J. Group Theory [redacted] (2008), 1–15
DOI 10.1515/JGT.2008.[redacted]

Journal of Group Theory

Graphs, free groups and the Hanna Neumann conjecture

Brent Everitt

(Communicated by M. R. Bridson)

Abstract. A new bound for the rank of the intersection of finitely generated subgroups of a free group is given, formulated in topological terms, and in the spirit of Stallings [19]. The bound is a contribution to the strengthened Hanna Neumann conjecture.

Introduction

This paper is about the interplay between graphs and free groups, with particular application to subgroups of free groups. This subject has a long history, where one approach is to treat graphs as purely combinatorial objects (as in for instance [11], [20], [21]), while another (for example [16]), is to treat them topologically by working in the category of 1-dimensional CW complexes.

We prefer a middle way, where to quote Stallings [19] who initiated it, graphs are "something purely combinatorial or algebraic", but also one may apply to them topological machinery, motivated by their geometrical realizations. We use this to give a new bound for the rank of the intersection of two finitely generated subgroups of a free group (Theorems 1 and 2), and to formulate graph-theoretic versions of some other classical results. The first section sets up the combinatorial–topological background; Section 2 studies graphs of finite rank; the topological meat of the paper is Section 3 and the group-theoretic consequences explored in Section 4.

1 Preliminaries from the topology of graphs

A *combinatorial* 1-*complex* or *graph* (see [6, §1.1] or also [2], [3], [17], [19]) is a set Γ with an involutory map $^{-1}: \Gamma \to \Gamma$ and an idempotent map $s: \Gamma \to V_\Gamma$, (i.e. $s^2 = s$), where V_Γ is the set of fixed points of $^{-1}$. Thus a graph has *vertices* V_Γ and *edges* $E_\Gamma := \Gamma \setminus V_\Gamma$ with

(i) $s(v) = v$ for all $v \in V_\Gamma$;

(ii) $v^{-1} = v$ for all $v \in V_\Gamma$, $e^{-1} \in E_\Gamma$ and $e^{-1} \neq e = (e^{-1})^{-1}$ for all $e \in E_\Gamma$.

The edge e has start vertex $s(e)$ and terminal vertex $t(e) := s(e^{-1})$; an arc is an edge/inverse edge pair; a pointed graph is a pair $\Gamma_v := (\Gamma, v)$ for $v \in \Gamma$ a vertex.

A map of graphs is a set map $f:\Gamma\to\Lambda$ with $f(V_\Gamma)\subseteq V_\Lambda$ that commutes with s and $^{-1}$, and preserves dimension if $f(E_\Gamma)\subseteq E_\Lambda$. An isomorphism is a dimension-preserving map, bijective on the vertices and edges. A map $f:\Gamma_v\to\Lambda_u$ of pointed graphs is a graph map $f:\Gamma\to\Lambda$ with $f(v)=u$.

A graph Γ has a functorial geometric realization as a 1-dimensional CW complex $B\Gamma$ (see, e.g. [6, §1.3]) with a graph map $f:\Gamma\to\Lambda$ inducing a regular cellular map $Bf:B\Gamma\to B\Lambda$ of CW complexes, in the sense of [13, §4]. Thus, one may transfer to graphs and their maps topological notions and adjectives (connected, fundamental group, homology, covering map, etc.) from their geometrical realizations.

If $\Lambda\hookrightarrow\Gamma$ is a subgraph, we will write Γ/Λ for the resulting quotient graph and quotient map $q:\Gamma\to\Gamma/\Lambda$. For a set $\Lambda_i\hookrightarrow\Gamma$ $(i\in I)$ of mutually disjoint subgraphs, we will write Γ/Λ_i for the graph resulting from taking successive quotients by the subgraphs Λ_i. The coboundary $\delta\Lambda$ of a subgraph consists of those edges $e\in\Gamma$ with $s(e)\in\Lambda$ and $t(e)\notin\Lambda$; equivalently, those edges $e\in\Gamma$ with $sq(e)$ the vertex $q(\Lambda)$ in the quotient graph $q:\Gamma\to\Gamma/\Lambda$. The real line graph $\mathbb{R}$ has vertices $V_{\mathbb{R}}=\{v_k\}_{k\in\mathbb{Z}}$ and edges $E_{\mathbb{R}}=\{e_k^{\pm1}\}_{k\in\mathbb{Z}}$ with $s(e_k)=v_k$, $s(e_k^{-1})=v_{k+1}$.

We have the obvious notion of path and in particular, a spur is a path that successively traverses both edges of an arc, and a path is reduced when it contains no spurs. A tree is a connected, simply connected graph and a forest a graph, all of whose connected components are trees. Any connected graph has a spanning tree $T\hookrightarrow\Gamma$ with the homology $H_1(\Gamma)$ free abelian on the set of *arcs* of Γ omitted by T, and the rank $\operatorname{rk}\Gamma$ of Γ (connected) defined to be $\operatorname{rk}_{\mathbb{Z}} H_1(\Gamma)$. If Γ has finite rank then $\operatorname{rk}\Gamma-1=-\chi(\Gamma)$, and if Γ is finite, locally finite, connected, then $2(\operatorname{rk}\Gamma-1)=|E_\Gamma|-2|V_\Gamma|$. If Γ is connected and $T_i\hookrightarrow\Gamma$ a set of mutually disjoint trees, then the fundamental group is unaffected by their excision: $\pi_1(\Gamma,v)\cong\pi_1(\Gamma/T_i,q(v))$ and so $\operatorname{rk}\Gamma=\operatorname{rk}\Gamma/T_i$.

If Λ is a connected graph and v a vertex, then the spine $\hat{\Lambda}_v$ of Λ at v is defined to be the union in Λ of all closed reduced paths starting at v. It is easy to show that $\hat{\Lambda}_v$ is connected with $\operatorname{rk}\hat{\Lambda}_v=\operatorname{rk}\Lambda$, that every closed reduced path starting at $u\in\hat{\Lambda}_v$ is contained in $\hat{\Lambda}_v$, and an isomorphism $\Lambda_u\to\Delta_v$ restricts to a isomorphism $\hat{\Lambda}_u\to\hat{\Delta}_v$ (so that spines are invariants of graphs).

If Λ_1, Λ_2 and Δ are graphs and $f_i:\Lambda_i\to\Delta$ maps of graphs, then the pullback $\Lambda_1\prod_\Delta\Lambda_2$ has vertices (resp. edges) the $x_1\times x_2$, $x_i\in V_{\Lambda_i}$ (resp. $x_i\in E_{\Lambda_i}$) such that $f_1(x_1)=f_2(x_2)$, and $s(x_1\times x_2)=s(x_1)\times s(x_2)$, $(x_1\times x_2)^{-1}=x_1^{-1}\times x_2^{-1}$ (see [19, p. 552]). Taking Δ to be the trivial graph gives the product $\Lambda_1\prod\Lambda_2$. Define maps $t_i:\Lambda_1\prod_\Delta\Lambda_2\to\Lambda_i$ to be the compositions $\Lambda_1\prod_\Delta\Lambda_2\hookrightarrow\Lambda_1\prod\Lambda_2\to\Lambda_i$, with the second map the projection $x_1\times x_2\mapsto x_i$. Then t_1, t_2 are dimension-preserving maps making the diagram below commute, and the pullback is universal with this property.

$$\begin{array}{ccc}
\Lambda_1\prod_\Delta\Lambda_2 & \xrightarrow[t_2]{} & \Lambda_2 \\
\Big\downarrow{\scriptstyle t_1} & & {\scriptstyle f_2}\Big\downarrow \\
\Lambda_1 & \xrightarrow{f_1} & \Delta
\end{array}$$

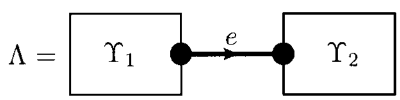
(AutoPDF V7 18/3/08 14:08) WDG (170×240mm) Tmath J-1900 JGT, PMU:I(CKN)18/3/2008 pp. 1–15 1900_17 (p. 2)

In general the pullback need not be connected, but if each $f_i : \Lambda_{u_i} \to \Delta_v$ is a pointed map then the *pointed* pullback $(\Lambda_1 \prod_\Delta \Lambda_2)_{u_1 \times u_2}$ is the connected component of the pullback containing the vertex $u_1 \times u_2$ (and we then have a pointed version of the diagram above).

There is a 'co'-construction, the pushout, for dimension-preserving maps $f_i : \Delta \to \Lambda_i$ of graphs, although it will play a lesser role for us (see [19, p. 552]). The principal example for us is the wedge sum $\Lambda_1 \vee_\Delta \Lambda_2$.

Graph coverings $f : \Lambda \to \Delta$ can be characterized combinatorially as dimension-preserving maps such that for every vertex $v \in \Lambda$, f is a bijection from the set of edges in Λ with start vertex v to the set of edges in Δ with start vertex $f(v)$. Graph coverings have the usual path and homotopy lifting properties (see [19, §4]), and from now on, all coverings will be maps between connected complexes unless stated otherwise, and we will write $\deg(\Lambda \to \Delta)$ for the degree of the covering. A covering is Galois if for all closed paths γ at v, the lifts of γ to each vertex of the fiber of v are either all closed or all non-closed.

Proposition 1. *Let Λ be a graph and $\Upsilon_1, \Upsilon_2 \hookrightarrow \Lambda$ subgraphs of the following form.*



(i) *If $f : \Lambda \to \Delta$ is a covering with Δ single-vertexed, then the real line is a subgraph $g : \mathbb{R} \hookrightarrow \Lambda$, with $g(e_0) = e$ and $fg(e_k) = f(e)$ for all $k \in \mathbb{Z}$.*

(ii) *If Υ_1 is a tree, $\Lambda \to \Delta$ and $\Gamma \to \Delta$ coverings, and $\Upsilon_2 \hookrightarrow \Gamma$ a subgraph, then there is an intermediate covering $\Lambda \to \Gamma \to \Delta$.*

(iii) *If Υ_1 is a tree, and $\Psi \to \Lambda$ a covering, then Ψ has the same form as Λ for some subgraphs $\Upsilon_1', \Upsilon_2' \hookrightarrow \Psi$ and with Υ_1' a tree.*

Proof. These are easy exercises using path lifting. For (i), build $\mathbb{R} \hookrightarrow \Lambda$ by taking successive lifts of the edge $f(e) \in \Delta$. For (ii), it suffices to find a map $\Lambda \to \Gamma$ commuting with the two coverings given. Let it coincide with $\Upsilon_2 \hookrightarrow \Gamma$ on Υ_2, and on Υ_1, project to Δ and then lift to Γ. For (iii), take Υ_1' to be the union of lifts of reduced paths from $t(e)$ to the vertices of Υ_1. □

If $f : \Lambda \to \Delta$ is a covering and $T \hookrightarrow \Delta$ a tree, then path and homotopy lifting give that $f^{-1}(T)$ is a forest such that if $T_i \hookrightarrow \Lambda$ $(i \in I)$ are the component trees, then f maps each T_i isomorphically onto T. There is then an induced covering $f' : \Lambda/T_i \to \Delta/T$, defined by $f'q' = qf$ where q, q' are the quotient maps, and such that $\deg(\Lambda/T_i \to \Delta/T) = \deg(\Lambda \to \Delta)$.

If $f : \Lambda_u \to \Delta_v$ is a covering then intermediate coverings $\Lambda_u \to \Gamma_x \to \Delta_v$ and $\Lambda_u \to \Upsilon_y \to \Delta_v$ are *equivalent* if and only if there is a isomorphism $\Gamma_x \to \Upsilon_y$ making the obvious diagram commute. The set $\mathscr{L}(\Lambda_u, \Delta_v)$ of equivalence classes of intermediate coverings is then a lattice with join $\Gamma_{x_1} \vee \Upsilon_{x_2}$ the pullback $(\Gamma \prod_\Delta \Upsilon)_{x_1 \times x_2}$, meet

$\Gamma_{x_1} \wedge \Upsilon_{x_2}$ the pushout $(\Gamma \coprod_\Lambda \Upsilon)_{g(x_i)}$ (where g is the covering $\Lambda_u \to \Gamma_{x_1}$), and with $\hat{0} = \Delta_v$ and $\hat{1} = \Lambda_u$. The repeated pointing of covers is annoying, but essential if one wishes to work with *connected* intermediate coverings and also have a lattice structure (both of which we do). The problem is the pullback: because it is not in general connected, we need the pointing to tell us which component to choose.

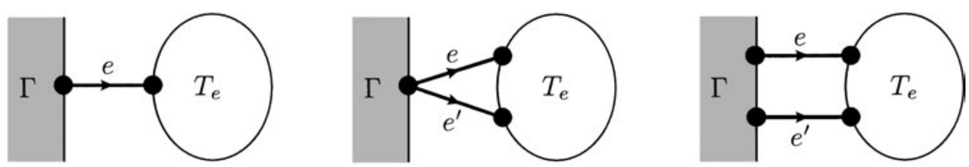
We end the preliminaries by observing that the excision of trees has little effect on the lattice $\mathscr{L}(\Lambda, \Delta)$. Let $f : \Lambda_u \to \Delta_v$ be a covering, $T \hookrightarrow \Delta$ a spanning tree, $T_i \hookrightarrow \Lambda$ the components of $f^{-1}(T)$, and $f : (\Lambda/T_i)_{q(u)} \to (\Delta/T)_{q(v)}$ the induced covering (where we have used q for both quotients and f for both coverings). One can then show (either by brute force, or using the Galois correspondence between $\mathscr{L}(\Lambda, \Delta)$ and the subgroup lattice of the group $\mathrm{Gal}(\Lambda, \Delta)$ of covering transformations), that there is a degree- and rank-preserving isomorphism of lattices $\mathscr{L}(\Lambda, \Delta) \to \mathscr{L}(\Lambda/T_i, \Delta/T)$, that sends Galois coverings to Galois coverings, and the equivalence class of $\Lambda_u \to \Gamma_x \xrightarrow{r} \Delta_v$ to the equivalence class of $\Lambda/T_i \to \Gamma/T_i' \to \Delta/T$, with $T_i' \hookrightarrow \Gamma$ the components of $r^{-1}(T)$. We will call this process *lattice excision*.

2 Graphs of finite rank

This section is devoted to a more detailed study of the coverings $\Lambda \to \Delta$ where $\mathrm{rk}\, \Lambda < \infty$.

Proposition 2. *Let Λ be a connected graph, $\Gamma \hookrightarrow \Lambda$ a connected subgraph and $v \in \Gamma$ a vertex such that every closed reduced path at v in Λ is contained in Γ. Then Λ has a wedge sum decomposition $\Lambda = \Gamma \bigvee_\Theta \Phi$ with Φ a forest and no two vertices of the image of $\Theta \hookrightarrow \Phi$ lying in the same component.*

Proof. Consider an edge e of $\Lambda \backslash \Gamma$ having at least one of its end vertices $s(e)$ or $t(e)$, in Γ. For definiteness we can assume, by relabeling the edges in the arc containing e, that it is $s(e)$ that is a vertex of Γ. If $t(e) \in \Gamma$ then by traversing a reduced path in Γ from v to $s(e)$, crossing e and a reduced path in Γ from $t(e)$ to v, we get a closed reduced path not contained in Γ, a contradiction. Thus $t(e) \notin \Gamma$. Let T_e be the union of all the reduced paths in $\Lambda \backslash \{e\}$ starting at $t(e)$, so we have the situation as in (a):



(a) (b) (c)

If γ is a non-trivial closed path in T_e starting at $t(e)$, then a path from v to $t(e)$, traversing γ, and going the same way back to v cannot be reduced. But the only place a spur can occur is in γ and so T_e is a tree. If e' is another edge of $\Lambda\backslash\Gamma$ with $s(e') \in \Gamma$ then we claim that neither of the two situations (b) and (c) above can occur, i.e. $t(e')$ is not a vertex of T_e. For otherwise, a reduced closed path in T_e from $t(e)$ to $t(e')$ will give a reduced closed path at v not in Γ. Thus, another edge e' yields a tree $T_{e'}$ defined like T_e, but disjoint from it. Each component of Φ is thus obtained this way. □

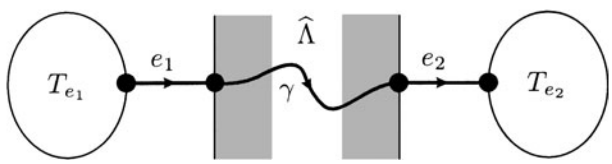
Corollary 1. *If Λ is connected then Λ is of finite rank if and only if for any vertex v the spine $\hat{\Lambda}_v$ is finite, locally finite.*

Proof. Proposition 2 gives the wedge sum decomposition $\Lambda = \hat{\Lambda}_v \bigvee_\Theta \Phi$, and by connectedness, any spanning tree $T \hookrightarrow \Lambda$ must contain the forest Φ as a subgraph. Thus if Λ has finite rank, then $\hat{\Lambda}_v$ is a tree with finitely many edges added, and hence finite. Conversely, a finite spine has finite rank and $\mathrm{rk}\,\Lambda = \mathrm{rk}\,\hat{\Lambda}_v$. □

Thus if $\mathrm{rk}\,\Lambda < \infty$ then the decomposition of Proposition 2 becomes,

$$\Lambda = \left(\cdots\left(\left(\hat{\Lambda}_v \bigvee_{\Theta_1} T_1\right) \bigvee_{\Theta_2} T_2\right)\cdots\right) \bigvee_{\Theta_k} T_k\right), \tag{1}$$

with $\hat{\Lambda}_v$ finite, Θ_i a single vertex such that $\Theta_i \hookrightarrow \hat{\Lambda}_v$ for each i, and the images $\Theta_i \hookrightarrow T_i$ incident with a single arc. Moreover, if $\Lambda \to \Delta$ is a covering with Δ single-vertexed and Λ of finite rank, then by Proposition 1(i), each tree T_i realizes an embedding $\mathbb{R} \hookrightarrow \Lambda$ of the real line in Λ, and as the spine is finite, the trees are thus paired



(‡)

with all e_i (and indeed all edges in the path $\mathbb{R} \hookrightarrow \Lambda$) in the same fiber of the covering. This pairing will play a key role in Section 3.

Corollary 2. *Let $\Lambda \to \Delta$ be a covering with Δ single-vertexed having non-empty edge set and* $\mathrm{rk}\,\Lambda < \infty$. *Then* $\deg(\Lambda \to \Delta) < \infty$ *if and only if* $\Lambda = \hat{\Lambda}_v$.

Proof. If Λ is more than $\hat{\Lambda}_v$ then one of the trees T_i in the decomposition (1) is non trivial and by Proposition 1(i) we get a real line subgraph $\mathbb{R} \hookrightarrow \Lambda$, with image in the fiber of an edge, contradicting the finiteness of the degree. The converse follows from Corollary 1. □

Proposition 3. *Let $\Lambda \to \Delta$ be a covering with*

(i) $\mathrm{rk}\,\Delta > 1$,

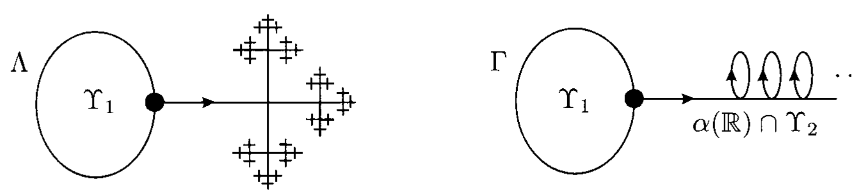
(ii) $\operatorname{rk}\Lambda < \infty$ *and*

(iii) $\operatorname{rk}\Gamma < \infty$ *for any intermediate covering* $\Lambda \to \Gamma \to \Delta$.

Then $\deg(\Lambda \to \Delta) < \infty$.

The covering $\mathbb{R} \to \Delta$ of a single-vertexed graph Δ of rank 1 by the real line shows why the condition that $\operatorname{rk}\Delta > 1$ cannot be dropped.

Proof. By lattice excision we may pass to the case where Δ is single-vertexed while preserving (i)–(iii). Establishing the degree here and passing back to the general Δ will give the result. If the degree of the covering $\Lambda \to \Delta$ is infinite for Δ single-vertexed, then by Corollary 2, in the decomposition (1) for Λ, one of the trees is non-empty and Λ has the form of the graph in Proposition 1 with this non-empty tree the union of the edge e and Υ_2. Let Γ be a graph defined as follows: take the union of Υ_1, the edge e and $\alpha(\mathbb{R}) \cap \Upsilon_2$, where $\alpha(\mathbb{R})$ is the embedding of the real line given by Proposition 1(i). At each vertex of $\alpha(\mathbb{R}) \cap \Upsilon_2$ place $\operatorname{rk}\Delta - 1$ edge loops:



(the picture depicting the $\operatorname{rk}\Delta = 2$ case). Then there is an obvious covering $\Gamma \to \Delta$ so that by Proposition 1(ii) we have an intermediate covering $\Lambda \to \Gamma \to \Delta$. Equally obviously, Γ has infinite rank, contradicting (iii). Thus, $\deg(\Lambda \to \Delta) < \infty$. □

Proposition 4. *Let* $\Psi \to \Lambda \to \Delta$ *be coverings with* $\operatorname{rk}\Lambda < \infty$, $\Psi \to \Delta$ *Galois, and* Ψ *not simply connected. Then* $\deg(\Lambda \to \Delta) < \infty$.

The idea is that if the degree is infinite, then Λ has a hanging tree in its spine decomposition, and so Ψ does too. But Ψ should look the same at every point, and hence is a tree.

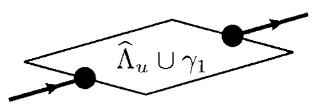
Proof. We apply lattice excision to $\mathscr{L}(\Psi, \Delta)$: as $\pi_1(\Psi, u)$ is unaffected by the excision of trees, we may assume that Δ is single-vertexed. If $\deg(\Lambda \to \Delta)$ is infinite, the spine decomposition for Λ has an infinite tree, and Λ has the form of Proposition 1. Thus Ψ does too, by part (iii) of this proposition, with subgraphs $\Upsilon_i' \hookrightarrow \Psi$, edge e' and Υ_1' a tree. Take a closed reduced path γ in Υ_2', and choose a vertex u_1 of Υ_1' such that the reduced path from u_1 to $s(e')$ has at least as many edges as γ. Project γ via the covering $\Psi \to \Delta$ to a closed reduced path, and then lift to u_1. The result is reduced, closed as $\Psi \to \Delta$ is Galois, and entirely contained in the tree Υ_1', hence trivial. Thus γ is also trivial, so that Υ_2' is a tree and Ψ is simply connected. □

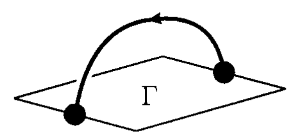
(AutoPDF V7 18/3/08 14:08) WDG (170×240mm) Tmath J-1900 JGT, PMU:I(CKN)18/3/2008 pp. 1–15 1900_17 (p. 6)

Proposition 5. *Let* $\Lambda_u \to \Delta_v$ *be a covering with* $\operatorname{rk}\Lambda < \infty$ *and* γ *a non-trivial reduced closed path at* v *lifting to a non-closed path at* u. *Then there is an intermediate covering* $\Lambda_u \to \Gamma_w \to \Delta_v$ *with* $\deg(\Gamma \to \Delta)$ *finite and* γ *lifting to a non-closed path at* w.

Stallings shows something very similar [19, Theorem 6.1] starting from a finite immersion rather than a covering. As the proof shows, the path γ in Proposition 5 can be replaced by finitely many such paths. Moreover, for $T \hookrightarrow \Lambda$ a spanning tree, recall that Schreier generators for $\pi_1(\Lambda, u)$ are the homotopy classes of paths through T from u to $s(e)$, traversing e and traveling back through T to u_1, for $e \in \Lambda \backslash T$. Then the intermediate Γ constructed has the property that any set of Schreier generators for $\pi_1(\Lambda, u)$ can be extended to a set of Schreier generators for $\pi_1(\Gamma, w)$.

Proof. If $T \hookrightarrow \Delta$ is a spanning tree and $q : \Delta \to \Delta/T$ then γ cannot be contained in T, and so $q(\gamma)$ is non-trivial, closed and reduced. If the lift of $q(\gamma)$ to Λ/T_i is closed then the lift of γ to Λ has start and finish vertices that lie in the same component T_i of $f^{-1}(T)$, mapped isomorphically onto T by the covering, and thus implying that γ is not closed. Thus we may apply lattice excision and pass to the single-vertexed case while maintaining γ and its properties. Moreover, the conclusion in this case gives the result in general as closed paths go to closed paths when excising trees. If the lift γ_1 of γ at u is not contained in the spine $\hat{\Lambda}_u$, then its terminal vertex lies in a tree T_{e_i} of the spine decomposition (‡). By adding an edge if necessary to $\hat{\Lambda}_u \cup \gamma_1$, we obtain a finite subgraph whose coboundary edges are paired, with the edges in each pair covering the same edge in Δ, as below left:





(if the lift is contained in the spine, take $\hat{\Lambda}_u$ itself). In any case, let Γ be $\hat{\Lambda}_u \cup \gamma_1$ together with a single edge replacing each pair as above right. Restricting the covering $\Lambda \to \Delta$ to $\hat{\Lambda}_u \cup \gamma_1$ and mapping the new edges to the common image of the old edge pairs gives a finite covering $\Gamma \to \Delta$, and hence an intermediate covering $\Lambda \xrightarrow{q} \Gamma \to \Delta$, with $q(\gamma_1)$ non-closed at $q(u)$. □

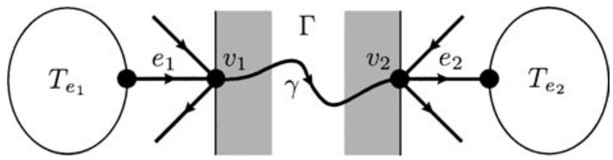
For the rest of this section we investigate the rank implications of the decomposition (1) and the pairing (‡) in a special case. Suppose that $\Lambda \to \Delta$ is a covering with Δ single-vertexed, $\operatorname{rk}\Delta = 2$, Λ non-(simply connected), and $\operatorname{rk}\Lambda < \infty$. Let $x_1^{\pm 1}$, $x_2^{\pm 1}$ be the edge loops of Δ and fix a spine, so we have the decomposition (1).

An *extended spine* for such a Λ is a connected subgraph $\Gamma \hookrightarrow \Lambda$ obtained by adding finitely many edges to a spine, so that every vertex of Γ is incident with either zero or three edges in its coboundary $\delta\Gamma$. It is always possible to find an extended spine: take the union of the spine $\hat{\Lambda}_u$ and each edge $e \in \delta\hat{\Lambda}_u$ in its coboundary. Observe that Γ is finite and the decomposition (1) gives $\operatorname{rk}\Gamma = \operatorname{rk}\hat{\Lambda}_u = \operatorname{rk}\Lambda$. Call a vertex of the

(AutoPDF V7 18/3/08 14:08) WDG (170×240mm) Tmath J-1900 JGT, PMU:I(CKN)18/3/2008 pp. 1–15 1900_17 (p. 7)

extended spine Γ *interior* (respectively *boundary*) when it is incident with zero (resp. three) edges in $\delta\Gamma$.

We have the pairing of trees (‡) for an extended spine, so that each boundary vertex v_1 is paired with another v_2,



(∗)

with e_1, e_2 and all the edges in the path $\gamma = \alpha(\mathbb{R}) \cap \Gamma$ covering an edge loop $x_i \in \Delta$. Call this an *x_i-pair*.

For two x_i-pairs (with the same i), the respective γ paths share no vertices in common, for otherwise there would be two distinct edges covering the same $x_i \in \Delta$ starting at such a common vertex. Moreover, γ must contain vertices of Γ apart from the two boundary vertices v_1, v_2, since otherwise Λ would be simply connected. These other vertices are incident with at least two edges of $\gamma \in \Gamma$, hence at most two edges of the coboundary $\delta\Gamma$, and thus must be interior.

Lemma 1. *If, for $i = 1, 2$, n_i is the number of x_i-pairs in an extended spine Γ for Λ, then the number of interior vertices is at least $\sum n_i$.*

Proof. The number of interior vertices is $|V_\Gamma| - 2\sum n_i$ and the number of edges of Γ is $4(|V_\Gamma| - 2\sum n_i) + 2\sum n_i$, and hence $\operatorname{rk}\Gamma - 1 = |V_\Gamma| - 3\sum n_i$. As Λ is not simply connected, $\operatorname{rk}\Lambda - 1 = \operatorname{rk}\Gamma - 1 \geqslant 0$, and thus $|V_\Gamma| - 2\sum n_i \geqslant \sum n_i$ as required. □

The lemma breaks down in the case $\operatorname{rk}\Delta > 2$. It will be helpful in Section 3 to have a pictorial description of the quantity rk − 1 for our graphs. To this end, a *checker* is a small plastic disk, as used in the eponymous boardgame (called *draughts* in British English). We place black checkers on some of the vertices of an extended spine Γ according to the following scheme: place black checkers on all the interior vertices of Γ; for each x_1-pair in (∗), take the interior vertex on the path γ that is closest to v_1 (i.e. is the terminal vertex of the edge of γ whose start vertex is v_1) and *remove* its checker; for each x_2-pair, we can find, by Lemma 1, an interior vertex with a checker still on it. Choose such a vertex and remove its checker also. We saw in the proof of Lemma 1 that $\operatorname{rk}\Lambda - 1 = \operatorname{rk}\Gamma - 1$ is equal to the number of interior vertices of Γ, less the number of x_i-pairs ($i = 1, 2$). Thus we have the following result.

Lemma 2. *With black checkers placed on the vertices of an extended spine for Λ as above, the number of black checkers is* $\operatorname{rk}\Lambda - 1$.

From now on we will only use the extended spine obtained by adding the coboundary edges to some fixed spine $\hat{\Lambda}_u$.

Let $p : \Lambda_u \to \Delta_v$ be a covering with $\operatorname{rk}\Delta = 2$, $\operatorname{rk}\Lambda < \infty$ and Λ not simply connected. A spanning tree $T \hookrightarrow \Delta$ induces a covering $\Lambda/T_i \to \Delta/T$ with Δ/T single-vertexed. Let $\mathscr{H}(\Lambda_u \to \Delta_v)$ be the number of vertices of the spine of Λ/T_i at $q(u)$ and $n_i(\Lambda_u \to \Delta_v)$ the number of x_i-pairs in the extended spine. The isomorphism class of Λ/T_i and the spine are independent of the spanning tree T, hence the quantities $\mathscr{H}(\Lambda_u \to \Delta_v)$ and $n_i(\Lambda_u \to \Delta_v)$ are too.

3 Pullbacks

Let $p_i : \Lambda_i := \Lambda_{u_i} \to \Delta_v$ $(i = 1, 2)$ be coverings and $\Lambda_1 \prod_\Delta \Lambda_2$ their (unpointed) pullback. If $\hat{\Lambda}_{u_i}$ is the spine at u_i then we can restrict the coverings to maps $p_i : \hat{\Lambda}_{u_i} \to \Delta_v$ and form the pullback $\hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2}$.

Proposition 6 (spine decomposition of pullbacks). *The pullback $\Lambda = \Lambda_1 \prod_\Delta \Lambda_2$ has a wedge sum decomposition $\Lambda = (\hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2}) \bigvee_\Theta \Phi$, with Φ a forest and no two vertices of the image of $\Theta \hookrightarrow \Phi$ lying in the same component.*

Proof. For $i = 1, 2$ let $\Lambda_i = \hat{\Lambda}_{u_i} \bigvee_{\Theta_i} \Phi_i$ be the spine decomposition and $t_i : \Lambda_1 \prod_\Delta \Lambda_2 \to \Lambda_i$ the covering provided by the pullback, and let Ω be a connected component of the pullback. If $\Omega \cap (\hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2}) = \varnothing$, then a reduced closed path $\gamma \in \Omega$ must map via one of the t_i to a closed path in the forest Φ_i. As the images under coverings of reduced paths are reduced, $t_i(\gamma)$ must contain a spur which can be lifted to a spur in γ. Thus Ω is a tree.

Otherwise choose a vertex $w_1 \times w_2$ in $\Omega \cap (\hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2})$ and let Γ be the connected component of this intersection containing $w_1 \times w_2$. If γ a reduced closed path at $w_1 \times w_2$ then $t_i(\gamma)$ is a reduced closed path at $w_i \in \hat{\Lambda}_{u_i}$ for $i = 1, 2$, hence $t_i(\gamma) \in \hat{\Lambda}_{u_i}$ and thus $\gamma \in \hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2}$. Applying Proposition 2, we have Ω a wedge sum of Γ and a forest of the required form. $\square$

Corollary 3 (Howsen–Stallings). *Let $p_i : \Lambda_i \to \Delta$ $(i = 1, 2)$ be coverings with $\operatorname{rk}\Lambda_i < \infty$ and $u_1 \times u_2$ a vertex of their pullback. Then* $\operatorname{rk}(\Lambda_1 \prod_\Delta \Lambda_2)_{u_1 \times u_2} < \infty$.

Proof. The component Ω of the pullback containing $u_1 \times u_2$ is either a tree or the wedge sum of a finite graph and a forest as described in Proposition 6. Both cases gives the result. $\square$

The remainder of this section is devoted to a proof of an estimate for the rank of the pullback of finite rank graphs in a special case. Let $p_j : \Lambda_j := \Lambda_{u_j} \to \Delta_v$ $(j = 1, 2)$ be coverings with $\operatorname{rk}\Delta = 2$, $\operatorname{rk}\Lambda_j < \infty$ and the graphs Λ_j not simply connected. Let $\mathscr{H}_j := \mathscr{H}(\Lambda_{u_j} \to \Delta_v)$ and $n_{ji} := n_i(\Lambda_{u_j} \to \Delta_v)$ be as at the end of Section 2.

Theorem 1. *For $i = 1, 2$,*

$$\sum_\Omega (\operatorname{rk}\Omega - 1) \leqslant \prod_j (\operatorname{rk}\Lambda_j - 1) + \mathscr{H}_1\mathscr{H}_2 - (\mathscr{H}_1 - n_{1i})(\mathscr{H}_2 - n_{2i}),$$

the sum being over all non-(simply connected) components Ω of the pullback $\Lambda_1 \prod_\Delta \Lambda_2$.

Proof. Lattice excision and the definition of the $\mathscr{H}_j$ and n_{ji} allow us to pass to the Δ single-vertexed case. Suppose then that Δ has edge loops $x_1^{\pm 1}$, $x_2^{\pm 1}$ at the vertex v, and extended spines $\hat{\Lambda}_{u_j} \hookrightarrow \Gamma_j \hookrightarrow \Lambda_j$. The covering $p_j : \Lambda_j \to \Delta_v$ can be restricted to maps $\Gamma_j \to \Delta_v$ and $\hat{\Lambda}_{u_j} \to \Delta_v$, and we form the three resulting pullbacks $\Lambda_1 \prod_\Delta \Lambda_2$, $\Gamma_1 \prod_\Delta \Gamma_2$ and $\hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2}$, with

$$\hat{\Lambda}_{u_1} \prod_\Delta \hat{\Lambda}_{u_2} \hookrightarrow \Gamma_1 \prod_\Delta \Gamma_2 \hookrightarrow \Lambda_1 \prod_\Delta \Lambda_2,$$

and $t_j : \Lambda_1 \prod_\Delta \Lambda_2 \to \Lambda_j$ the resulting covering maps.

Place black checkers on the vertices of the extended spines Γ_j as in Section 2 and place a black checker on a vertex $v_1 \times v_2$ of $\Gamma_1 \prod_\Delta \Gamma_2$ precisely when both $t_j(v_1 \times v_2) \in \Gamma_j$ $(j = 1, 2)$ have black checkers on them. By Lemma 2, and the construction of the pullback for Δ single-vertexed, the number of vertices in $\Gamma_1 \prod_\Delta \Gamma_2$ with black checkers is equal to $\prod(\operatorname{rk} \Lambda_j - 1)$.

Let Ω be a non-(simply connected) component of the pullback $\Lambda_1 \prod_\Delta \Lambda_2$ and $\Upsilon = \Omega \cap (\Gamma_1 \prod_\Delta \Gamma_2)$. If $v_1 \times v_2$ is the start vertex of at least one edge in the coboundary $\delta\Upsilon$, then at least one of the v_j must be incident with at least one, hence three, edges of the coboundary $\delta\Gamma_j$. Lifting these three via the covering t_j to $v_1 \times v_2$ gives at least three edges starting at $v_1 \times v_2$ in the coboundary $\delta\Upsilon$. Four coboundary edges starting here would mean that Ω was simply connected, hence every vertex of Υ is incident with either zero or three coboundary edges.

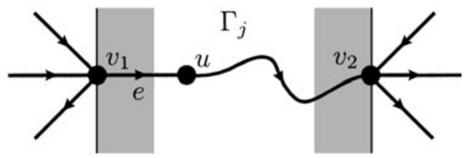
We can thus extend the interior/boundary terminology of Section 2 to the vertices of Υ, and observe that a vertex of Υ covering, via either of the t_j, a boundary vertex in Γ_j, must itself be a boundary vertex. The upshot is that Υ is an extended spine in Ω and by Proposition 6, $\operatorname{rk} \Omega - 1 = \operatorname{rk} \Upsilon - 1$. Now place red checkers on the vertices of Υ as in Section 2 and do this for each non-(simply connected) component Ω. The number of red checkered vertices is $\sum_\Omega(\operatorname{rk} \Omega - 1)$.

The result is that $\Gamma_1 \prod_\Delta \Gamma_2$ has vertices with black checkers, vertices with red checkers, vertices with red checkers on top of black checkers, and vertices that are completely uncheckered. Thus,

$$\sum_\Omega(\operatorname{rk} \Omega - 1) \leqslant \prod(\operatorname{rk} \Lambda_j - 1) + N,$$

where N is the number of vertices of $\Gamma_1 \prod_\Delta \Gamma_2$ that have a red checker but no black checker.

It remains then to estimate the number of these 'isolated' red checkers. Observe that a vertex of $\Gamma_1 \prod_\Delta \Gamma_2$ has no black checker precisely when it lies in the fiber, via at least one of the t_j, of a checkerless vertex in Γ_j. Turning it around, we investigate the fibers of the checkerless vertices of both Γ_j. Indeed, in an x_1-pair, the vertices v_1, v_2 and u are checkerless, while v_1, v_2 are also checkerless in an x_2-pair. We claim that no vertex in the fiber, via t_j, of these five has a red checker. A vertex of Υ in the fiber of the boundary vertices v_1, v_2 is itself a boundary vertex, hence contains no red checker. If $v_1 \times v_2 \in \Upsilon$ is in the fiber of u and is a boundary vertex of Υ then it carries no red checker either. If instead $v_1 \times v_2$ is an interior vertex then the lift to $v_1 \times v_2$ of e^{-1} cannot be in the coboundary $\delta\Upsilon$, hence the terminal vertex of this lift is in Υ also and covers v_1. Thus, this terminal vertex is a boundary vertex for an x_1-pair of Υ, and $v_1 \times v_2$ is the interior vertex from which a red checker is removed for this pair.



The only remaining checkerless vertices of the Γ_j unaccounted for are those interior vertices chosen for each x_2-pair. Let $S_1 = \{u_1, \dots, u_{n_{12}}\} \subset \Gamma_1$ and $S_2 = \{w_1, \dots, w_{n_{22}}\} \subset \Gamma_2$ be these sets of vertices. The result of the discussion above is that if $v_1 \times v_2$ has an isolated red checker then it must be contained in $(S_1 \times V_{\Gamma_2}) \cup (V_{\Gamma_1} \times S_2)$, the vertices of $\Gamma_1 \prod_\Delta \Gamma_2$ in the fiber of a u_i or a w_i. If $u_i \times y \in S_1 \times V_{\Gamma_2}$ with y a boundary vertex of Γ_2, then $u_i \times y$ is a boundary vertex of $\Gamma_1 \prod_\Delta \Gamma_2$, hence has no red checker. Similarly a $x \times w_i$ with x a boundary vertex of Γ_1 has no red checker, and so N is at most the number of vertices in the set $(S_1 \times V_2) \cup (V_1 \times S_2)$, with V_i the vertices of the spine $\hat{\Lambda}_{u_i}$. As $S_i \subset V_i$, the two sets in this union intersect in $S_1 \times S_2$, and so we have

$$N \leqslant |S_1 \times V_2| + |V_1 \times S_2| - |S_1 \cap S_2| = n_{12}\mathscr{H}_2 + n_{22}\mathscr{H}_1 - n_{12}n_{22};$$

hence the result holds for $i = 2$. Interchanging the checkering scheme for the x_i-pairs gives the result for $i = 1$. $\square$

4 Free groups and the topological dictionary

A group F is *free of rank* $\operatorname{rk} F$ if and only if it is isomorphic to the fundamental group of a connected graph of rank $\operatorname{rk} F$. If Γ_1, Γ_2 are connected graphs with $\pi_1(\Gamma_1, v_1) \cong \pi_1(\Gamma_2, v_2)$, then $H_1(\Gamma_1) \cong H_1(\Gamma_2)$ and thus $\operatorname{rk} \Gamma_1 = \operatorname{rk} \Gamma_2$.

The free groups so defined are of course the standard free groups and the rank is the usual rank of a free group. At this stage we appeal to the existing (algebraic) theory of free groups, and in particular, the fact that by applying Nielsen transformations, a set of generators for a free group can be transformed into a set of free generators whose cardinality is no greater. Thus, a finitely generated free group has finite rank (the converse being obvious). From now on we use the (topologically more tractable) notion of finite rank as a synonym for finitely generated.

Let F be a free group and $\varphi : F \to \pi_1(\Delta, v)$ an isomorphism for Δ connected. We call a topological realization, and the 'topological dictionary' is the loose term used to describe the correspondence between algebraic properties of F and topological properties of Δ. The non-abelian groups F correspond to the graphs Δ with $\operatorname{rk} \Delta > 1$. A subgroup $A \subset F$ corresponds to a covering $f : \Lambda_u \to \Delta_v$ with

$f_*\pi_1(\Lambda, u) = \varphi(A)$, and hence $\operatorname{rk} A = \operatorname{rk} \Lambda$ (here, f_* is the homomorphism induced by p using the functorality of π_1). Thus finitely generated subgroups correspond to finite rank graphs Λ and normal subgroups to Galois coverings. Inclusion relations between subgroups correspond to covering relations, indices of subgroups to degrees of coverings, trivial subgroups to simply connected coverings, conjugation to change of basepoint, and so on.

Applying the topological dictionary to the italicized results below we recover some classical facts (see also [18], [19]).

(1) *Proposition* 3. If a finitely generated subgroup A of a non-abelian free group F is contained in no subgroup of infinite rank, then A has finite index in F; see [7], [12].

(2) *Proposition* 4. If a finitely generated subgroup A of a free group F contains a non-trivial normal subgroup of F, then it has finite index in F; see [7].

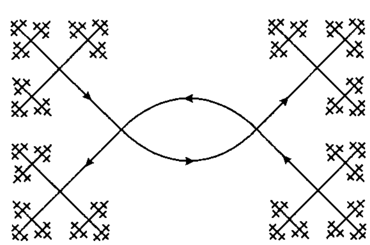
(3) *Proposition* 5 (and the comments following it). Let F be a free group, X a finite subset of F, and A a finitely generated subgroup of F disjoint from X. Then A is a free factor of a group G, of finite index in F and disjoint from X; see [1], [8].

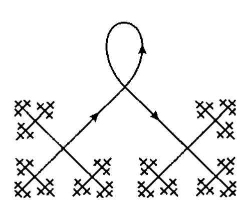
(4) *Corollary* 3. If A_1, A_2 are finitely generated subgroups of a free group F, then the intersection of conjugates $A_1^{g_1} \cap A_2^{g_2}$ is finitely generated for any $g_1, g_2 \in F$; see [10].

If Δ is a graph, $\operatorname{rk} \Delta = 2$, and $A \subset F = \pi_1(\Delta, v)$, then we define $\mathscr{H}(F, A) := \mathscr{H}(\Lambda_u \to \Delta_v)$ and $n_i(F, A) := n_i(\Lambda_u \to \Delta_v)$, where $f : \Lambda_u \to \Delta_v$ is the covering with $f_*\pi_1(\Lambda, u) = A$. For an arbitrary free group F realized via $\varphi : F \to \pi_1(\Delta, v)$, define $\mathscr{H}^\varphi(F, A)$ and $n_i^\varphi(F, A)$ to be $\mathscr{H}(\varphi(F), \varphi(A))$ and $n_i(\varphi(F), \varphi(A))$.

The appearance of φ in the notation is meant to indicate that these quantities, unlike rank, are realization dependent. This can be both a strength and a weakness: a weakness because it seems desirable for algebraic statements to involve only algebraic invariants, and a strength if we have the freedom to choose the realization, especially if more interesting results are obtained when this realization is not the 'obvious' one.

For example, if F is a free group with free generators x and y, and Δ is single-vertexed with two edge loops whose homotopy classes are a and b, then the subgroup $A = \langle xy \rangle \subset F$ corresponds to the Λ below left under the obvious realization $\varphi_1(x) = a$, $\varphi_1(y) = b$, and to the right-hand graph via $\varphi_2(x) = a$, $\varphi_2(y) = a^{-1}b$.





Thus, $\mathscr{H}^{\varphi_1}(F,A)=2$, $n_i^{\varphi_1}(F,A)=1$ $(i=1,2)$, whereas

$$\mathscr{H}^{\varphi_2}(F,A)=1, \quad n_1^{\varphi_2}(F,A)=1, \quad n_2^{\varphi_2}(F,A)=0.$$

We now apply the topological dictionary to Theorem 1. Let $\varphi: F\to\pi_1(\Delta,v)$, let A_1, A_2 be finitely generated non-trivial subgroups, and $f_j:\Lambda_{u_j}\to\Delta_v$ a covering with $\varphi(A_j)=f_{j*}\pi_1(\Lambda,u_j)$ for $j=1,2$. Each non-(simply connected) component Ω of the pullback corresponds to some non-trivial intersection of conjugates $A_1^{g_1}\cap A_2^{g_2}$. As observed in [14], these in turn correspond to the conjugates $A_1\cap A_2^g$ for g from a set of double coset representatives for $A_2\backslash F/A_1$.

Theorem 2. *Let F be a free group of rank 2 and A_1, A_2 finitely generated non-trivial subgroups. Then for any realization $\varphi: F\to\pi_1(\Delta,v)$ and $i=1,2$,*

$$\sum_g(\operatorname{rk}(A_1\cap A_2^g)-1)\leqslant\prod_j(\operatorname{rk}A_j-1)+\mathscr{H}_1\mathscr{H}_2-(\mathscr{H}_1-n_{1i})(\mathscr{H}_2-n_{2i}),$$

the sum being over all double coset representatives g for $A_2\backslash F/A_1$ with $A_1\cap A_2^g$ non-trivial, and where $\mathscr{H}_j=\mathscr{H}^{\varphi}(F,A_j)$ and $n_{ji}=n_i^{\varphi}(F,A_j)$.

This theorem should be viewed in the context of attempts to prove the so-called *strengthened Hanna Neumann conjecture*: namely, that if A_1, A_2 are finitely generated, non-trivial, subgroups of an arbitrary free group F, then

$$\sum_g(\operatorname{rk}(A_1\cap A_2^g)-1)\leqslant\prod_j(\operatorname{rk}A_j-1)+\varepsilon,$$

where $\varepsilon=0$ and the sum is over all double coset representatives g for $A_2\backslash F/A_1$ with $A_1\cap A_2^g$ non-trivial. In the existing results, ε is an error term having a long history. A selection of estimates for in chronological order is as follows:

$(\operatorname{rk}A_1-1)(\operatorname{rk}A_2-1)$; see [15];

$\max\{(\operatorname{rk}A_1-2)(\operatorname{rk}A_2-1),(\operatorname{rk}A_1-1)(\operatorname{rk}A_2-2)\}$; see [1];

$\max\{(\operatorname{rk}A_1-2)(\operatorname{rk}A_2-2)-1,0\}$; see [20];

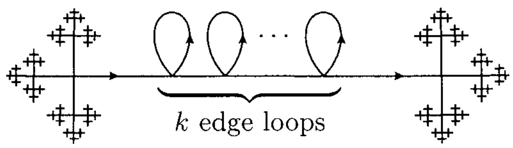
$\max\{(\operatorname{rk}A_1-3)(\operatorname{rk}A_2-3),0\}$; see [4].

The original, unstrengthened conjecture [15] involved just the intersection of the two subgroups, rather than their conjugates, and the first two expressions for ε were proved in this restricted sense; the strengthened version was formulated in [14], and the Neumann and Burns estimates for ε were improved to the strengthened case there.

Observe that as the join $\langle A_1,A_2\rangle$ of two finitely generated subgroups is finitely generated, and every finitely generated free group can be embedded as a subgroup of the free group of rank two, we may replace the ambient free group in the conjecture with the free group of rank two.

It is hard to make a precise comparison between the terms ε provided by Theorem 2 and those above. Observe that if $A_j \subset F$, with F free of rank two, then with respect to a topological realization we have $\operatorname{rk} A_j = \mathscr{H}_j - (n_{j1} + n_{j2}) + 1$. It is straightforward to find infinite families $A_{1k}, A_{2k} \subset \pi_1(\Delta, v)$ $(k \in \mathbb{N})$, for which the error term in Theorem 2 is less than those listed above for all but finitely many k, or even for which the strengthened Hanna Neumann conjecture is true by Theorem 2, for instance,

$A_{1k} = A_{2k} =$



$\mathcal{H}_i = k$

$n_{11} = n_{21} = 0$

$n_{12} = n_{22} = 1$

but where the error terms above are quadratic in k.

Acknowledgements. Some of the results of this paper were obtained while the author was visiting the Department of Mathematics, University of Adelaide, Australia. He is grateful for their hospitality. He is also grateful to the referee, whose careful reading and numerous suggestions have significantly improved (and shortened) the exposition of this paper.

References

[1] R. G. Burns. A note on free groups. *Proc. Amer. Math. Soc.* **23** (1969), 14–17.

[2] D. E. Cohen. *Combinatorial group theory*: *a topological approach*. London Math. Soc. Student Texts 14 (Cambridge University Press, 1989).

[3] D. J. Collins and H. Zieschang. Combinatorial group theory and fundamental groups. In *Algebra*, *VII*, Encyclopaedia Math. Sci. 58 (Springer-Verlag, 1993), pp. 1–166, 233–240.

[4] W. Dicks and E. Formanek. The rank three case of the Hanna Neumann conjecture. *J. Group Theory* **4** (2001), 113–151.

[5] W. Dicks. Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture. *Invent. Math.* **117** (1994), 373–389.

[6] S. M. Gersten. Intersections of finitely generated subgroups of free groups and resolutions of graphs. *Invent. Math.* **71** (1983), 567–591.

[7] L. Greenberg. Discrete groups of motions. *Canad. J. Math.* **12** (1960), 415–426.

[8] M. Hall, Jr. Subgroups of finite index in free groups. *Canad. J. Math.* **1** (1949), 187–90.

[9] A. Hatcher. *Algebraic topology* (Cambridge University Press, 2002).

[10] A. G. Howson. On the intersection of finitely generated free groups. *J. London Math. Soc.* **29** (1954), 428–434.

[11] W. Imrich. Subgroup theorems and graphs. In *Combinatorial mathematics*, *V*, *Proc. Fifth Austral. Conf.*, *Roy. Melbourne Inst. Tech.* (*Melbourne 1976*), Lecture Notes in Math. 622 (Springer-Verlag, 1977), pp. 1–27.

[12] A. Karrass and D. Solitar. On finitely generated subgroups of a free group. *Proc. Amer. Math. Soc.* **22** (1969), 209–213.

[13] A. T. Lundell and S. Weingram. *The topology of CW complexes* (Van Nostrand Reinhold, 1969).

[14] W. D. Neumann. On intersections of finitely generated subgroups of free groups. In *Groups—Canberra 1989*, Lecture Notes in Math. 1456 (Springer-Verlag, 1990), pp. 161–170.
[15] H. Neumann. On the intersection of finitely generated free groups. *Publ. Math. Debrecen* **4** (1956), 186–189.
[16] P. Scott and T. Wall. Topological methods in group theory. In *Homological group theory* (*Proc. Sympos.*, *Durham*, *1977*), London Math. Soc. Lecture Note Ser. 36 (Cambridge University Press, 1979), pp. 137–203.
[17] J.-P. Serre. *Trees*, corrected 2nd printing (Springer-Verlag, 2003).
[18] B. Servatius. A short proof of a theorem of Burns. *Math. Z.* **184** (1983), 133–137.
[19] J. R. Stallings. Topology of finite graphs. *Invent. Math.* **71** (1983), 551–565.
[20] G. Tardos. Towards the Hanna Neumann conjecture using Dicks' method. *Invent. Math.* **123** (1996), 95–104.
[21] G. Tardos. On the intersection of subgroups of a free group. *Invent. Math.* **108** (1992), 29–36.

Received 5 October, 2006; revised 4 May, 2007

Brent Everitt, Department of Mathematics, University of York, York YO10 5DD, United Kingdom
E-mail: bje1@york.ac.uk