$ 

$\delta_W(w_1, w_2) = s_1 s_3 s_1 s_2 s_3$
$(= s_3 s_1 s_2 s_3 s_2 = s_3 s_1 s_3 s_2 s_3$
$= \cdots)$

Lecture 4 Buildings

• A building type (W,S) = chamber system Δ (over I if $S=\{s_i\}_{i\in I}$)

s.t. (1). every panel $\supseteq$ at least two chambers.

(2). Δ has W-valued "metric" $\delta : \Delta\times\Delta \to W$ s.t. if

$s_{i_1}\cdots s_{i_k}$ reduced word then $\delta(c,c') = s_{i_1}\cdots s_{i_k} \iff \exists$

a gallery $c \longrightarrow_f c'$ of type $f = i_1\cdots i_k$.

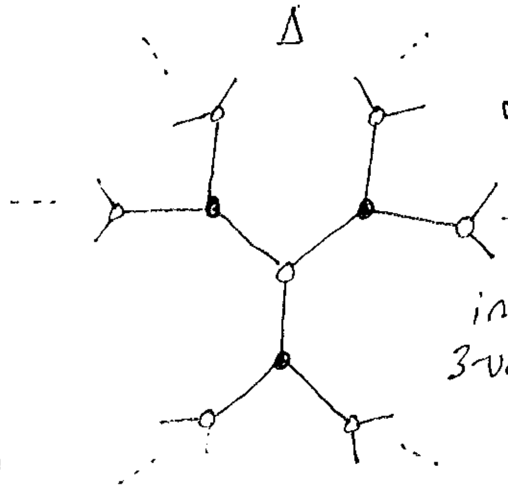
(call $|S|$ the rank of Δ).

• Eg1: Coxeter complex Δ_W with $\delta_W(w_1,w_2) = w_1^{-1}w_2$

(Every panel $\supseteq$ precisely two chambers; call such buildings thin).

• Eg2: building type $\underset{s_1}{\circ}\overset{\infty}{\text{———}}\underset{s_2}{\circ}$

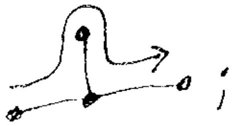
Euclidean or affine building (= "the" affine building of $SL_2\mathbb{Q}_2$)



$\bullet$ = 1-incident
$\circ$ = 2-incident

infinite 3-valent tree

to define $\delta(c,c')$: recall that in a tree there is a unique path ~~[illegible]~~ between edges without backtracking

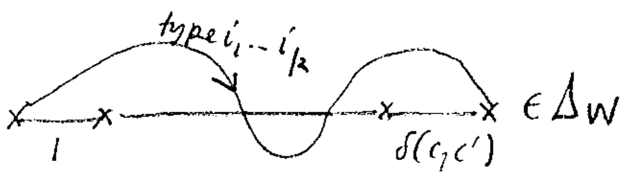


; $\times\!\overset{c}{—}\!\underset{}{\overset{i_1}{\times}}\!\text{————}\!\overset{i_k}{\times}\!\overset{c'}{—}\!\times \in \Delta$ (unique path)

define $\delta(c,c') := \delta_W(1,w) = w$.

$\times\!\overset{1}{—}\!\underset{i_1}{\times}\!\text{————}\!\underset{i_k}{\times}\!\overset{w}{—}\!\times \in \Delta_W$

(recall: $\Delta_W = \cdots \circ \overset{s_2s_1}{\text{---}} \bullet \overset{s_2}{\text{---}} \circ \overset{1}{\text{---}} \bullet \overset{s_1}{\text{---}} \circ \overset{s_1s_2}{\text{---}} \bullet \cdots$)

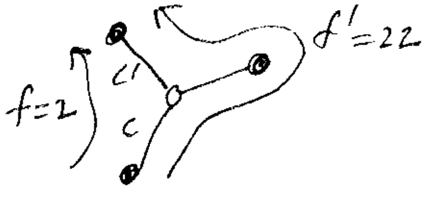


if $\delta(c,c') = s_{i_1} \dots s_{i_k}$ then there is a gallery type $i_1 \dots i_k$ in Δ_W from $1 \to \delta(c,c')$; it differs from unique minimal gallery only by backtracks $\Rightarrow$ the gallery type $i_1 \dots i_k$ starting at $c \in \Delta$ ends at c'.

$\Rightarrow$ there is a gallery $c \to_f c'$ in Δ ($f = i_1 \dots i_k$).

If there is a gallery $c \to_f c'$ ($f = i_1 \dots i_k$) and $s_{i_1} \dots s_{i_k}$ reduced $\Rightarrow$ $i_j \neq i_{j+1}$ $\Rightarrow$ gallery has no backtracks $\Rightarrow$ is the unique such gallery from $c \to c'$ $\Rightarrow$ $\delta(c,c') = w = s_{i_1} \dots s_{i_k}$.

[note: a gallery $c \to_f c'$ with $s_{i_1} \dots s_{i_k}$ not reduced $\not\Rightarrow$ $\delta(c,c') = s_{i_1} \dots s_{i_k}$



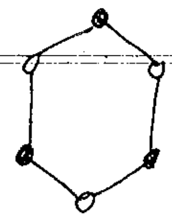
but $\delta(c,c') = s_2$ ($\neq s_2 s_2 = 1$)]

• Eg3: building type $\underset{s_1}{\circ} \text{---} \underset{s_2}{\circ}$ (a spherical building)

chambers = flags $V_1 \subset V_2$ ($\dim V_i = i$) and

$(V_1 \subset V_2) \sim_i (V_1' \subset V_2')$ $\iff$ $V_j = V_j'$ for $j \neq i$

i.e.: $V_1 \subset V_2 \supset V_1'$ 1-adjacent

$V_2 \supset V_1 \subset V_2'$ 2-adjacent define $\delta(C,C')$ by situating in  = Coxeter complex of (W,S)

and taking $\delta_W(C,C')$.

How do we see that (i). δ well-defined and (ii) does what it should?

[• In Eg's 2-3 every panel $\supseteq$ at least three chambers (in Eg. 3, if $|k|=q$ then a panel $\supseteq$ $q+1$ chambers; if $|k|=\infty$ then $\supseteq$ ∞-ly many chambers). Such buildings are said to be thick.]

- A (set) map $\alpha : X \to Y$ where $X \subset (\Delta,\delta)$, $Y \subset (\Delta',\delta')$ buildings, is an isometry $\iff \delta'(\alpha(C),\alpha(C')) = \delta(C,C')$ for all $C,C' \in X$.

- Eg: $w_0 \in (W,S)$, then $w \mapsto w_0 w$ an isometry $(\Delta_W,\delta_W) \to (\Delta_W,\delta_W)$.

- Theorem: any isometry $X \to \Delta$ for $X \subset \Delta_W$ extends to an isometry $\Delta_W \to \Delta$.

- Δ building type (W,S). An apartment in Δ is an isometric image $\alpha(\Delta_W)$ of Δ_W in Δ.

- Corollary (of Th$^{\text{m}}$) (1). any two chambers C, C' are $\subseteq$ some apartment.

[proof: $\delta(C,C') = w \in W \Rightarrow \left.\begin{matrix}1\\ w\end{matrix}\right\} \overset{\alpha}{\mapsto} \left\{\begin{matrix}C\\ C'\end{matrix}\right.$ isom. $\Rightarrow \Delta_W \overset{\alpha}{\mapsto} \Delta$]

(2). If $C, C' \subseteq$ apartments A_1, A_2 then there is an isometry

$A_1 \to A_2$ fixing chambers in $A_1 \cap A_2$

[proof: $A_i = \alpha_i(\Delta_W) \Rightarrow \alpha_2\alpha_1^{-1}: A_1 \to A_2$ isometry; let $c_0 \in A_1 \cap A_2$; pre-compose with isometries $\Delta W \to \Delta W$ if necessary so that $c_0 = \alpha_i(1)$, $i = 1,2$; $\Rightarrow \alpha := \alpha_2\alpha_1^{-1}(c_0) = c_0$; let $c \in A_1 \cap A_2$ so that $\delta_\Delta(c_0, c) = \delta_\Delta(\alpha(c_0), \alpha(c)) = \delta_\Delta(c_0, \alpha(c))$; in an apartment there is a unique chamber a distance $w \in W$ from a given chamber (Exercise) $\Rightarrow c = \alpha(c)$]

- Ex: Δ, Δ' chamber systems (over same I); a morphism $\alpha: \Delta \to \Delta'$ is a map preserving i-incidence for all i, i.e.: $c \sim_i c' \Rightarrow \alpha(c) \sim_i \alpha(c')$.

An isomorphism = bijective morphism whose inverse a morphism.

Show: (i). $\alpha: (\Delta, \delta) \to (\Delta', \delta')$ isometry $\Leftrightarrow \alpha: \Delta \to \Delta'$ injective morphism.

(ii). α surjective isometry $\Leftrightarrow$ α isomorphism.

Theorem: Δ chamber system containing subsystems (called apartments) over same I, each isomorphic to the Coxeter cx. type (W,S) and s.t.

(i). any two chambers $\in$ common apartment (ii). if chambers $c, c' \in$ apartments A_1, A_2 then there is an isomorphism $A_1 \to A_2$ fixing A_1, A_2. Then Δ a building with $\delta(c,c') = \delta_W(\alpha(x), \alpha(y))$ for $\alpha: \Delta_W \to A \ni c, c'$ isomorphism.