Brent Everitt

Combinatorial Topology and Group Theory

– Monograph –

October 15, 2008
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.8</td>
<td>Four definitions of free group</td>
<td>86</td>
</tr>
<tr>
<td>6.9</td>
<td>Interlude: the ( p )-adic numbers</td>
<td>87</td>
</tr>
<tr>
<td>6.10</td>
<td>Interlude: Lie groups</td>
<td>87</td>
</tr>
<tr>
<td>6.11</td>
<td>Ihara's theorem</td>
<td>87</td>
</tr>
<tr>
<td>6.12</td>
<td>Notes on Chapter 6</td>
<td>88</td>
</tr>
<tr>
<td>7</td>
<td>Coverings and groups</td>
<td>89</td>
</tr>
<tr>
<td>7.1</td>
<td>Regular trees and the Cayley complex of a free group</td>
<td>89</td>
</tr>
<tr>
<td>7.2</td>
<td>Nielsen-Schreier and the ranks of subgroups of free groups</td>
<td>90</td>
</tr>
<tr>
<td>7.3</td>
<td>Normal forms</td>
<td>92</td>
</tr>
<tr>
<td>7.4</td>
<td>Local properties and Marshall Hall's theorem</td>
<td>92</td>
</tr>
<tr>
<td>7.5</td>
<td>Some theorems of Greenberg</td>
<td>93</td>
</tr>
<tr>
<td>7.6</td>
<td>Howsen's theorem and the Hanna Neumann conjecture</td>
<td>94</td>
</tr>
<tr>
<td>7.7</td>
<td>The Reidemeister-Schreier theorem</td>
<td>96</td>
</tr>
<tr>
<td>7.8</td>
<td>Cayley complexes of quotients</td>
<td>96</td>
</tr>
<tr>
<td>7.9</td>
<td>Monodromy representations and Miller's theorem</td>
<td>97</td>
</tr>
<tr>
<td>7.10</td>
<td>Notes on Chapter 7</td>
<td>103</td>
</tr>
<tr>
<td>8</td>
<td>Amalgams</td>
<td>105</td>
</tr>
<tr>
<td>8.1</td>
<td>Type I amalgams</td>
<td>105</td>
</tr>
<tr>
<td>8.2</td>
<td>Type II amalgams</td>
<td>108</td>
</tr>
<tr>
<td>8.3</td>
<td>Coverings</td>
<td>110</td>
</tr>
<tr>
<td>8.4</td>
<td>The Kurosh theorem</td>
<td>111</td>
</tr>
<tr>
<td>8.5</td>
<td>Title?</td>
<td>111</td>
</tr>
<tr>
<td>8.6</td>
<td>Graphs of groups and their amalgams</td>
<td>115</td>
</tr>
<tr>
<td>8.7</td>
<td>Virtually free groups</td>
<td>117</td>
</tr>
<tr>
<td>8.8</td>
<td>Notes on Chapter 8</td>
<td>123</td>
</tr>
<tr>
<td>9</td>
<td>Serre's Arboreal Dictionary</td>
<td>125</td>
</tr>
<tr>
<td>9.1</td>
<td>Representation varieties</td>
<td>127</td>
</tr>
<tr>
<td>9.2</td>
<td>Notes on Chapter 9</td>
<td>127</td>
</tr>
</tbody>
</table>
Combinatorial Complexes

Of course, one has to face the question, what is the good category of spaces in which to do homotopy theory? –John Frank Adams.

For us the question is: “what is the good category in which to do combinatorial group theory?”. This chapter introduces it, and studies some of the important constructions one can do in it.

1.1 1-Complexes (ie: Graphs)

1.1.1 The category of graphs

Definition 1.1 (1-complex: first go). A 1-complex or graph is a non-empty set $X$ with an involutary map $-1 : X \rightarrow X$ and an idempotent $s : X \rightarrow X^0$, where $X^0$ is the set of fixed points of $-1$.

Thus a graph has 0-cells or vertices $X^0$ and 1-cells or edges $X^1 = X \setminus X^0$. One says that the edge $e \in X^1$ has start vertex $s(e)$ and terminal vertex $s(e^{-1})$ and thinks of the inverse edge $e^{-1}$ as just $e$, but traversed in the reverse direction (or with the reverse orientation). The edge $e$ is incident with the vertex $v$ if $e \in s^{-1}(v)$. We draw pictures like Figure 1.1, but it is important to keep in mind that they are purely for illustrative purposes. For instance, if the vertex set has cardinality that of the power set of the continuum, then there are not enough points on a piece of paper for a picture to fit. Definition 1.1 is a little terse, and sometimes it is useful to spell it out a little more:

Definition 1.2 (1-complex: second go). A 1-complex or graph $X$ is composed of two disjoint non-empty sets $X^0$ and $X^1$, together with two incidence maps and an inverse map,

$s, t : X \rightarrow X^0$ and $^{-1} : X \rightarrow X$,

such that, (i). $s(v) = v$ for all $v \in X^0$; (ii). $v^{-1} = v$ for all $v \in X^0$, $e^{-1} \in X^1$ and $e^{-1} \neq e = (e^{-1})^{-1}$ for all $e \in X^1$; (iii). $t(x) = s(x^{-1})$ for all $x \in X$.

More terminology: an arc is an edge/inverse edge pair, and an orientation $\mathcal{O}$ for $X$, is a set $\mathcal{O}$ of edges containing exactly one edge from each arc. Write $\mathcal{P}$ for the arc containing the...
edge $e$ (so that $e^{-1} = e$). The graph $X$ is finite when $X^0$ is finite and locally finite when the set $s^{-1}(v)$ is finite for every $v \in X^0$. Thus a finite graph may have infinitely many edges, a situation that possibly differs from that in combinatorics. The cardinality of the set $s^{-1}(v)$ is the valency of the vertex $v$. A pointed graph is a pair $X_v := (X, v)$ for $v \in X$ a vertex.

**Exercise 1.3** (1-complex: third go). Here is one more, recursive definition of graph that makes clearer the connection with the 2-complexes of §1.2. Firstly, a 0-complex is a set $X$; a map of 0-complexes is a map $f : X \to Y$ of sets; a pointed 0-complex is a pair $(X, v)$ consisting of a 0-complex $X$ and $v \in X$, and a map of pointed 0-complexes $f : (X, v) \to (Y, u)$ is a set map $f : X \to Y$ with $f(v) = u$; finally, the 0-sphere $S^0$ is the set with two elements.

With these preliminaries out of the way, a graph $X$ is a graded set $X = X^0, X^1$ with $X^k \neq \emptyset$, such that

1. there is an involutory map $-1 : X \to X$ with fixed point set $X^0$;
2. $X^0$ is 0-complex;
3. each $\alpha \in X^1$ has boundary $\partial \alpha = (e^\alpha, f^\alpha)$ with $e^\alpha$ the 0-sphere $S^0$ and $f^\alpha : e^\alpha \to X^0_u$ a pointed mapping of 0-complexes.

Here is the exercise: show that definitions 1.1, 1.2 and this version are all equivalent.

The trivial graph has a single vertex and no edges. Figure 1.2 shows some more examples of graphs, including some with countably many edges. These will tend to be more interesting than finite graphs.

![Fig. 1.2. Examples of graphs: the $S^1$ graph at left and two graphs with a countably infinite number of vertices and edges.](image)

We want graphs to form the objects of a category, so a map of graphs is a set map $f : X \to Y$ with $f(X^0) \subseteq Y^0$, such that the diagram on the left of Figure 1.3 commutes, where $\sigma_X$ is one of the maps $s_X$ or $-1$ for $X$, and $\sigma_Y$ similarly, i.e. $f s_X(x) = s_Y f(x)$ and $f(x^{-1}) = f(x)^{-1}$. Notice that a map is allowed to squash edges down to vertices: we call a map dimension preserving if in addition to the above we have $f(X^1) \subseteq Y^1$. A map $f : X_v \to Y_u$ of pointed graphs is a graph map $f : X \to Y$ with $f(v) = u$.

The commuting of $f$ with $s$ and $-1$ is meant to be a combinatorial version of continuity. Note in particular that if $x$ is a vertex then $f s_X(x) = s_Y f(x)$ is vacuous, but if $x$ is an edge mapped by $f$ to a vertex as on the right in Figure 1.3, then the condition becomes $f s_X = f$, $f^{-1}$. 

![Fig. 1.3. Graph mapping $f : X \to Y$](image)
so that \( f(v) \) is the image of \( s_X(e) \); the condition also ensures that \( t_X(e) \) is mapped to \( f(e) \), and not left “hanging”.

For a fixed vertex \( v \in X^0 \), and \( s_X^{-1}(v) \in X^1 \) the edges with start vertex \( v \), it is easy to see that a map \( f : X \to Y \) induces a map \( s_X^{-1}(v) \to s_Y^{-1}(f(v)) \) to the set of edges starting at \( f(v) \). We call this the local continuity of \( f \) at the vertex \( v \).

A map \( f : X \to Y \) is an isomorphism if it is dimension preserving and a bijection on the vertex and edge sets. Write \( X \cong Y \).

**Exercise 1.4.** Show that if \( f : X \to Y \) is an isomorphism then the map \( f^{-1} : Y \to X \), which is the inverse of \( f \) on \( Y^0 \) and \( Y^1 \), is also a graph isomorphism. Show that the set \( \text{Aut}(X) \) of graph isomorphisms \( X \to X \) forms a group under composition.

A group action is a homomorphism \( G \xrightarrow{\phi} \text{Aut}(X) \), and we abbreviate \( \varphi(g)(x) \) to \( g(x) \). It preserves orientation if there is an orientation \( \emptyset \) for \( X \) with \( g(\emptyset) = \emptyset \) for all \( g \in G \).

**Exercise 1.5.** An inversion of a graph \( X \) is a map \( f : X \to X \) which interchanges the edges of some arc, ie: \( f(e) = e^{-1} \) for some \( e \in X^1 \). A group action is said to be without inversions iff no \( g \in G \) acts as an inversion on \( X \). Show that an action by a group \( G \) preserves orientation if and only if it acts without inversions.

A group \( G \) acts freely iff the action is free on the vertices, ie: for any \( g \in G \) and \( v \) a vertex, \( g(v) = v \) implies that \( g \) is the identity element.

**Exercise 1.6.** If \( G \) acts freely and orientation preservingly on a graph, then show that the action is free on the edges too.

Graph isomorphisms are pretty rigid, and it is sometimes useful to have a relation with a bit more slack. Thus, a subdivision of an edge \( e \) in a graph \( X \) replaces it by two new edges and a new vertex as in Figure 1.4, or is the reverse of this process. Write \( X \leftrightarrow X' \) when two graphs differ by the subdivision of a single edge. Two graphs \( X \) and \( Y \) are then homeomorphic, written \( X \cong Y \), when there is a finite sequence \( X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_k = Y \) of subdivisions connecting them. It is easy to see that homeomorphism is an equivalence relation for graphs. A property that is invariant under homeomorphism, ie: if \( X \cong Y \) are homeomorphic, then \( X \) has the property if and only if \( Y \) does, is called a topological invariant.

### 1.1.2 Quotients and subgraphs

The most useful construction in the category of graphs is the quotient:

**Definition 1.7 (quotient relations and quotient graphs).** If \( X \) is a graph, then a quotient relation is an equivalence relation \( \sim \) on \( X \) such that

(i). \( x \sim y \Rightarrow s(x) \sim s(y) \) and \( x^{-1} \sim y^{-1} \)  
(ii). \( x \sim x^{-1} \Rightarrow [x] \cap X^0 \neq \emptyset \),

where \([x]\) is the equivalence class under \( \sim \) of \( x \). If \( \sim \) is a quotient relation on a graph \( X \) then define \( s_{X/\sim} \) and \( ^{-1} \) on the set of equivalence classes \( X/\sim \) by

(i). \( s_{X/\sim}[x] = [s_X(x)] \)  
(ii). \([x]^{-1} = [x^{-1}]\).
Notice that edges can be equivalent to vertices, but if an edge is equivalent to its inverse then it must also be equivalent to a vertex. This ensures that \([e] \not= [e]^{-1}\) in the quotient.

**Proposition 1.8.** If \(\sim\) is a quotient relation then \(X/\sim\) with the maps \(s_{X/\sim}\) and \(\sim^{-1}\) defined above is a graph, and the quotient map \(q : X \to X/\sim\) given by \(q(x) = [x]\) is a map of graphs.

The proof is a straightforward exercise. In particular, the fixed points in \(X/\sim\) of the new \(\sim^{-1}\) map are precisely those equivalence classes \([x]\) where \(x \sim v\) for some \(v \in X^0\). Thus the quotient has vertices the \([e]\) for \(v \in X^0\) (and these classes may well include some of the edges of the old graph \(X\)) and edges those \([e]\) with \([e] \cap X^0 = \emptyset\).

The two main examples of graph quotients arise by factoring out the action of a group, or by squashing a subgraph down to a vertex. For the first we have the following.

**Proposition 1.9.** Let \(\sim\) be the equivalence relation on \(X\) given by the orbits of a group action. Then \(\sim\) is a quotient relation if and only if the group action is orientation preserving, and we write \(X/G := X/\sim\) for the quotient of the graph by the action of the group.

A subgraph is a graph mapping \(X \hookrightarrow Y\) that is an isomorphism onto its image. Equivalently, it is a subset \(X \subseteq Y\), such that the maps \(s\) and \(\sim^{-1}\) give a graph when restricted to \(X\).

Let \(X \hookrightarrow Y\) be a subgraph and define \(\sim\) on \(Y\) by \(x \sim y\) iff \(x = y\) or both \(x\) and \(y\) lie in \(Y\). Then this is a quotient relation and we write \(Y/X\) for \(Y/\sim\), the quotient of \(Y\) by the subgraph \(X\). It is what results by squashing \(X\) to a vertex. Extending this a little, if \(X_i, (i \in I)\) is a family of disjoint subgraphs in \(Y\) then define \(\sim\) by \(x \sim y\) iff \(x = y\) or \(x\) and \(y\) lie in the same \(X_i\), and write \(Y/X\) for the corresponding quotient. Note the difference between this, where each \(X_i\) has been squashed to a distinct vertex \(v_i\), and \(Y/(\coprod X_i)\), where the whole union is squashed to just the one vertex.

The coboundary \(\delta X\) of a subgraph \(X \hookrightarrow Y\) consists of those edges \(e \in Y\) with \(s(e) \in X\) and \(t(e) \not\in X\).

**Exercise 1.10.** Let \(X_1, X_2\) and \(Y\) be graphs and \(f : Y \to X_i\) dimension preserving maps of graphs. Let \(\sim\) on the disjoint union \(X_1 \coprod X_2\) be the equivalence relation generated by the \(x \sim y\) iff there is a \(z \in Y\) with \(x = f_1(z)\) and \(y = f_2(z)\). Thus, \(x \sim y\) iff there are \(x_0, x_1, \ldots, x_k\) with \(x_0 = x\) and \(x_k = y\), and for each \(j\), there is \(z \in Y\) with \(x_j = f_i(z)\), \(x_{j+1} = f_{i+1 \mod 2}(z)\). Show that \(\sim\) is a quotient relation if and only if there are orientations \(O, O_i\) for \(Y, X_i\) with \(f_i(O) \subseteq O_i\).

### 1.1.3 Balls, spheres, paths and homotopies

Two particular graphs are the \(I^1\)-graph and \(S^1\)-graph as shown in Figure 1.5. A graph \(X\) is

![Fig. 1.5. The \(I^1\) and \(S^1\)-graphs.](image)

then a 1-ball either if it is the trivial graph or is homeomorphic to \(I^1\), and a 1-sphere if it is homeomorphic to \(S^1\).

It is easy to see that the vertices of a 1-ball can be labelled as \(v_0, \ldots, v_n\) and the edges as \(e^1_1, \ldots, e^1_n\) with \(s(e_i) = v_{i-1}\) and \(s(e_i^{-1}) = v_i\). Thus a 1-ball has end vertices \(v_0, v_n\) in an obvious sense. The following is also easily proved by induction:
Lemma 1.11. A graph \( X \) is a 1-sphere if and only if either \( X = S^1 \), or there are non-trivial 1-balls \( B_i, (i = 1, 2) \) with end vertices \( v_{11}, v_{21} \), such that \( X = B_1 \coprod B_2 / \sim \), where the equivalence classes of \( \sim \) are \( \{v_{11}, v_{21}\}, \{v_{12}, v_{22}\} \) and the \( \{x\} \) for all other cells \( x \in X_1 \coprod X_2 \).

A path in \( X \) is a graph mapping \( B \to X \) with \( B \) a 1-ball. It is convenient not to insist that the map preserve dimension, but by Exercise 1.12 below, we can always replace the 1-ball by another so that the map is dimension preserving if necessary. In any case, the image in \( X \) is a sequence of edges \( e_1 \ldots e_\ell \), that are consecutively incident in the obvious way: \( s(e_{-1}) = s(e_{\ell+1}) \), and there is no harm in thinking about paths in terms of their images. We will use both points of view interchangeably as convenient. A path joins the vertices \( s(e_1), s(e_\ell^{-1}) \) that are the images of the end vertices of the 1-ball, and is closed if these end vertices have the same image.

Exercise 1.12. Let \( f : B \to X \) be a path with edges labelled \( e_1^1 \ldots e_\ell^1 \) as in the comments before Lemma 1.11, and image edges \( e_1^2 \ldots e_\ell^2 \) in \( X \). Show there are \( 1 \leq i_2 \leq i_1 \leq \cdots \leq i_\ell \leq n \) with \( f(e_{i_j}) = e_{i_j}^2 \) and all other edges mapped to vertices. Thus, \( B \) can be replaced by a 1-ball \( B' \) and dimension preserving map \( f' : B' \to X \) having the same image path.

Exercise 1.13. Show that a closed path \( B \to X \) is the same thing as a mapping \( S \to X \) for some \( S \simeq S^1 \).

If \( X \to Y \) is a graph map and \( B \to X \) a path, then there is an induced path in \( Y \) given by the composition \( B \to X \to Y \). Thus, a graph mapping sends paths to paths.

The graph \( X \) is connected if any two vertices can be joined by a path. The connected component of \( X \) containing the vertex \( v \) consists of those vertices that can be joined to \( v \), together with all their incident edges. A connected graph has finitely many edges if and only if it is finite and locally finite.

A path \( e_1 \ldots e_\ell \) contains a spur if \( e_{i+1} = e_i^{-1} \) for some \( i \), ie: the path consecutively traverses an edge and its inverse. An elementary homotopy of a path, \( e_1 \ldots e_i e_{i+1} \ldots e_\ell \leftrightarrow e_1 \ldots e_i (e_i^{-1}) e_{i+1} \ldots e_\ell \), inserts or deletes a spur as in Figure 1.6. Two paths are freely homotopic iff there is a finite sequence of elementary homotopies taking one to the other.

Paths homotopic to a trivial path consisting of a single vertex and no edges are said to be homotopically trivial. We leave it as an exercise to formulate these notions for a path \( B \to X \) thought of as a mapping.

Exercise 1.14. Show that homotopic paths have the same start and end vertices, and thus homotopically trivial paths are necessarily closed. Show that homotopy is an equivalence relation on the paths with common fixed endpoints.
1.1.4 Forests and Trees

A path in a graph $X$ is reduced when it contains no spurs. By removing spurs we can ensure that if there exists a path between two vertices, then there must exist a reduced path; indeed for any two vertices, there is a reduced path between them if and only if they lie in the same component.

**Proposition 1.15.** The following are equivalent for a graph $X$:

1. There is at most one reduced path joining any two vertices;
2. any closed path is homotopically trivial;
3. any non-trivial closed path contains a spur.

A graph satisfying any of the conditions of Proposition 1.15 is called a forest, and a connected forest is a tree.

**Proof.** 1 $\Rightarrow$ 2: a closed path is necessarily contained in a component of the graph, thus by assumption there is a unique reduced path connecting any two of its vertices. The path cannot contain just a single vertex $u$ and edge $e$ (which it circumnavigates some number of times), for if so, then the edge $e$ and the trivial path at $u$ are distinct reduced paths from $u$ to $u$. Thus any closed path contains at least two distinct vertices. We show the path is homotopic to the trivial path based at one of them, say $u$. Let $v$ be another vertex of the path and $w$ the reduced path running from $u$ to $v$. Then the path decomposes into two parts, $w_1$ running from $u$ to $v$ and $w_2$ running from $v$ to $u$. If $w_1 \neq w$ then it cannot be reduced, hence must contain a spur. Removing it and continuing, we have a series of homotopies that reduces the path $w_1$ to the trivial path as in the left of Figure 1.7. Similarly for $w_2$ and $w^{-1}$.

![Fig. 1.7. homotoping a closed path to the trivial path](image)

(Which is the unique reduced path from $v$ to $u$) as on the right of Figure 1.7. Thus our path is homotopic to $ww^{-1}$, which in turn can be reduced by the removal of spurs to the trivial path based at $u$.

2 $\Rightarrow$ 1: if $u$ and $v$ lie in different components then there are obviously no reduced paths connecting them. Otherwise, if $w_1$ and $w_2$ are reduced paths running from $u$ to $v$ then the path $w_1w_2^{-1}$ is homotopically trivial, hence contains a spur. As the $w_i$ are reduced, the spur must be at the beginning or the end of the closed path, i.e., involve the first edges of $w_1$ and $w_2$, or the last edges as in Figure 1.8. Shifting attention to the reduced subpaths not involving this spur and continuing, we get $w_1 = w_2$. We leave the equivalence of parts 2 and 3 as an Exercise. $\square$
1.2 The category of 2-complexes

Exercise 1.16. Let $X$ be a finite graph, remembering that this only means that the vertex set $X^0$ is finite. If each vertex has valency at least two, show that $X$ contains a homotopically non-trivial closed path. Deduce that if $T$ is a finite tree, then $|T^1| = 2(|T^0| - 1)$.

We’ll have more to say about trees later. We finish this section by considering how to approximate a graph by a tree: if $X$ is a connected graph, then a spanning tree is a subgraph $T \hookrightarrow X$ that is a tree and contains all the vertices of $X$ (i.e. $X^0 = T^0$). The following exercise shows that under some mild set-theoretic assumptions, spanning trees always exist.

Exercise 1.17. Recall the well-ordering principle from set theory: any set $X$ can be given an order $\leq$ (see Definition 3.40) so that for any elements $x, y \in X$, either $x \leq y$ or $y \leq x$, and for any $S \subseteq X$ there is a $s \in S$ with $s \leq x$ for all $x \in X$. In particular, the edge set $X^1$ of a graph can be well-ordered. Choose a basepoint vertex $v_0$, and consider those vertices at distance one from $v_0$, i.e. the $v \neq v_0$ with $s(e) = v_0, s(e^{-1}) = v$ for some edge $e$. For each such, choose an edge $e_0$ that is minimal in the well-ordering amongst the edges joining $v_0$ to $v$. Let $T_1$ be the subgraph consisting of $v_0$, its distance 1 neighbours and the chosen edges.

1. Show that $T_1$ is a tree. Continue the construction inductively: at step $k$, take the tree $T_{k-1}$ constructed at the end of step $k - 1$, and for each vertex $v$ of $X$ a distance 1 from a vertex of $T_{k-1}$, choose a minimal edge as above. Let $T_k$ be the subgraph consisting of $T_{k-1}$ together with the distance 1 vertices and minimal edges. Show that $T_k$ is a tree.

2. Show that $T$, the union of all the $T_k$, is the required spanning tree.

In a graph the vertex and edge sets are arbitrary, so the edge set can have a wildly different cardinality from the vertex set, causing difficulties with some arguments. This shortcoming is avoided by spanning trees, which have a number of edges that is roughly the same as the number of vertices of the graph they span:

Proposition 1.18. Let $X$ be a connected graph and $T \hookrightarrow X$ a spanning tree. Then

$$|T^1| = \begin{cases} 2|X^0| - 1, & \text{if } X \text{ is finite,} \\ |X^0|, & \text{if } X \text{ is infinite.} \end{cases}$$

Proof. The result for finite graphs is the content of Exercise 1.16. If $X$ is an infinite graph with spanning tree $T$, then the edge set of $T$ must be infinite, as a finite edge set only spans $|T^1| + 1$ vertices by the first part. Then $X^0 = T^0 = \bigcup_{e \in T} \{s_T(e), s_T(e^{-1})\}$ has the cardinality that of $T^1$. \qed

Exercise 1.19. Let $T_i \hookrightarrow X$ be a family of mutually disjoint trees in a connected graph $X$. Show there is a spanning tree $T \hookrightarrow X$ containing the $T_i$ as subgraphs, and such that $q(T)$ is a spanning tree for $X/T_i$, where $q : X \rightarrow X/T_i$ is the quotient map.

A spanning forest is a subgraph $\Phi \hookrightarrow X$ that is a forest and contains all the vertices of $X$. By considering $q^{-1}(T')$ for some spanning tree $T'$ of the (connected) graph $X/\Phi$, show that any spanning forest can be extended to a spanning tree.

1.2 The category of 2-complexes

Now to the category of combinatorial 2-complexes, where we mimic the constructions in the previous section as much as possible.
1.2.1 2-complexes

**Definition 1.20 (2-complex).** A combinatorial 2-complex $X$ is a graded set $X = X^0, X^1, X^2$ with $X^k \neq \emptyset$, and such that

1. if $X^{(1)} := X^0 \amalg X^1$, there is an involutory map $^{-1} : X^{(1)} \to X^{(1)}$ with fixed points $X^0$ and an idempotent $s : X^{(1)} \to X^0$, making $X^{(1)}$ a graph;
2. each $σ ∈ X^2$ has boundary $∂σ = (X^σ, σ^σ)$ with $X^σ ≈ S^1$ and $σ^σ : X^σ_σ \to X^σ_0$ a pointed dimension preserving map of graphs.

The elements of the set $X^2$ are the 2-cells or faces of the complex with the pair $(X^σ, σ^σ)$ the boundary of the face $σ ∈ X^2$. The map $σ^σ$ is the attaching map of the face (see Figure 1.9). The underlying graph $X^{(1)}$ is called the 1-skeleton. The edges of a 2-complex are distinguished by the map $^{-1}$, whereas the vertices and faces are not. Partly this helps with the accounting later on, but it also reflects the motivating topology: 1-spheres are connected whereas the 0-sphere is not, so the $^{-1}$ map allows us to choose a connected boundary for edges in some sense.

![Fig. 1.9. Boundary of a face consisting of a 1-sphere and an attaching map, which may wrap the sphere around the face several times. The vertex $v$ appears in the boundary of $σ$ and the red path wrapping twice around is a boundary path of $σ$ starting at $v$.](image)

One thinks of a face as a disc attached to the 1-skeleton with boundary a closed path as in Figure 1.9, and again, although these pictures provide topological motivation, they do not carry as much information as the definition. For example, the image $σ^σ(X^σ)$ of a face boundary can be a path that wraps around the same set of edges several times. A example of this phenomenon is the “torus” complex below, and we will encounter an important example later where a face has boundary with $X^σ$ having many edges, but the image $σ^σ(X^σ)$ a single loop.

The pointing of the attaching maps will prove to be essential later on, allowing us in particular to distinguish faces using the 1-skeleton. Call $v = σ^σ(x)$ the distinguished vertex of the face $σ$.

We say that the vertex $v$ appears in the boundary of the face $σ$ whenever $(σ^σ)^{-1}(v) \neq \emptyset$. Thus there is a vertex $x$ of $X^σ$ mapping to $v$ via the attaching map $σ^σ$, and indeed there may be several of them. We will call the vertices in $(σ^σ)^{-1}(v) \subseteq X^σ$ the appearances of $v$ in the boundary of the face $σ$. For such a vertex $x$, if we take a path $γ$ circumnavigating the 1-sphere $X^σ$ in some direction as shown in Figure 1.9, then we call its image a boundary path of $σ$ starting at $v$. The vertex $v$ appears a total of $|(σ^σ)^{-1}(v)|$ times in the boundary of $σ$, and each appearance gives rise to a pair of boundary paths starting at $v$.

**Exercise 1.21.** Formulate an equivalent version of definition 1.20 so that instead of the face attaching maps, the boundary of a face is a closed path $e_1 \ldots e_ℓ$ with a distinguished vertex in the underlying graph.

Many of the concepts and adjectives of graphs can be applied directly to 2-complexes by applying them to the 1-skeleton. Thus, we have the obvious notions of a finite 2-complex, connected 2-complexes, paths in 2-complexes, etc.
1.2 Examples

Figure 1.10 gives three different versions of a 2-complex that we will call the 2-sphere. The first version (top left) is a straight pictorial version of definition 1.20 in this case: the 1-skeleton in the middle is a graph with two vertices and two edges; meanwhile there are two faces with the attaching maps shown, and each face has distinguished vertex one of the vertices in $X^{(1)}$. This version is both the most accurate and the most cumbersome.

In the second version (bottom left) we have adopted the convention that parts of the complex with the same label give the same cell and drawn the complex “face-centrically” with the faces thought of as discs sewn onto the 1-skeleton. Carrying out the identifications suggested by this picture gives the third version on the right.

![Fig. 1.10. 2-sphere: pictorial version of definition 1.20 (top left); face-centric version with faces sewn on (bottom left) and topologically suggestive version (right).](image)

Figure 1.11 gives a very similar complex, differing only in that both faces now have the same distinguished vertex. For reasons that are a little obscure at the moment, we very much prefer the complex of Figure 1.10 to that of Figure 1.11. For now we will content ourselves with the following comparison: both complexes have the same 1-skeleton, and once maps of 2-complexes are defined in §1.2.3, we will see that the only isomorphism of Figure 1.10 that restricts to the identity on the 1-skeleton is the identity isomorphism. In Figure 1.11 on the other hand, there will be a non-trivial isomorphism restricting to the identity on the 1-skeleton. This distinction will have ramifications in Chapter 4.

We call Figure 1.12 the (real) projective plane $\mathbb{R}P^2$, a combinatorial model for the disc with antipodal points on the boundary identified. We have again drawn the complex both face-centrically and *ala* Definition 1.20. Similarly, Figure 1.13 shows various versions of the torus complex.

1.2.3 Maps of 2-complexes

To complete the definition of the category of 2-complexes we need to define maps between them. The principle is the same as that for maps of graphs in §1.1.1: continuity is captured
by making the presumptive map commute with the various attaching maps of the cells. The definition is complicated by the fact that a map may squash a face down to a path in the 1-skeleton.

**Definition 1.22 (maps of 2-complexes).** A map \( f : X \to Y \) of 2-complexes is a map \( f : X^{(1)} \to Y^{(1)} \) of the underlying graphs, and for each face \( \sigma \in X^2 \) we have either \( f(\sigma) \) is a face \( \tau \in Y^2 \) or \( f(\sigma) \) is a closed path in the graph \( Y^{(1)} \) such that,

1. if \( \partial \sigma = (X^\sigma, a^\sigma) \) with \( a^\sigma : X^\sigma_u \to X^{(1)}_v \), and \( f(\sigma) = \tau \) a face with \( \partial \tau = (Y^\tau, a^\tau) \) and \( a^\tau : Y^\tau_x \to Y^{(1)}_w \), then there is an isomorphism \( \varepsilon_\sigma : X^\sigma_u \to Y^\tau_x \) making the diagram below left commute;

\[
\begin{array}{c}
X^\sigma_u \xrightarrow{\sim} Y^\tau_x \\
\alpha^\sigma \downarrow \quad \downarrow \alpha^\tau \\
X^{(1)}_v \xrightarrow{f} Y^{(1)}_w
\end{array}
\]

2. if \( f(\sigma) \) is the closed path \( S \to Y \) with \( S \approx S^1 \), then there is a map \( X^\sigma_u \to S_x \) making the diagram above right commute, and moreover, the path \( S \to X \) is homotopically trivial.

The first part is just saying that when a face \( \sigma \) is mapped to a face \( \tau \), then the attaching map \( a^\tau \) of \( \tau \) is the composition \( Y^\tau \to X^\sigma \to X^{(1)} \to Y^{(1)} \), going anti-clockwise around the diagram. Alternatively, replacing \( Y^\tau \) by the isomorphic copy \( X^\sigma \), \( \tau \) has boundary \( \partial f(\sigma) = (X^\sigma, f a^\sigma) \), with \( f a^\sigma : X^\sigma_u \to X^{(1)}_v \to Y^{(1)}_{f(v)} \) a pointed map of graphs as in Figure 1.14. If \( f(\sigma) \) a path, then the edges forming the boundary \( a^\sigma(X^\sigma) \) of \( \sigma \) are mapped by \( f \) to edges of this path.

With Exercise 1.21 in mind, it is an easy exercise to reformulate Definition 1.22 as follows: if \( f(\sigma) \) is a face \( \tau \), then the closed path forming the boundary of \( \sigma \) is mapped via \( f \) to the closed path forming the boundary of \( \tau \).
Thus, the boundaries of faces are mapped to the boundaries of faces, and the distinguished vertex of a face is mapped to the distinguished vertex of the image face. If a face is mapped to a path in the 1-skeleton, then this path must be the image of the boundary path of the face (hence closed) and is homotopically trivial. The motivating example is squashing a face flat: the boundary gets squashed too, into a path of the form \(e_1 \cdots e_k e_1^{-1} \cdots e_1^{-1}\).

In this example the result is clearly homotopically trivial, but we will really require this condition in Chapter 2 for the fundamental group \(\pi_1\) to be a functor.

We now establish a number of conventions regarding maps that will hold throughout this book. They all involve the face isomorphisms that appear in part 1 of Definition 1.22. The first relates to the question, “when are two maps the same?” Suppose that \(X\) and \(Y\) are 2-complexes and \(f, g : X \to Y\), maps of 2-complexes. Then \(f\) and \(g\) are identical when \(f(x) = g(x)\) for all cells \(x \in X\) (obviously), and if \(f(\sigma) = g(\sigma) = \tau\) for faces \(\sigma \in X\) and \(\tau \in Y\), the associated isomorphisms \(\varepsilon_\sigma\) and \(\varepsilon'_\sigma : X^\sigma \to Y^{\tau}\) of Definition 1.22 part 1 are identical.

Our next convention concerns the formation of compositions. Suppose that \(X \xrightarrow{f} Y \xrightarrow{g} Z\), are maps of 2-complexes. Then the composition \(gf\) is formed in the usual way, with the additional proviso that if \(\sigma \in X\) is a face such that \(gf(\sigma)\) is a face of \(Z\), the isomorphism \(X^\sigma \to Z^{gf(\sigma)}\) is the composition of the isomorphisms,

\[
X^\sigma \xrightarrow{\varepsilon_\sigma} Y^{f(\sigma)} \xrightarrow{\varepsilon'_\tau} Z^{gf(\sigma)}.
\]

These two conventions have ramifications for commuting diagrams of complexes and maps, which are after all, just statements about two maps being the same. For example, when we say that the diagram of maps and complexes on the left commutes,

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y_1 \\
& \downarrow{f_2} & \downarrow{g_2} \\
Y_2 & \xrightarrow{g_1} & Z \\
\end{array}
\]

then for \(i = 1, 2\), the compositions \(g_i f_i\) are the same as the dotted map across the middle. If \(\tau\) is a face of \(Y\) that maps to a face \(g_i f_i(\tau)\) of \(Z\), then the diagram of isomorphisms on the right must also commute.

Let \(f : X \to Y\) be a map of 2-complexes, \(u, v\) vertices with \(f(u) = v\), and \(\tau\) a face in \(Y\). For a face \(\sigma\) of \(X\) that maps to \(\tau\) let \(\varepsilon_\sigma : X^\sigma \to X^{\tau}\) be the isomorphism making the diagram in Definition 1.22 part 1 commute. This isomorphism then induces a map \(\varepsilon_\sigma : (a^\sigma)^{-1}(u) \to (a^\tau)^{-1}(v)\) between the appearances of \(u\) in the boundary of \(\sigma\) and the appearances of \(v\) in the boundary of \(\tau\). Since this is true for all the \(\sigma\) mapping to \(\tau\), we can formulate the,
Definition 1.23 (local continuity). If \( f : X \to Y \) is a map of 2-complexes, \( u, v \) vertices with \( f(u) = v \), and \( \tau \) a face in \( Y \), then the local continuity map associated to this triple is

\[
\prod \varepsilon_\sigma : \prod_{f(\sigma) = \tau} (a^{\sigma})^{-1}(u) \to (a^{\tau})^{-1}(v),
\]

where the disjoint union is over the faces \( \sigma \) mapping to \( \tau \), and \( \prod \varepsilon_\sigma \) is the disjoint union of the isomorphisms \( \varepsilon_\sigma \).

An example is given in Figure 1.15 with the mapping of the “plane” complex to the torus.

Exercise 1.24. Show that if the right hand side of the set in Definition 1.23 is empty, then so is the left hand side.

A map is dimension preserving if the map of the underlying graphs preserves dimension and \( f(X^2) \subset Y^2 \). It is an isomorphism if it is dimension preserving, and a bijection on the vertex, edge and face sets. In this case one easily sees that, as for graphs, the inverse map \( f^{-1} \) is also an isomorphism \( f^{-1} : Y \to X \) (just reverse the horizontal arrows in the left commuting diagram of Definition 1.22(1)!) so that the set of automorphisms \( f : X \to X \) forms a group \( \text{Aut}(X) \) under composition. A group action \( G \overset{\varphi}{\to} \text{Aut}(X) \) then preserves orientation, is without inversion and free, when the underlaying map of graphs has these properties.

A subcomplex is a mapping \( X \hookrightarrow Y \) of 2-complexes that is an isomorphism onto its image.

Exercise 1.25. Formulate an equivalent version of the definition of a subcomplex \( X \) of \( Y \), with the sets of \( k \)-cells \( X^k \) of \( X \) as subsets of the sets of \( k \)-cells \( Y^k \) of \( Y \).

1.2.4 Homotopies and homeomorphisms

As with graphs in §1.1.3 we can deform paths in a combinatorial manner, mimicking the continuous homotopies of paths in topology.

Let \( e_1 \ldots e_\ell \) be a path in the 2-complex \( X \). An elementary homotopy either inserts or deletes a spur as in §1.1.3 or inserts or deletes the boundary of a face in the following sense. If \( \sigma \in X^2 \) is a face with boundary \( \partial \sigma = (X^\sigma, a^\sigma) \), then by Exercise 1.13, \( a^\sigma(X^\sigma) \)

\[
\text{Fig. 1.15. local continuity for a map of 2-complexes: the infinite plane complex on the left maps via } f \text{ to the torus on the right; } \tau \text{ is the single face of the torus and for the single vertex } v, \text{ the set } (a^{\tau})^{-1}(u) \in Y^\tau \text{ consists of the four vertices marked with little red and blue circles and squares. A fixed vertex } u \text{ mapping to } v \text{ appears in the boundary of the four faces } \sigma, \text{ with } (f^{\sigma})^{-1}(u) \text{ consisting of a single vertex in each case. For all other faces } \sigma \text{ we have } (a^{\sigma})^{-1}(u) = \emptyset. \text{ The red and blue circles and squares on the left map via the local continuity map of Definition 1.23 to the corresponding ones on the right.}
\]

A map is dimension preserving if the map of the underlying graphs preserves dimension and \( f(X^2) \subset Y^2 \). It is an isomorphism if it is dimension preserving, and a bijection on the vertex, edge and face sets. In this case one easily sees that, as for graphs, the inverse map \( f^{-1} \) is also an isomorphism \( f^{-1} : Y \to X \) (just reverse the horizontal arrows in the left commuting diagram of Definition 1.22(1)!) so that the set of automorphisms \( f : X \to X \) forms a group \( \text{Aut}(X) \) under composition. A group action \( G \overset{\varphi}{\to} \text{Aut}(X) \) then preserves orientation, is without inversion and free, when the underlaying map of graphs has these properties.

A subcomplex is a mapping \( X \hookrightarrow Y \) of 2-complexes that is an isomorphism onto its image.

Exercise 1.25. Formulate an equivalent version of the definition of a subcomplex \( X \) of \( Y \), with the sets of \( k \)-cells \( X^k \) of \( X \) as subsets of the sets of \( k \)-cells \( Y^k \) of \( Y \).

1.2.4 Homotopies and homeomorphisms

As with graphs in §1.1.3 we can deform paths in a combinatorial manner, mimicking the continuous homotopies of paths in topology.

Let \( e_1 \ldots e_\ell \) be a path in the 2-complex \( X \). An elementary homotopy either inserts or deletes a spur as in §1.1.3 or inserts or deletes the boundary of a face in the following sense. If \( \sigma \in X^2 \) is a face with boundary \( \partial \sigma = (X^\sigma, a^\sigma) \), then by Exercise 1.13, \( a^\sigma(X^\sigma) \)
is a closed path, say \(e'_1 \ldots e'_k\). The homotopy then inserts into (or deletes from) the path the result of completely traversing the closed path, starting at one of its vertices, so that all the incidences match up in the obvious way, i.e.: so that \(s(e'_j) = t(e'_{j-1})\) is the vertex \(t(e_i) = s(e_{i+1})\).

We make two remarks before proceeding. The first is that the pointing of the face attaching maps plays no role: if \(a^\sigma: X^u_\sigma \rightarrow X^v_\sigma\) is the attaching map then the vertex \(v\) lies in the closed path \(a^\sigma(X^\sigma)\), but we do not insist that the traversal of the boundary start and finish at this vertex. The other is that pictures such as the right hand side of Figure 1.16 should be approached with care. The entire boundary path of \(\sigma\) must be traversed, including any repetitions. Later we will have faces with boundary a closed path that travels \(n\) times, say, around an edge loop. Any homotopy involving this boundary must then travel the full \(n\) time around the loop.

Two paths are \textit{homotopic} if and only if there is a finite sequence of elementary homotopies, taking one to the other. A path homotopic to the trivial path based at one of the vertices it passes through is said to be \textit{homotopically trivial}. For example, two paths running different ways around a face are homotopic as shown in Figure 1.17. To get from the first picture to the second, insert the boundary of the face; to get from the second to the third, remove the obvious spurs.

**Exercise 1.26.** Show that homotopic paths have the same start and end vertices, and thus homotopically trivial paths are necessarily closed. Show that homotopy is an equivalence relation on the paths with common fixed endpoints.

We can also subdivide 2-complexes to get homeomorphic ones, although this will play less of a role than it does with graphs (graphs homeomorphic to \(S^1\) were essential to the definition of 2-complex). What we want to do is summarized by Figure 1.18: replace an existing face \(\sigma\) by two new faces \(\sigma_1, \sigma_2\) by creating a new arc running between vertices of \(\sigma\). We are then free to choose new distinguished vertices for the \(\sigma_i\).

**Exercise 1.27.** Formulate the definition of subdividing a face in terms of Definition 1.20 by using the description of 1-spheres given by Lemma 1.11.
Write \( X \leftrightarrow X' \) when the two complexes differ by the subdivision of an edge or face, so that \( X \) and \( Y \) are then homeomorphic, written \( X \approx Y \), when there is a finite sequence \( X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_k = Y \) of subdivisions (of either type) connecting them. It is easy to see that homeomorphism is an equivalence relation and we have topological invariants for 2-complexes as well as graphs.

Figure 1.19 shows a series of subdivisions of the 2-sphere.

1.3 Aside: comparison with “proper” topology

1.3.1 The category of CW complexes

1.4 Quotients of 2-complexes

Often we want to squash unwanted parts of a complex away, glue complexes together, factor out the action of a group, and so on. In otherwords, we want to be able to take quotients of complexes. We do this by defining an equivalence relation on the cells of the complex, and then defining a new complex whose cells are the equivalence classes of the relation. It turns out that there are a number of subtleties complicating the exposition, arising if we want to be able to identify cells of different dimensions.

1.4.1 Quotients in general

All quotients start with an equivalence relation:

**Definition 1.28 (quotient relation on a 2-complex).** If \( X \) is a 2-complex, then a quotient relation is an equivalence relation on the vertices, edges, paths and faces of \( X \) such that

1. \( \sim \) restricted to the 1-skeleton \( X^{(1)} \) is a graph quotient relation as in Definition 1.7, with quotient map of graphs \( q : X^{(1)} \rightarrow X^{(1)}/\sim \);
2. if \( \Omega \) is an equivalence class of \( \sim \) with \( \Omega \subset X^2 \), then there is a \((X^\Omega, a^\Omega)\) with \( X^\Omega \approx S^1 \) and \( a^\Omega : X^\Omega \rightarrow (X^{(1)})_/\sim \), dimension preserving, such that for all faces \( \sigma \in \Omega \) with \( \partial \sigma = (X^\sigma, a^\sigma) \), there is an isomorphism \( X^\sigma \rightarrow X^\Omega \) making the diagram below left commute:

\[
\begin{array}{ccc}
X^\sigma & \xrightarrow{\cong} & X^\Omega \\
a^\sigma & \downarrow & a^\Omega \\
X^{(1)} & \rightarrow & (X^{(1)})_/\sim \\
\end{array}
\]

\[
\begin{array}{ccc}
X^\sigma & \rightarrow & S \\
a^\sigma & \downarrow & q \\
X^{(1)} & \rightarrow & (X^{(1)})_/\sim \\
\end{array}
\]

3. if \( \Omega \) is an equivalence class containing a face but with \( \Omega \not\subset X^2 \), then there is a homotopically trivial path \( S \rightarrow (X^{(1)})_/\sim \) such that for all faces \( \sigma \in \Omega \) there is a map \( X^\sigma \rightarrow S \) making the diagram above right commute.

Part 2 is just saying that equivalent faces have boundaries that fold up in the quotient graph to give the same thing. Similarly, if a face is to be identified with a closed path then part 3 forces the boundary of the face to be identified with it as well. The homotopically trivial condition is a little obscure at the moment: it’s role will become clearer in Chapter 2.

**Definition 1.29 (quotient 2-complex).** If \( \sim \) is a quotient relation on the 2-complex \( X \) then define the quotient \( X/\sim \) as follows

1. the 1-skeleton \((X/\sim)^{(1)}\) is the quotient graph \((X^{(1)})_/\sim\) with quotient map of graphs \( q : X^{(1)} \rightarrow (X^{(1)})_/\sim\);
2. the faces \((X/\sim)^2\) consist of those equivalence classes \( \Omega \) with \( \Omega \subset X^2 \). Such a face has boundary \( \partial \Omega = (X^\Omega, a^\Omega) \) as given by Definition 1.28 part 2.

**Proposition 1.30.** If \( \sim \) is a quotient relation then \( X/\sim \) is a 2-complex and the quotient map \( q : X \rightarrow X/\sim \) given by \( q(x) = [x] \) is a map of 2-complexes.

**Proof.** That the quotient is a 2-complex is immediate from definition 1.29, and the commuting diagrams given there are precisely the definition of \( q \) being a map of 2-complexes. \( \square \)

### 1.4.2 Group actions and their quotients

When a group \( G \) acts on a 2-complex \( X \) we can replace \( X \) by a complex on which the action of \( G \) is trivial, ie: every element of \( G \) acts as the identity. Thus we may factor out group actions, and we do this by forming a quotient. Recall from §1.2.3 the group \( \text{Aut}(X) \) of automorphisms of the 2-complex \( X \), and a group action is a homomorphism \( G \xrightarrow{\phi} \text{Aut}(X) \).

Let \( \sim \) be the equivalence relation on \( X \) given by the orbits of the action, so that \( x \sim y \) iff \( y = g(x) \) for some \( g \in G \) (and then necessarily \( x \) and \( y \) have the same dimension).

**Proposition 1.31 (quotient by a group action).** Let \( \sim \) be the equivalence relation on the 2-complex \( X \) given by the orbits of a group action. Then \( \sim \) is a quotient relation if and only if the group action is orientation preserving.

We write \( X/G \) for the quotient complex \( X/\sim \). Thus, although one may in principle consider group actions that don’t preserve orientation, as the primary purpose of such actions is to form quotients, we will only consider orientation preserving actions. Compare this with Proposition 1.9, noting how the 2-complex structure imposes no new conditions for \( \sim \) to be a quotient relation.
Proof. We have a quotient relation on the 1-skeleton $X^{(1)}$ if and only if the action preserves orientation by Proposition 1.9. This leaves us with the second part of Definition 1.28 to worry about. If $\Omega \subset X^2$ is an equivalence class of faces, fix a face $\sigma \in \Omega$, and let $(X^{(1)}, a^{(1)}) = (X^{(1)}, qa^{(1)})$, where $q$ is the graph quotient map. If $\tau \sim \sigma$ then $\sigma = g(\tau)$ for some $g \in G$, so Definition 1.22 part 1 gives an isomorphism $X^{(1)} \to X^{(1)}$ with the diagram below left commuting:

\[
\begin{array}{ccc}
X^\tau & \cong & X^\sigma \\
\downarrow a^\tau & & \downarrow a^\sigma \\
X^{(1)} & \cong & X^{(1)} \\
\end{array}
\]

The triangular diagram above right commutes by the nature of the quotient map: if $y = g(x)$ then $[x] = [y]$ so that $q(x) = q(y)$. Now stitch the two diagrams together, so that $(X^{(1)}, a^{(1)}) = (X^{(1)}, qa^{(1)})$ does the job. $\square$

Consider as an example Figure 1.20, where a Euclidean plane complex is rolled into an infinite tube by the action of the integers $\mathbb{Z}$.

![Diagram](image)

**Fig. 1.20.** Let $\mathbb{Z}$ act on the $X$ (left) by mapping $1 \in \mathbb{Z}$ to the automorphism that translates $X$ one step to the right, as shown by the red arrow: The quotient $X/\mathbb{Z}$ is an infinite rolled up tube. If $m \in \mathbb{Z}$, then its effect on $X$ is to translate $m$ steps to the right, whereas its effect on $X/\mathbb{Z}$ is to rotate a cell $m$ times around the tube, bringing it back to itself.

**Exercise 1.32.** In the definition of the quotient $X/G$ we took $(X^{(1)}, a^{(1)}) = (X^{(1)}, qa^{(1)})$ for some fixed $\sigma \in \Omega$. Show that we are free to choose instead a different face from $\Omega$: if $\tau \sim \sigma$ and we take $(X^{(1)}, a^{(1)}) = (X^{(1)}, qa^{(1)})$ instead, then this new version of $X/G$ is isomorphic to the old one by the identity map.

1.4.3 Quotients by a subcomplex

Now for a quotient that involves some serious squashing: if $Y \hookrightarrow X$ is a subcomplex we define a new complex where $Y$ has been compacted down to a single vertex.

Define $\sim$ on $X$ to be the equivalence relation with the following equivalence classes: (i). all the cells in $Y$ (of whatever dimension) form one class; (ii). every other class has the form $[x] = \{x\}$. Thus, we have $x \sim y$ if and only if either $x = y$ or both $x$ and $y$ lie in the subcomplex $Y$.

**Proposition 1.33.** The relation $\sim$ defined above is a quotient relation.

Write $X/Y$ for the corresponding quotient, the **quotient of $X$ by the subcomplex $Y$**: it is what results from collapsing $Y$ to a vertex and propagating the effects of this on the incidence of cells throughout $X$, but otherwise leaving the cells of $X$ unaffected.
Proof. ~ is clearly a graph quotient relation on the 1-skeleton. If Ω is an equivalence class containing faces then it either contains a single face σ (and so let (X^ω, a^ω) = (X^σ, a^σ)) or is all of Y (in which case Definition 1.28 part 3 kicks in, so let S → (X/Y)^{(1)} consist of a single vertex for S mapping to the vertex that is Y). □

A typical quotient by a subcomplex arises when T ↪ X is a spanning tree for the 1-skeleton and we form X/T as in Figure 1.21.

![Figure 1.21. squashing a spanning tree down to a vertex](image)

**1.4.4 Pushouts**

The pushout is a rather general construction which arises whenever a pair of complexes are glued together across a common subcomplex. Before giving the definition we need some preliminary notions about equivalence relations which may or may not be well known to the reader. We place them in an exercise:

**Exercise 1.34.** Recall that the formal definition of an equivalence relation on a set X is a subset S ⊂ X such that, (i) S contains the diagonal, (x, x) ∈ S for all x ∈ X, (ii) S is symmetric, (x, y) ∈ S ⇒ (y, x) ∈ S, and (iii) (x, y), (y, z) ∈ S ⇒ (x, z) ∈ S. Show that if S_1, S_2 are equivalence relations on X then so is S_1 ∩ S_2, and hence if Y is any subset of X we may define the **equivalence relation generated by Y** to be the intersection of all equivalence relations S with Y ⊂ S.

**Definition 1.35 (pushout).** Let X_1, X_2 and Y be 2-complexes and g_i : Y → X_i maps of 2-complexes. Let ~ on the disjoint union X_1 ∐ X_2 be the equivalence relation generated by the x ~ y if there is a z ∈ Y with x = g_1(z) and y = g_2(z). If ~ is a quotient relation then call the quotient X_1 ∐ X_2/~ the **pushout** of the data g_i : Y → X_i, and denote it by X_1 ∐_Y X_2.

Figure 1.22 illustrates the schematic setup:

**Exercise 1.36.** If ~ is the relation described in Definition 1.35, show that x ~ z iff there are x_0, x_1, . . . , x_k ∈ X_1 ∐ X_2 with x_0 = x and x_k = z, and y_1, . . . , y_k ∈ Y, such that g_1(y_1) = x_0, g_2(y_1) = x_1, g_2(y_2) = x_1, g_1(y_2) = x_2, . . . and so on.

Define t_i : X_i → X_1 ∐ Y X_2 to be the compositions X_i ↪ X_1 ∐ X_2 ↪ X_1 ∐ X_2/~ of the inclusion of X_i in the disjoint union and the quotient map.

**Proposition 1.37.** If Y ≠ ∅, the X_i are connected and the ~ of Definition 1.35 is a quotient relation, then the pushout is connected, and the maps t_i make the diagram on the left commute.
Moreover if the $g_i$ are dimension preserving, the pushout is universal in that if $Z, t'_1, t'_2$ are a 2-complex and maps making such a square commute, then there is a map $h : X_1 \coprod_Y X_2 \to Z$ making the diagram above right commute.

The proposition gives a clue as to why the name pushout: the data $g_i : Y \to X_i$ forming the input to the construction gives the two sides of the commutative square top left, and $X_1 \coprod_Y X_2$ “pushes out” these two sides to complete the square. The proof of the proposition, while routine, is a good test of all the definitions so far, so we go through it in some detail.

Proof. Let $v$ be a vertex of $Y$ and $v_1 = g_i(u) \in X_i$. By the connectedness of the $X_i$, vertices in either one of them can be joined by a path to the appropriate $v_i$. Suppose that $[x_i]$ are vertices in the pushout with $x_i \in X_i$. Then the image under $t_1$ of a path from $x_1$ to $v_1$ finishes at the vertex $[v_1]$ and joins up with the image of a path under $t_2$ from $v_2$ to $x_2$, since $[v_1] = [v_2]$, and so the pushout is connected. If $y \in Y$ then its image under the two maps $Y \to X_i \hookrightarrow X_1 \coprod_Y X_2$ are equivalent by the definition of $\sim$. Thus $t_1 g_1 (y) = t_2 g_2 (y)$ and the square commutes.

Suppose now that we have complexes and maps $Z, t'_i$ making the outside square on the right hand side commute. After a moments thought it is obvious what the map $h : X_1 \coprod_Y X_2 \to Z$ ought to be: every cell of the pushout has the form $[x]$ for $x$ some cell of one of the $X_i$. Define $h[x] = t'_i(x)$ when $x \in X_i$. We leave it to the reader to show that we have a well defined map, and also a map of graphs.

To finish the verification that $h$ is a map, it is generally easier, rather than showing Definition 1.22 directly, to use the reformulation where boundaries of faces are closed paths in the 1-skeleton and a map must send boundaries to boundaries. If $\Omega = [\sigma]$ is a face of the pushout with $\sigma \in X_1$ say, then $t_1$ maps $\sigma$ to $\Omega$. But then the boundary of $\sigma$ must map to the boundary of $t'_1(\sigma)$, and as the bottom triangle in the diagram above commutes, this forces the boundary of $\Omega$ to be mapped to the boundary of $t'_1(\sigma)$ as well.

But we also show $h$ is a map via Definition 1.22, to satisfy ourselves that such things are possible (and as a consequence, to see why we won’t tend to in the future!). Because the $g_i$ are dimension preserving, the image under $h$ of a face $\Omega = [\sigma]$ is always a face, so we just have part 1 of Definition 1.22 to check. Suppose for specificness that $\sigma$ is a face of $X_1$, let $\tau = t'_1(\sigma)$, and consider the following diagram:

$$
\begin{align*}
X^- & \cong X^\sigma \cong X^\Omega \\
\downarrow & \quad \downarrow \\
Z^{(1)} & X_1^{(1)} \to (X_1 \coprod_Y X_2)^{(1)}
\end{align*}
$$

The lefthand square commutes and is given to us courtesy of the map $t'_i$ and Definition 1.22 part 1; the righthand square commutes and is a product of Definition 1.28 part 2 applied to
the quotient that is the pushout. The map \( h \), goes from the bottom right to the bottom left complex, and it is, by definition, the map making the triangle formed with the maps along the bottom edge, commute. Define a map \( \varepsilon \) from the top right to the top left complexes to be the composition of one of the isomorphisms with the inverse of the other. We leave it as an exercise in diagram chasing to see that the large square around the outside commutes, i.e: that \( a^T \varepsilon = h f^T \). This is exactly what we need for \( h \) to be a map.

Exercise 1.38 (pointed pushout). Suppose the \( g_i : Y \rightarrow (X_i)_{x_i} \) are maps of pointed complexes. If \( q : X_1 \coprod X_2 \rightarrow (X_1 \coprod X_2)/\sim \) is the quotient map, let \( x = q(x_1) = q(x_2) \), and call \( (X_1 \coprod Y \coprod X_2)_{x} \) the pointed pushout. Show that we have a Proposition analogous to Proposition 1.37 with all complexes and maps pointed.

We now come to the slightly delicate matter of the actual existence of pushouts. Given the setup of Definition 1.35 the pushout cannot always be formed, precisely because the quotient cannot always be formed. This is true even for graphs, as Figure 1.23 illustrates.

Fig. 1.23. Data \( g_i : Y \rightarrow X_i \) for which the pushout does not exist. Here \( X_1 = X_2 = X \) and \( Y \) are all graphs with a single edge joining two vertices and the maps \( g_1 \) and \( g_2 \) send the edge of \( Y \) to the edge \( e \) and its inverse \( e^{-1} \) respectively. The resulting equivalence relation on \( X \) has \( e \sim e^{-1} \) and so is not a quotient relation.

A complete solution to the existence of pushouts when \( Y \) is a graph is given in Exercise 1.39.

Exercise 1.39. Show that if \( Y \) is a graph and the \( g_i \) are dimension preserving, then the pushout exists if and only if there are orientations \( O, O_i \) for \( Y, X_i \) with \( g_i(O) \subseteq O_i \). Thus in particular, if the graphs \( g_1(Y) \) and \( g_2(Y) \) are disjoint, then the pushout can always be formed.

A simple example of the situation of Example 1.39 is the Stallings fold, shown in Figure 1.24. Here the graph \( Y \) is a single edge joining two vertices. An even simpler one, shown in Figure 1.25, is the wedge of a pair of complexes: \( Y \) is now the trivial complex consisting of just a single vertex.

Fig. 1.24. Stallings fold: here \( Y \) is a single edge \( e \) joining two distinct vertices, and \( X_1 = X_2 = X \), \( g_i : Y \rightarrow X, (i = 1, 2) \) with \( g_1(s(e)) = g_2(s(e)) \) and \( g_1(e) \neq g_2(e)^{-1} \).

Things are less simple when the \( X_i \) are 2-complexes. Nevertheless, pushouts exist most of the time:

Proposition 1.40. If \( X_1, X_2 \) and \( Y \) are 2-complexes and \( g_i : Y \rightarrow X_i \) dimension preserving maps, with orientations \( O, O_i \) for \( Y, X_i \) such that \( g_i(O) \subseteq O_i \), then the pushout exists.


Proof. The relation ~ of Definition 1.35 is a quotient relation on the 1-skeleton by Exercise 1.39. As the \( g_i \) are dimension preserving, all the cells in an equivalence class have the same dimension, and we are thus left with part 2 of Definition 1.28 to do. Let \( \Omega \) be an equivalence class of faces, fix a \( \tau \in \Omega \), and take \((X^\Omega, f^\Omega) = (X^\tau, qa^{\tau})\), with \( qa^{\tau} \) dimension preserving as the \( g_i \) are. Suppose that \( \sigma \in \Omega \) and that we are in the special case that \( \sigma = g_1(x) \), \( \tau = g_2(x) \) for some face \( x \) in \( Y \). Then we get a commutative diagram very similar to the one in the proof of Proposition 1.37,

\[
\begin{array}{c}
X_1^\sigma \xrightarrow{\sim} Y^\tau \xrightarrow{\sim} X_2^\tau \\
(X_1 \coprod X_2)^{(1)} \xrightarrow{Y^{(1)}} (X_1 \coprod X_2)^{(1)}
\end{array}
\]

by splicing together the diagrams supplied by the maps \( g_1, g_2 \) and a diagram for the maps \( g'_i : Y \to X_i \leftarrow X_1 \coprod X_2 \) where \( qg'_1 = qg'_2 \). The bottom left and righthand complexes both map to the pushout, making a commuting square that attaches to the bottom of the diagram, and we take the composition of the two isomorphisms to give an isomorphism \( X_1^\tau \to X_2^\tau \). The reader can then check that the outside circuit of this large diagram is what we need to verify part 2 of Definition 1.28.

Fig. 1.26. Initial data for a typical pushout of Chapter 3: \( Y \) has six vertices, edges and faces, \( X_1 \) has three of everything and \( X_2 \) has two of everything. The boundaries of all faces are hexagonal, but we’ve only shown one in each case; the others are identical, with the distinguished vertices exhausting the six, three and two possibilities respectively. The attaching maps wrap around the 1-skeletons as shown.

In general, when \( \sigma \sim \tau \) we have \( \sigma = \sigma_0 = g_1(x_1), \sigma_1 = g_2(x_1), \ldots, \sigma_{k-1} = g_1(x_k), \tau = \sigma_k = g_2(x_k) \), and the requirements for a quotient relation can be verified by repeatedly applying the process of the previous paragraph. \( \square \)
Pushouts will really prove their mettle in Chapters 3-4, where the \( g_i \) will be coverings. Figure 1.26 illustrates the kind of initial set-up we will have, and Figure 1.27 the resulting pushout.

![Diagram](image1.png)

**Fig. 1.27.** Pushout resulting from the set-up in Figure 1.26. There is a single vertex, edge and face in the quotient, and the face has a hexagonal boundary with the attaching map wrapping it around the 1-skeleton six times as shown.

Our final example of a pushout arises when one of the maps \( g_i \) is just an inclusion, so that the initial data consists of two complexes \( X_1 \) and \( X_2 \), and a map \( g \) from a subcomplex of \( X_1 \) to \( X_2 \). The resulting pushout (when it exists) is the result of gluing \( X_1 \) and \( X_2 \) together via the attaching map \( g \). See Figure 1.28.

![Diagram](image2.png)

**Fig. 1.28.** Glueing complexes together via an attaching map.

### 1.5 Pullbacks and Higman composition

We now come to a pair of constructions which both start with the roughly the same kind of input data: a complex \( Y \), a family of complexes \( X_i \), and a family of maps \( g_i : X_i \to Y \). The first of these, the pullback, is dual to the pushout, or, to use categorical terminology, is a “co”-pushout. It is essentially what we get if we reverse the directions of all the maps in the description of the pushout. Pullbacks, like pushouts, will play a crucial role in the theory of coverings of complexes in Chapters 3-4; pullbacks will act like a kind of “union” of complexes and pushouts like a kind of “intersection”. In §3.4.3 we will be able to be much more precise about what we mean by this.

The other construction, Higman composition, is less well known, and can be performed only in very special circumstances. Nevertheless, when possible, it will prove extremely powerful, and this makes its inclusion worthwhile.

#### 1.5.1 Pullbacks

It is easier to do graphs first, then extend to 2-complexes proper:
**Definition 1.41 (pullbacks of graphs).** Let $X_1$, $X_2$ and $Y$ be graphs and $g_i : X_i \to Y$ maps of graphs. The **pullback** $X_1 \prod_Y X_2$ has vertices (respectively edges) the $x_1 \times x_2$, $x_i \in X_i^0$ (resp. $x_i \in X_i^1$), such that $g_1(x_1) = g_2(x_2)$. The incidence maps are $s(e_1 \times e_2) = s(e_1) \times s(e_2)$, and $(e_1 \times e_2)^{-1} = e_1^{-1} \times e_2^{-1}$. See Figure 1.29.

![Fig. 1.29. Construction of the pullback for graphs: the vertices $u_i$ and $v_i$ map via the $g_i$ to $u$ and $v$, and the edges $e_i$ map via the $g_i$ to $e$. Thus in the pullback we get vertices $u_1 \times u_2$ and $v_1 \times v_2$ joined by an edge $e_1 \times e_2$.](image)

We now set up for the pullback of 2-complexes by seeing how the boundaries of faces in the $X_i$ behave when we pullback the 1-skeletons using this recipe. Suppose that the $g_i$ are dimension preserving, and that $\sigma$ is a face of $Y$ and $\sigma_i$ faces of the $X_i$, mapping to $\sigma$ via the $g_i$. We get a by now familiar commuting diagram:

\[
\begin{array}{ccc}
X_1^{\sigma_1} & \cong & Y^{\sigma} & \cong & X_2^{\sigma_2} \\
\alpha^{\sigma_1} & \varepsilon_1 & \hspace{1cm} & \varepsilon_2 & \alpha^{\sigma_2} \\
X_1^{(1)} & \hspace{1cm} & Y^{(1)} & \hspace{1cm} & X_2^{(2)}
\end{array}
\]

Suppose also that as we move clockwise around the sphere $Y^\sigma$ the edges are $e_1, e_2, \ldots, e_k$. Then the edges of the $X_i^{\sigma_i}$ can be labelled $e_{i1}, e_{i2}, \ldots, e_{ik}$, with $e_j = \varepsilon_i(e_{ij})$ and the closed path $\alpha^{\sigma_1} e_{i1}, \alpha^{\sigma_1} e_{i2}, \ldots, \alpha^{\sigma_1} e_{ik}$ the set of boundary edges for $\sigma_i$ with $\alpha^{\sigma_1} e_{ij}$ mapping via $g_i$ to $\alpha^\sigma e_j$ in $Y$.

The upshot is that the pullback contains a path of edges

\[
a^{\sigma_1} e_{11} \times a^{\sigma_2} e_{21}, a^{\sigma_1} e_{12} \times a^{\sigma_2} e_{22}, \ldots, a^{\sigma_1} e_{1k} \times a^{\sigma_2} e_{2k},
\]

(1.1)

and since $s a^{\sigma_1} e_{11} = ta^{\sigma_1} e_{1k}$ and $s a^{\sigma_2} e_{21} = ta^{\sigma_2} e_{2k}$, this path in the pullback is closed. The idea is to “sew a face” into the 1-skeleton having boundary this closed path, by taking $Y^\sigma$ and using the attaching map $a^{\sigma_1} \varepsilon_1^{-1} \times a^{\sigma_2} \varepsilon_2^{-1}$ that sends $e_j$ to $a^{\sigma_1} e_{1j} \times a^{\sigma_2} e_{2j}$. Moreover, all the pointings are respected, so that if the pointed maps $a^{\sigma_1}$ send vertices $u_i$ to $v_i$, and $a^{\sigma_2}$ sends $u$ to $v$, then the attaching map in the pullback sends $u$ to $v_1 \times v_2$.

**Definition 1.42 (pullbacks of 2-complexes).** Let $X_1$, $X_2$ and $Y$ be 2-complexes and $g_i : X_i \to Y$ dimension preserving maps of 2-complexes. The **pullback** $X_1 \prod_Y X_2$ has $\ell$-dimensional cells the $x_1 \times x_2$, for $x_i \in X_i^\ell$, such that $g_1(x_1) = g_2(x_2)$. The incidence maps are $s(e_1 \times e_2) = s(e_1) \times s(e_2)$, $(e_1 \times e_2)^{-1} = e_1^{-1} \times e_2^{-1}$ and

\[
\partial(\sigma_1 \times \sigma_2) = (Y^\sigma, a^{\sigma_1} \varepsilon_1^{-1} \times a^{\sigma_2} \varepsilon_2^{-1}),
\]

where $g_1(\sigma_1) = \sigma = g_2(\sigma_2)$, and the $\varepsilon_i$ are the isomorphisms in the diagram above.
We get a schematic like Figure 1.30 whenever \( g_1(\sigma_1) = g_2(\sigma_2) \). For \( i = 1, 2 \) we now define maps

\[
t_i : X_1 \prod_Y X_2 \rightarrow X_i,
\]

as follows. Each cell of the pullback has the form \( x_1 \times x_2 \) with the \( x_i \in X_i \), so let \( t_i(x_1 \times x_2) = x_i \). If \( \sigma_1 \times \sigma_2 \) is a face of the pullback, then writing \( X = X_1 \prod_Y X_2 \), we have \( X^{\sigma_1 \times \sigma_2} = Y^\sigma \) for the boundary of this face, where \( g_1(\sigma_1) = \sigma = g_2(\sigma_2) \in Y \). For the isomorphism

\[
X^{\sigma_1 \times \sigma_2} \xrightarrow{\cong} X_1^{\sigma_1},
\]

we take \( \varepsilon_i^{-1} : Y^\sigma \rightarrow X_1^{\sigma_1} \). We leave it as an exercise for the reader to show that the diagram of Definition 1.22 part 1 commutes, so that the \( t_i \) are indeed mappings of 2-complexes.

**Proposition 1.43.** The \( t_i \) are dimension preserving maps making the diagram below left commute,

Moreover, the pullback is universal in that if \( Z, t'_1, t'_2 \) are a 2-complex and maps making such a square commute, then there is a map \( Z \rightarrow X_1 \prod_Y X_2 \) making the diagram above right commute.

**Proof.** We have \( g_1t_1(x_1 \times x_2) = g_2t_2(x_1 \times x_2) \), and writing \( X = X_1 \prod_Y X_2 \), the isomorphism \( X^{\sigma_1 \times \sigma_2} \rightarrow Y^\sigma \) for both \( g_i t_i \) is the composition,

\[
X^{\sigma_1 \times \sigma_2} \xrightarrow{\varepsilon_1^{-1}} X_1^{\sigma_1} \xrightarrow{\varepsilon_1} Y^\sigma,
\]

hence identical (to the identity map). If \( z \) is a cell of \( Z \), then the commuting of the large square on the outside of the righthand diagram gives that \( g_1 t'_1(z) = g_2 t'_2(z) \), so that \( t'_1(z) \times t'_2(z) \) is a cell of the pullback. Define \( Z \rightarrow X_1 \prod_Y X_2 \) by \( z \mapsto t'_1(z) \times t'_2(z) \), and for the isomorphism \( Z^\sigma \rightarrow X_1^{t'_1(\sigma)} \times X_2^{t'_2(\sigma)} = Y^{g_i t'_i(\sigma)} \), take the composition

\[
Z^\sigma \xrightarrow{\cong} X_1^{t'_1(\sigma)} \xrightarrow{\cong} Y^{g_i t'_i(\sigma)},
\]

(necessarily identical for \( i = 1, 2 \) as the square commutes) provided by the maps \( t'_i \) and \( g_i \). We leave it to the reader to check that this is a map of 2-complexes with the required properties. \( \square \)

Comparison with the pushout reveals good news and bad news. The good news is that the construction of the pullback is built into the definition, so there is never any question about the existence of pullbacks. Pullbacks always exist. The bad news is that pullbacks are not in general connected. The following goes some way to alleviating this:
Exercise 1.44 (pointed pullbacks). Suppose the \( g_i(X_i)_x \rightarrow Y \) are maps of pointed complexes. Then \( x = x_1 \times x_2 \) is a vertex of the pullback. Write \( (X_1 \coprod_Y X_2)_x \) for the connected component of the pullback containing \( x_1 \times x_2 \). Show that we have a Proposition analogous to Proposition 1.43 for the pointed pullback, with all complexes and maps pointed.

Fig. 1.31. Initial data for a typical pullback of Chapters 3-4: \( Y \) has six vertices, edges and faces, \( X_1 \) has three of everything and \( X_2 \) has two of everything. The boundaries of all faces are hexagonal, but we’ve only shown one in each case; the others are identical, with the distinguished vertices exhausting the six, three and two possibilities respectively. The attaching maps wrap around the 1-skeletons as shown.

Fig. 1.32. Pullback resulting from the set-up in Figure 1.26. There is a single vertex, edge and face in the quotient, and the face has a hexagonal boundary with the attaching map wrapping it around the 1-skeleton six times as shown.

If the reader needed any more convincing about the duality between pullbacks and pushouts then we can run the example of §1.4.4 backwards and use the pullback to get back to where we started. Thus in Figure 1.31 we have the same two complexes \( X_1, X_2 \) as in the pushout example, but this time \( Y \) is the end result of that example and the maps \( g_i \) are the \( t_i \) from Proposition 1.37. The resulting pullback, shown in Figure 1.32 is the starting point of the pushout example, and the maps \( t_i \) given by Proposition 1.43 are the starting \( g_i \) from that example. Notice how the number of faces proliferates (rather than declines as it does in the pushout): \( X_1 \) has three, each of which can be paired up with the two in \( X_2 \), giving six faces in the pullback. As the distinguished vertices cycle through the three vertices of \( X_1 \) (resp. two in \( X_2 \)), their products cycle through the six vertices of \( X_1 \coprod_Y X_2 \).

1.5.2 Higman composition

1.6 Notes on Chapter 1
Fig. 1.33. handle
Coverings

We return to the combinatorial topology of $2$-complexes and develop their covering space theory.

3.1 Basics

3.1.1 Coverings

**Definition 3.1 (covering of $2$-complexes).** A map $f : Y \to X$ of $2$-complexes is a covering if and only if

1. $f$ preserves dimension;
2. for every pair of vertices $u$ and $v$, with $f(u) = v$, the local continuity (see §1.1.1) of $f$ at $u$,
   $$s_Y^{-1}(u) \to s_X^{-1}(v),$$
   is a bijection.
3. for every pair of vertices $u$ and $v$, with $f(u) = v$, and face $\tau$ of $X$, the local continuity (see Definition 1.23) of $f$ at $v$,
   $$\prod_{f(\sigma) = \tau} \prod_{\sigma} (a^\tau)^{-1}(u) \to (a^\tau)^{-1}(v)$$
   is a bijection.

The terminology cover and lift is used for images and pre-images of a covering map. If $f(y) = x$, then one says that $y$ covers $x$, or that $x$ lifts to $y$. The set of all lifts of $x$, or alternatively the set $f^{-1}(x)$ of all cells covering $x$, is its fiber.

Note that each of the $\varepsilon_\sigma$ is an isomorphism, so the local continuity map is a disjoint union of a set of maps, each of which is the restriction of a bijection. In any case, each is individually an injection.

The last part of each definition express “local isomorphism” properties of covering maps: if $u$ is a vertex covering $v$, then $Y$ looks the same near $u$ as $X$ does near $v$. Specifically, the configuration of edges around a vertex looks the same both upstairs and downstairs (Figure 3.1 left): for every vertex $u$ of $Y$, the covering $f$ is a bijection from the set of edges in $Y$ with start vertex $u$ to the set of edges in $X$ with start vertex $f(u)$.

Given a vertex $v$ and a face $\tau$ downstairs containing the vertex in its boundary, this face looks the same near $v$ as its lifts do near any vertex $u$ covering $v$: if $\tau$ contains $u$ in its boundary $k$ times, so there are $k$ “wedge-shaped” pieces of $\tau$ fitting together around $v$, then there are $k$ wedge-shaped pieces fitting together around $u$, where these wedges belong to faces $\sigma$ that cover $\tau$ (see Figure 3.1 right). Note that the wedges upstairs don’t necessarily belong to distinct faces.
Part 3 of the definition gives in particular that
\[ \sum_{f(\sigma) = \tau} |(a^\sigma)^{-1}(u)| = |(a^\tau)^{-1}(v)|, \]
so that \( v \) appears the same number of times in the boundary of \( \tau \) as \( u \) does in the boundaries of all the faces \( \sigma \) in the fiber of \( \tau \).

One commonly sees the assumption that in a covering, both the covering complex \( Y \) and the covered complex \( X \) are connected, but we won’t assume this at the moment. Indeed, we will find it positively useful in some situations to not assume that a covering complex be connected.

**Exercise 3.2.** Let \( f : Y \to X \) be a covering and \( Y^\circ \) a connected component of \( Y \). Show that restricting \( f \) to \( Y^\circ \) gives a covering. Show that we may not restrict a covering to an arbitrary subcomplex and still get a covering.

**Fig. 3.2.** A simple graph covering: the two vertices of \( Y \) cover the single vertex of \( X \) and the two arcs of \( Y \) similarly.

**Fig. 3.3.** The graph covering of Figure 3.2 extended to a covering of 2-complexes: \( Y \) is now the 2-sphere with the face \( \sigma_i \) having \( u_i \) as its distinguished vertex. The \( \sigma_i \) cover the single face of \( X \), which, unlike in Figure 3.2, has been drawn face-centrally.

Figure 3.2 shows a very simple graph covering with the red path on the left covering the red path on the right. Figure 3.3 extends this to a covering of 2-complexes.
Exercise 3.3. Call a map \( f : Y \to X \) of graphs an immersion when it preserves dimension and the local continuity maps are injections; call a map of 2-complexes an immersion if the restriction to the 1-skeletons is an immersion and the local continuity maps in Definition 3.1 part 3 are injections. Give examples of immersions that are not coverings.

Exercise 3.4. Show that for any \( X \) the identity map \( X \to X \) is a covering.

3.1.2 Lifting

When we have a covering \( f : Y \to X \), the complexes \( X \) and \( Y \) look the same as long as we restrict our attention to small pieces. If two complexes look the same as each other then we should be able to pull back parts of \( X \) to find parts of \( Y \) mapping to them. Putting these together, when we have a covering we should be able to pull back small pieces of \( X \) and get small pieces of \( Y \) covering them. This process is called lifting. The small pieces turn out to be paths and faces.

Proposition 3.5 (path and spur lifting). Let \( f : Y \to X \) be a covering with \( f(u) = v \) vertices.

1. If \( \gamma \) is a path in \( X \) starting at \( v \) then there is a path \( \tilde{\gamma} \) in \( Y \) starting at \( u \) and covering \( \gamma \). Moreover, if \( \gamma_1, \gamma_2 \) are paths in \( Y \) starting at \( u \) and covering the same path in \( X \), then \( \gamma_1 = \gamma_2 \).

2. A path in \( Y \) covering a spur is itself a spur. Consequently, two paths in \( Y \) covering freely homotopic paths are freely homotopic.

Part 1 is called path lifting and part 2 is spur lifting. Call \( \tilde{\gamma} \) the lift of \( \gamma \) at \( u \). Thus a path can always be lifted to one starting at any vertex covering its initial vertex and this lift is unique. As with so many such results, it is the uniqueness of the lift, rather than the existence, that turns out to be most useful.

Proof. The existence of \( \tilde{\gamma} \) is easily seen, as in Figure 3.4, since if \( \gamma = e_1 \ldots e_n \), there is an edge \( e'_1 \) covering \( e_1 \) under the bijection induced by \( f \) of the edges with start vertex \( u \) and those with start vertex \( v \). This edge \( e'_1 \) must end at a vertex that covers the end vertex of edge \( e_1 \), as coverings (being maps of complexes) preserve vertex-edge incidences. The process can be repeated starting at this new vertex to give \( \tilde{\gamma} \). For the uniqueness, the first edges of the \( \gamma_i \) both have initial vertex \( u \) and map to \( e_1 \), hence must be the same edge. Continuing in this manner along the two paths gives their equality. For the second part, the path \( \gamma \in Y \) must have the form \( e_1 e_2 \), where the “middle vertex” is the start of the edges \( e_1^{-1} \) and \( e_2 \). Use the injectivity of the cover on the edges starting at this vertex to deduce that \( e_1^{-1} = e_2 \). \( \Box \)
Exercise 3.6. Let \( f : Y \to X \) be a covering and \( \gamma = \alpha_1 \alpha_2 \) a path in \( X \) (hence \( t(\alpha_1) = s(\alpha_2) \)). Show that the lift at a vertex \( u \in Y \) of \( \gamma \) is the path \( \tilde{\alpha}_1 \tilde{\alpha}_2 \) consisting of the lift \( \tilde{\alpha}_1 \) of \( \alpha_1 \) at \( u \) followed by the lift of \( \alpha_2 \) at \( t(\tilde{\alpha}_1) \).

Exercise 3.7. Show that paths cannot necessarily be lifted by an immersion, but when they can, they are unique. Show that spur lifting is a property also enjoyed by immersions.

The first spin-off of path lifting justifies the usage of the word “cover”, and is not a priori obvious from the definition:

**Proposition 3.8 (surjectivity of coverings).** If \( f : Y \to X \) is a covering with \( X \) connected then \( f \) is a surjective map of 2-complexes, i.e: every cell of \( X \) is the image under \( f \) of some cell of \( Y \).

**Proof.** Path lifting gives the surjectivity on the vertices and the local continuity maps gives it on the edges and faces: let \( u \) be a vertex of \( Y \), and by connectedness, we can join any vertex \( v' \) of \( X \) to \( f(u) \) by a path. Lift this path to \( u \), so that its terminal vertex in \( Y \) maps via \( f \) to \( v' \). For any edge or face \( x \) of \( X \), take a vertex \( v \) lying in its boundary (which can just be the start vertex of the edge if we have an edge), so that the sets \( s^{-1}_X(v) \) or \( (a^x)^{-1}(v) \) are non-empty, and let \( u \) be a vertex mapping to \( v \). The bijectivity of the local continuity maps then gives an edge or face mapping via \( f \) to the one we started with. \( \square \)

Exercise 3.9. Illustrate by an example why the connectedness of \( X \) is necessary for Proposition 3.8.

**Proposition 3.10 (face lifting).** Let \( f : Y \to X \) be a covering, \( \tau \) a face of \( X \), \( v \) a vertex that appears in its boundary and \( \gamma \) a boundary path of \( \tau \) starting at \( v \). Let \( u \) be a vertex covering \( v \) and \( \tilde{\gamma} \) the lift of \( \gamma \) at \( u \). Then there is a face \( \sigma \) of \( Y \) that covers \( \tau \), contains \( u \) in its boundary and has \( \tilde{\gamma} \) as a boundary path starting at \( u \).

Thus the boundaries of faces lift to the boundaries of faces as shown schematically in Figure 3.5.

![Fig. 3.5. Face lifting](image)

**Proof.** Suppose that \( \tau \) has boundary \( \partial \tau = (X^\tau, a^\tau) \), so that \( (a^\tau)^{-1}(v) \neq \emptyset \) as \( v \) lies in its boundary. Hence there is a vertex \( x \) of the 1-sphere \( X^x \) that maps via the attaching map \( a^x \) to \( v \). Definition 3.1 part 3 gives a vertex \( y \) of \( Y \) lying in \( \bigsqcup_{f(\sigma)=\tau} (a^\sigma)^{-1}(u) \) mapping to \( x \) via the local continuity of \( f \), i.e: there is a face \( \sigma \) of \( Y \) covering \( \tau \), containing \( u \) in its boundary, and with the diagram,

\[
\begin{array}{ccc}
Y^\sigma_y & \cong & X^\tau_x \\
\downarrow a^\sigma & & \downarrow a^\tau \\
Y^1_u & \cong & X^1_v \\
f & & \\
\end{array}
\]

commuting. In particular there is a boundary path of \( \sigma \) starting at \( u \) that covers \( \gamma \), and by the uniqueness of lifts, this must be the path \( \tilde{\gamma} \). \( \square \)
The result of being able to find pre-images of paths and faces by a covering is that homotopies of paths can also be “pulled back” through a covering:

**Corollary 3.11 (homotopy lifting).** Let $Y \to X$ be a covering. Then two paths that cover homotopic paths are themselves homotopic.

**Proof.** The homotopy between the covered paths is realised by a finite sequence of insertions or deletions of spurs and face boundaries. By spur and face lifting, those sections of the covering paths mapping to these spurs and face boundaries are themselves spurs and face boundaries, while by uniqueness of path lifting, the remaining pieces are identical. Thus the same sequence of elementary homotopies can be realised between the covering paths as between the covered ones. □

**Exercise 3.12.** Let $f : Y \to X$ be a covering and $\gamma_1, \gamma_2$ paths in $X$ related by an elementary homotopy, i.e. $\gamma_2$ is what results from inserting a spur or a face boundary into $\gamma_1$. Let $\tilde{\gamma}_1$ be the lift of $\gamma_1$ at some vertex $v$ of $Y$ and $\gamma$ the result of lifting to the appropriate vertex the elementary homotopy and performing it on $\tilde{\gamma}_1$. Show that $\gamma$ is the lift $\tilde{\gamma}_2$ of $\gamma_2$ at $u$.

Another spin-off of homotopy lifting (and Exercise 3.12) is the following characterisation of the image of the induced homomorphism between fundamental groups:

**Corollary 3.13.** Let $f : Y \to X$ be a covering with $f(v) = v$ and $f_* : \pi_1(Y,v) \to \pi_1(X,v)$, the induced homomorphism. Then $f_*$ is injective and a closed path $\gamma$ at $v$ represents an element of $f_*\pi_1(Y,v)$ if and only if the lift $\tilde{\gamma}$ of $\gamma$ at $u$ is closed.

The injectivity of the induced homomorphism is probably the single most important property of coverings: it means that the fundamental group of the covering space can be indentified with a subgroup of the fundamental group of the space that is being covered. The appropriate context in which to develop this idea properly will be the Galois theory of coverings in Chapter 4, so we will make a bigger deal of it then.

**Proof.** If two elements of $\pi_1(Y,v)$ map to the same element of $\pi_1(X,v)$ then they are represented by closed paths at $u$ covering homotopic paths in $X$. Homotopy lifting thus gives that the closed paths themselves are homotopic, and so the two elements of the fundamental group coincide, thus establishing the injectivity of the homomorphism. For the second part, if $\tilde{\gamma}$ is closed then its homotopy class maps via $f_*$ to the homotopy class of $\gamma$. Conversely, if $\gamma$ represents an element in the image of the homomorphism then there is a closed path $\gamma_1$ at $v$, homotopic to $\gamma$, and with $f(\gamma_1) = \gamma_1$ for $\gamma_1$ closed at $u$. By Exercise 3.12, the lift of $\gamma$ at $u$ is what results by lifting these homotopies; in particular, $\tilde{\gamma}_1$ and the lift are homotopic, so that the lift is closed. □

**Exercise 3.14.** Let $f : Y \to X$ be a covering with $f(u) = v$ and $u'$ the terminal vertex of a path $\mu \in Y$ starting at $u$. Show that $f_*\pi_1(Y,v) = h f_*\pi_1(Y,u') h^{-1}$, where $h$ is the homotopy class of $f(\mu)$.

This thread of ideas culminates in the following general lifting result.

**Proposition 3.15 (map lifting).** If $f : Y_u \to X_v$ is a (pointed) covering and $g : Z_x \to X_v$ a map with $Z$ connected, then there is a map $\tilde{g} : Z_x \to Y_u$ making the diagram

\[
\begin{array}{ccc}
Y_u & \xrightarrow{f} & X_v \\
\downarrow \quad \tilde{g} & & \\
Z_x & \xrightarrow{g} & X_v 
\end{array}
\]

commute if and only $g_*\pi_1(Z,x) \subseteq f_*\pi_1(Y,u)$. If $\tilde{g}$ exists then it is unique.
One way to think of this result is as a generalisation of path lifting: if \( Z \) is a 1-ball then the map \( g : Z \to X \) is a path in \( X \) starting at \( v \). As the fundamental group of a 1-ball is trivial, the condition \( g_\# \pi_1(Z, v) \subset g_\# \pi_1(Y, u) \) is trivially satisfied, as the left hand side is the identity subgroup. The resulting map \( \tilde{g} : Z \to Y \) is a path in \( Y \) starting at \( u \), and the commuting of the diagram just says that this new path covers the old one.

**Proof.** The “only if” part can be dispensed with quickly: it follows as the commuting of the diagram just says that this new path covers the old one. Suppose we have a map \( g \) having the required properties: if \( z \) is a vertex of \( Z \), then by connectedness there is a path joining it to \( x \). Take the image of this path by \( g \) and then lift the result via the covering \( f \) to a path at \( u \). Define \( \tilde{g}(z) \) to be the end vertex of the resulting path in \( Y \).

Edges and faces are similar: take a path in \( f \) of the single vertex and the fiber of the single edge contain two cells. Extending this covering to one of \( \pi_1 \) is a trivial, the condition \( f_\# \tilde{g}_\# \pi_1(Z, x) \subset f_\# g_\# \pi_1(Y, u) \).

As the fundamental group of a 1-ball is trivial, the condition \( f_\# \tilde{g}_\# \pi_1(Z, x) \subset f_\# g_\# \pi_1(Y, u) \).

We now come to an important numerical invariant that can be attached to a covering. Looking back at the example of Figure 3.2, we have a graph covering of \( X \) where both the fiber of the single vertex and the fiber of the single edge contain two cells. Extending this covering to one of 2-complexes as in Figure 3.3, the fiber of the single face also contains two cells. This is no coincidence.

**Proposition 3.17 (covering degree).** If \( f : Y \to X \) is a covering with \( X \) connected, then any two fibers have the same cardinality.

This common cardinality of the fibers is called the *degree* of the covering, written \( \deg(Y \to X) \) or \( \deg(Y/X) \).

The connectedness of \( X \) is essential, for if \( X \) has components \( X_1 \) and \( X_2 \) say, and \( f_i : Y_i \to X_i \) are coverings of different degree then we can cobble together a new covering \( f : Y = Y_1 \coprod Y_2 \to X \coprod X_2 \) with \( f|_{Y_1} = f_1 \). The cardinality of the fibers now depends on which component of \( X_i \) they lie above. Anyway, the connectedness of \( X \) is used explicitly in the proof:
Proof. If \( v, u \) are vertices of \( X \) and \( \gamma \) a path from \( v \) to \( u \), then lifting \( \gamma \) to a path \( \tilde{\gamma} \) at any vertex of the fiber of \( v \) and taking its end vertex \( l(\tilde{\gamma}) \), gives a (set) mapping from the fiber of \( v \) to the fiber of \( u \). By the uniqueness of the lift of the path \( \gamma^{-1} \) at any vertex in the fiber of \( u \), this mapping is injective. Interchanging the roles of \( v \) and \( u \) gives an injective map in the reverse direction, hence the fibers of \( v \) and \( u \) have the same cardinality. Similarly, by the second part of the definition of covering and path lifting, if \( e \) is an edge of \( X \) then there is a bijection between the fiber of \( e \) and the fiber of its start vertex \( s(e) \). This establishes the degree for graphs.

Now to the fiber of a face. Let \( \tau \) be a face of \( X \) and \( v \) be a vertex in its boundary with \(|(a^*)^{-1}(v)| = m > 0\). We will show that this face and vertex have fibers with the same cardinality, ie: \(|f^{-1}(\tau)| = |f^{-1}(v)|\). These \( m \) vertices are marked by little red circles in Figure 3.6 (for \( m = 2 \)). Let \( u_1, \ldots, u_k \) be the vertices in the fiber of \( v \) and consider those faces \( \sigma_i \) in the fiber of \( \tau \) containing one of the vertices \( u_j \) in their boundary. Then in fact, *every* face in the fiber of \( \tau \) must be in this collection of faces, by the incidence preserving property Definition 1.22 part 1, of the map \( f \). Thus, this set of faces \( \sigma_i \) is the fiber of \( \tau \). Now, each \( \sigma_i \) in this fiber has \( X^{\sigma_i} \) isomorphic to \( X^{\tau} \), and so \( X^{\sigma_i} \) has \( m \) vertices that correspond to the \( m \) red vertices under this isomorphism. Consider this set of \( m|f^{-1}(\tau)| \) vertices, marked in blue in Figure 3.6. As each \( \sigma_i \) in the fiber maps to \( \tau \), each of these blue vertices is sent by the attaching maps to one of the \( u_j \) in the fiber of \( v \). Thus, the various attaching maps \( a^{\sigma_i} \) map this set of blue vertices to the fiber of \( v \).

We haven’t used the covering yet! Here is where we do: fix a vertex \( u_j \) in the fiber of \( v \), so that the definition of covering applied to the triple \( u_j, v \) and \( \tau \) gives \( m \) blue vertices mapping via the various attaching maps to \( u_j \). Thus, the \( m|f^{-1}(\tau)| \) blue vertices are mapped in an \( m \)-to-1 fashion onto the fiber of \( v \), ie:

\[
m|f^{-1}(\tau)| = m|f^{-1}(v)|,
\]

and we are done. \( \square \)

Degree plays a similar role for coverings as does dimension for vector spaces or index for groups. In particular, “if \( H \) is a subgroup of index one in a group \( G \) then \( H = G^* \), is an argument whose combinatorial topology version is,

**Corollary 3.18.** A degree one covering of a connected complex is an isomorphism.

The Corollary follows immediately from the surjectivity of coverings Proposition 3.8, and the definition of degree. This simple little result will play a crucial role in the proof of the Galois correspondence of §4.3.1. Notice that this is the second time in as many sections that subgroups have made a natural appearance in covering space theory (the other time was Corollary 3.13).
Exercise 3.19 (degree of tree lifting). Show that in Exercise 3.16 part 3 we have
\[ \deg(Y \to X) = \deg(Y/T_i \to X/T). \]

3.2 Actions, intermediate and universal covers

3.2.1 Group actions

Recall from §1.2.3 that a group acts freely on a 2-complex precisely when it acts freely on the vertices, i.e.: for any \( g \in G \) and vertex \( v \), if \( g(v) = v \) then \( g \) is the identity. It turns out that such group actions give us a plentiful supply of coverings:

**Proposition 3.20.** If a group \( G \) acts orientation preservingly and freely on a 2-complex \( X \) then the quotient map \( q : X \to X/G \) is a covering.

**Proof.** is an exercise in not getting yourself confused! It may help to remember that the cells of \( X/G \) are the equivalence classes of the action and the fiber under \( q \) of any cell consists of those cells of \( X \) in the equivalence class. Suppose for definiteness that \( G = \{ g_i, i \in I \} \).

That \( q \) is a covering of the underlying graphs is straight-forward: if \( E \) is an equivalence class of edges then \( E = \{ g_i(e), i \in I \} \) for some edge \( e \) of \( X \), and in \( X/G \) it has start vertex the equivalence class \( V = \{ g_i(v), i \in I \} \) for \( v = s(e) \). Then for the vertex \( g_i(v) \) covering \( V \), the quotient maps the edge \( g_i(e) \) onto \( E \). Two different edges starting at \( g_i(v) \) cannot lie in the same class, for then a non-trivial element of \( G \) would fix this vertex, contradicting the freeness of the action.

Suppose \( \Sigma \) is a face of the quotient with \( V \) a vertex lying in its boundary. Thus there are \( v \in V \) and \( \sigma \in \Sigma \) with \( V = \{ g_i(v), i \in I \} \), \( \Sigma = \{ g_i(\sigma), i \in I \} \) and \( g_i(\sigma) \) lying in the boundary of \( g_i(\sigma) \) as in Figure 3.7 (where for illustrative purposes \( G \) is finite of order \( k \)). We are free, by Exercise 1.32, to choose the boundary of \( \Sigma \) from amongst the faces in \( \Sigma \),

![Fig. 3.7. A face of the quotient X/G and a vertex lying in its boundary](image)

so we choose \( \partial \Sigma = (X^\sigma, g_0 \sigma) \). Then the elements of \( (a^\Sigma)^{-1}(V) \) are those vertices of \( X^\sigma \) that are sent by the attaching map \( a^\sigma \) to a vertex of the equivalence class \( V \). If \( v \in V \) is some vertex of this class, then the elements of \( \coprod_{g_i(\sigma) \in \Sigma} (a^\sigma)^{-1}(v) \) are the vertices of the various \( X^{g_i(\sigma)} \) that are sent via their attaching maps \( a^{g_i(\sigma)} \) to \( v \). Our job is to show that the local continuity map \( \coprod g_i(\sigma) \) is a bijection between these two sets, where \( \varepsilon_{g_i(\sigma)} \) is the isomorphism \( X^{g_i(\sigma)} \to X^\sigma \) induced by \( g_i^{-1} \).

For injectivity, we know that the \( \varepsilon_{g_i(\sigma)} \) are individually injective; if for \( i \neq j \) there are vertices \( x_i \in X^{g_i(\sigma)} \) and \( x_j \in X^{g_j(\sigma)} \) sent via their attaching maps to \( v \) then \( g_i g_j^{-1} \) is not the identity but nevertheless fixes \( v \), a contradiction. Thus the local continuity maps must be injective.

Let \( x \) be a vertex in \( (a^\Sigma)^{-1}(V) \), hence sent by the attaching map \( a^\sigma \) to some \( g_i(v) \). There is a vertex \( y \) of \( X^{g_i(\sigma)} \) sent to \( x \) by the isomorphism \( \varepsilon_{g_i(\sigma)} \), and since the attaching map of the face \( g_i(\sigma) \) is the composition
3.2 Actions, intermediate and universal covers

\[ X^{g_i(\sigma)} \xrightarrow{\epsilon_{g_i(\sigma)}} X \xrightarrow{a^{\sigma}} X^{(1)} \xrightarrow{\theta^{-1}} X^{(1)}, \]

we get that \( y \) is sent via this attaching map to \( v \), and so \( y \in \bigsqcup_{g_i(\sigma) \in \Sigma} (a^{\sigma})^{-1}(v) \). The local continuity map is thus a surjection. \( \square \)

### 3.2.2 Intermediate covers

**Lemma 3.21 (intermediate graph coverings).** Let \( Y \xrightarrow{q} X \xrightarrow{r} Z \) be dimension preserving maps of graphs and let \( p = rq \). If any two of \( p, q \) and \( r \) are coverings, then so is the third.

**Proof.** Given three sets and three set maps forming a commutative triangle, then any two of them a bijection implies that the third is also a bijection. This simple fact, applied to the sets of edges starting at a triple of vertices \( x, y, z \) with \( x = q(y) \) and \( z = r(x) \), gives the result. \( \square \)

**Lemma 3.22 (intermediate coverings of 2-complexes).** Let \( Y \xrightarrow{q} X \xrightarrow{r} Z \) be dimension preserving maps of 2-complexes and let \( p = rq \). Then,

1. if \( q, r \) are coverings then so is \( p \), and
2. if \( p, q \) are coverings then so is \( r \).

**Proof.** We do 1 and leave 2, which is similar, to the reader. First, we need to know about the fiber under \( p \) of a face \( \tau \in Z \):

\[ p^{-1}(\tau) = \bigsqcup_{\sigma \in r^{-1}(\tau)} q^{-1}(\sigma). \]

Let \( z \in Z \) be a vertex and \( y \in Y \) a vertex covering it via the (graph) covering \( p \). Let \( x = q(y) \) so that \( z = r(x) \). We need to show that the local continuity map is a bijection from the appearances of \( y \) in the faces of \( p^{-1}(\tau) \) and the appearances of \( z \) in \( \tau \). Local continuity of the covering \( r \) gives a bijection between the latter and the appearances of \( x \) in the faces of \( r^{-1}(\tau) \), which is a disjoint union over the appearances of \( x \) in each individual face \( \sigma \in r^{-1}(\tau) \). For each \( \sigma \in r^{-1}(\tau) \), local continuity of the covering \( q \) gives a bijection between the appearances of \( x \) in \( \sigma \) and the appearances of \( y \) in the faces of \( q^{-1}(\sigma) \). The disjoint union of these appearances of \( y \) in the faces of \( q^{-1}(\sigma) \) as \( \sigma \) varies over \( r^{-1}(\tau) \) gives the appearances of \( y \) in the faces of \( p^{-1}(\tau) \), and since the local continuity associated to \( p \) is the composition of the local continuities of \( q \) and \( r \), we are done. \( \square \)

Why do we not have a result that says that if \( p \) and \( r \) are coverings then so is \( q \), like we do for graphs? The problem is that for coverings of 2-complexes, the fibers of a cell play a central role: we need to know about the fiber of a face in order to verify that a map is a covering. Now, in the two cases of Lemma 3.22, we can describe the fiber of a face under the map we are interested in, in terms of the two maps that we already know about. This is not so when we know about \( p \) and \( r \) and are interested in \( q \).

If \( Y \xrightarrow{q} X \xrightarrow{r} Z \) are coverings, so that the composition \( Y \xrightarrow{p} Z \) is a covering, then call \( q \) and \( r \) coverings intermediate to \( p \). It turns out that the set of coverings intermediate to a fixed covering \( Y \xrightarrow{\theta} Z \) carries a great deal of structure, and we will explore this more fully in §3.4.

**Exercise 3.23.** Let \( Y \) be a graph and \( Y_1, Y_2 \subseteq Y \) subgraphs of the form,

\[
\begin{array}{c}
\text{\includegraphics[width=1cm]{graph_diagram.png}}
\end{array}
\]
1. If \( p \colon Y \to X \) is a covering with \( X \) single vertexed, then the real line is a subgraph \( \alpha : \mathbb{R} \hookrightarrow Y \), with \( \alpha(e_0) = e \) and \( p\alpha(e_k) = p(e) \) for all \( k \in \mathbb{Z} \).

2. If \( Y_1 \) is a tree, \( p : Y \to X \), \( r : Z \to X \) coverings, and \( \alpha : Y_2 \hookrightarrow Z \) an isomorphism onto its image, then there is an intermediate covering \( Y \xrightarrow{\beta} Z \xrightarrow{\tau} X \).

3. If \( W \to Y \) is a covering and \( Y_1 \) a tree, then \( W \) also has the form (†) for some subgraphs \( Y_1', Y_2' \hookrightarrow W \), with \( Y_1' \) a tree.

### 3.2.3 Universal covers

Much of the discussion of coverings so far has been in the abstract: we haven’t seen many actual covers! By Exercise 3.4 we know that at the very least, a complex is a cover of itself, but that is rather trivial. In this section we show that a complex always has another cover which is at the other extreme, in that it is as “big” as possible.

**Definition 3.24 (universal covers).** A covering \( f : Y \to X \) is universal if and only if for any covering \( q : Z \to X \) there is a covering \( p : Y \to Z \) making the diagram,

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & Z \\
\downarrow{f} & & \downarrow{q} \\
X & \xrightarrow{q} & X \\
\end{array}
\]

commute.

Equivalently, \( Y \to X \) is universal when any other covering \( Z \to X \) of \( X \) is intermediate to it.

**Exercise 3.25.** Show that if \( Y_1, Y_2 \to X \) are universal covers then there is an isomorphism \( Y_1 \cong Y_2 \). Thus, universal covers, if they exist, are unique.

But universal covers do exist! To construct them we mimic a standard construction in topology: let \( X \) be connected and fix a reference point vertex \( v_0 \) of \( X \) and let \( \tilde{X} \) be the following complex. Its vertices \( \tilde{v} \) are the homotopy classes of paths in \( X \) starting at \( v_0 \). Suppose we have two such vertices, \( \tilde{v}_1 \) and \( \tilde{v}_2 \), the homotopy classes of the paths \( \gamma_1 \) and \( \gamma_2 \). Then there is an edge \( \tilde{e} \) of \( \tilde{X} \) connecting \( \tilde{v}_1 \) to \( \tilde{v}_2 \) if and only if there is an edge \( e \) of \( X \) such that the path \( \gamma_1 e \) is homotopic to \( \gamma_2 \). Figure 3.8 shows the kind of schematic set-up we have.

**Fig. 3.8.** How an edge of \( X \) gives an edge of \( \tilde{X} \): there is an edge \( \tilde{e} \) with start vertex the homotopy class of \( \gamma \) and finish vertex the homotopy class of \( \gamma \tilde{e} \).

**Exercise 3.26.** Let \( \tilde{v}_1 \) and \( \tilde{v}_2 \) be vertices of \( \tilde{X} \) corresponding to the homotopy classes of the paths \( \gamma_1 \) and \( \gamma_2 \). If there is a path \( \tilde{\gamma} = \tilde{e}_1 \ldots \tilde{e}_k \) in \( \tilde{X} \) from \( \tilde{v}_1 \) to \( \tilde{v}_2 \), then there is a path \( \gamma = e_1 \ldots e_k \) in \( X \) with the edge \( \tilde{e}_i \) arising from the edge \( e_i \) as above, and \( \gamma_1 \gamma \) homotopic to \( \gamma_2 \).
Lemma 3.27. The graph $\tilde{X}$ is connected. Define $f : \tilde{X} \to X^{(1)}$ by $f(\tilde{v}) = \text{the terminal vertex of } \gamma$, where $\tilde{v}$ is the homotopy class of the path $\gamma$, and $f(\tilde{e}) = e$, where the edge $\tilde{e}$ arises from $e$ as above. Then $f$ is a graph covering.

Proof. If $\tilde{v}$ is a vertex of $\tilde{X}$ corresponding to the path $\gamma = e_1 \ldots e_k$, then $\tilde{e}_1 \ldots \tilde{e}_k$ is a path connecting $\tilde{v}$ to the homotopy class of the empty path, and so $\tilde{X}$ is connected. To see that $f$ is a covering we need that it is a dimension preserving map (which we leave to the reader) and that for every pair of vertices $\tilde{v}$ and $v$ with $f(\tilde{v}) = v$, it induces a bijection from the edges starting at $\tilde{v}$ to the edges starting at $v$. Suppose then that $\tilde{v}$ is a vertex corresponding to the class of the path $\gamma$ and $\tilde{e}_1, \tilde{e}_2$ are edges connecting $\tilde{v}$ to vertices $\tilde{v}_1$ and $\tilde{v}_2$. Let $v$ be the terminal vertex of $\gamma$ (so that $f(\tilde{v}) = v$) and $e$ an edge starting at $v$ with $f(\tilde{e}_1) = f(\tilde{e}_2) = e$. Then the vertices $\tilde{v}_1$ and $\tilde{v}_2$ both correspond to the homotopy class of the path $\gamma e$, and so $\tilde{v}_1 = \tilde{v}_2$. The edges $\tilde{e}_1, \tilde{e}_2$ both arise by applying the construction described above to the pair of vertices $\tilde{v}$ and $\tilde{v}_1 = \tilde{v}_2$, and as only one edge can arise this way, we have $\tilde{e}_1 = \tilde{e}_2$. The local continuity maps are thus injective. For an edge $e$ starting at $v$, and a vertex $\tilde{v}$ in its fiber corresponding to the path $\gamma$ from $v_0$ to $v$, there is, by definition, an edge $\tilde{e}$ connecting $\tilde{v}$ and the vertex corresponding to the path $\gamma e$. This gives the surjectivity of the local continuity maps. \[\Box\]

![Fig. 3.9. Three complexes. Note that $X_2$ and $X_3$ have the same 1-skeleton.](image)

![Fig. 3.10. The 1-skeletons of the complex $\tilde{X}$ for the $X$ of Figure 3.9.](image)

Figure 3.9 gives three complexes and the graphs $\tilde{X}$ are given in Figure 3.10. If $X$ is the $S^1$-graph on the left, then there is a 1-1 correspondence between the homotopy classes of paths and paths of the form $e \ldots e$ ($k$ times) or $e^{-1} \ldots e^{-1}$ ($k$ times). Thus $\tilde{X}$ has vertices $\tilde{v}_k$ for $k \in \mathbb{Z}$. There is an edge connecting the vertex of the path $e \ldots e$ ($k$ times) to the vertex of the path $e \ldots e$ ($k+1$ times), to give the infinite 2-valent tree at left in Figure 3.10. Similarly the single-vertexed graph with two edges has $\tilde{X}$ the 4-valent infinite tree.

The last of the three complexes is the torus, which has exactly the same 1-skeleton as $X_2$, but the presence of a face drastically changes the graph $\tilde{X}$. In $X_2$ paths of the form $\gamma$ and $\gamma e_1 e_2 e_1^{-1} e_2^{-1}$ give distinct vertices in $\tilde{X}_2$, but the same vertex in $\tilde{X}_3$.

One more thing before proceeding with the definition of $\tilde{X}$: the graph $\tilde{X}$ gives a covering of the graph $X$, but not necessarily a universal one. We can see this in the examples above.
where $X_2$ and $X_3$ have the same 1-skeleton but quite different $\tilde{X}$. Indeed, it is not hard to construct a graph covering $\tilde{X}_2 \to \tilde{X}_3$, so $\tilde{X}_3$ is most definitely not universal.

Now to the faces of $\tilde{X}$, which arise out of the faces of $X$, much as the edges do. First, we urge the reader to do the following exercise.

**Exercise 3.28.** In the graph covering $f : \tilde{X} \to X$, show that the boundaries of faces in $X$ lift to closed paths in $\tilde{X}$. More precisely, let $u$ be a vertex of $X$, $\sigma$ a face containing $u$ in its boundary and $\gamma$ a boundary path of $\sigma$ starting at $u$. If $v$ is a vertex covering $u$ via the (graph) covering $f$ and $\tilde{\gamma}$ is the lift of $\gamma$ at $v$, then $\tilde{\gamma}$ is a closed path in $\tilde{X}$.

Thus the boundaries of faces in $X$ give rise to closed paths in $\tilde{X}$, and the idea behind the construction of the 2-skeleton of $\tilde{X}$ is to “sew” faces into these closed paths (actually we saw this phenomenon when we constructed the graph $\tilde{X}_3$ in Figure 3.10).

Fix a triple consisting of a vertex $u$ of $X$, a vertex $v$ of $\tilde{X}$ covering it, and a face $\sigma$ of $X$. Suppose $\partial \sigma = (X^\sigma, a^\sigma)$ and that $u$ appears $k$ times in the boundary of $\sigma$. In particular, there are $k$ paths circumnavigating $X^\sigma$ clockwise, and mapping via the attaching map $a^\sigma$ to $k$ boundary paths of $\sigma$ starting at $u$.

Let $x$ be one of the $k$ appearances of $u$ in the boundary of $\sigma$; $\gamma$ the path circumnavigating $X^\sigma$ clockwise starting at $x$ and $a^\sigma(\gamma)$ the resulting boundary label of $\sigma$ starting at $u$. Lift the label $a^\sigma(\gamma)$ to $v$ (via the graph covering $f : \tilde{X} \to X$), to get by Exercise 3.28, a closed path in $\tilde{X}$. If there is already a face of $\tilde{X}$ covering $\sigma$ and with boundary the lift of $a^\sigma(\gamma)$ to $v$, then do nothing. Otherwise, sew in a new face $\tilde{\sigma}$ with this boundary in the following way: let $\tilde{\partial} \tilde{\sigma} = (X^\tilde{\sigma}, a^\tilde{\sigma})$ where $a^\tilde{\sigma}$ is the composition of $a^\sigma$ and the lift of $a^\sigma(\gamma)$ to $\tilde{X}$. Point $a^\tilde{\sigma}$ using the vertices $x \in X^\sigma$ and $v \in \tilde{X}$.

Repeat the procedure above for all such triples $u, v, \sigma$, remembering not to sew in a new face if there is already one covering $\sigma$ with the lifted boundary label.

Figure 3.11 shows the result of performing this process with the complex $X_3$ of Figure 3.9, sewing faces onto the 1-skeleton of Figure 3.10. Figure 3.12 shows the effect on $\tilde{X}$ of an extra face in $X$.

**Proposition 3.29.** Define $f : \tilde{X} \to X$ on the 1-skeleton as in Lemma 3.27, and for each face $\sigma$ arising as above from a face $\tilde{\sigma}$ of $X$, define $f(\tilde{\sigma}) = \sigma$. Then $f$ is a covering of 2-complexes.

**Corollary 3.30.** The 2-complex $\tilde{X}$ is simply connected.

**Proof.** Let $\tilde{\gamma}$ be a closed path in $\tilde{X}$ based at the vertex $\tilde{v}$ corresponding to the empty path at $v_0$. Then by Exercise 3.26 it covers a path $\gamma$ at $v_0$ that is homotopically trivial. Homotopy lifting via the covering $f$ of Proposition 3.29 gives that $\tilde{\gamma}$ is homotopically trivial too, and hence that $\tilde{X}$ is simply connected. □
3.2 Actions, intermediate and universal covers

This leaves one last piece of remaining business:

**Proposition 3.31.** The covering \( f : \tilde{X} \to X \) is universal.

**Proof.** Given a covering \( q : Y \to X \) we construct a covering \( \tilde{X} \to Y \) by “cover and lift”. \( \square \)

**Exercise 3.32.** Show, using universal coverings, that if \( X \) is a graph and \( \gamma_1, \gamma_2 \) are reduced homotopic paths in \( X \) with the same start vertex, then \( \gamma_1 = \gamma_2 \). [hint: lift the paths to \( \tilde{X} \) and use properties of reduced paths in trees to deduce that these lifts are identical.]

### 3.2.4 Monodromy

When \( f : Y \to X \) is a covering we will eventually get an action of two groups on \( Y \), or at least on parts of \( Y \). The more important of these is the Galois group of the covering, which acts on the whole complex \( Y \), and forms the principle subject of Chapter 4.

If \( v \) is a vertex of \( X \) then the lesser of these two actions is that of the fundamental group \( \pi_1(X, v) \) on the fiber \( f^{-1}(v) \) of the vertex \( v \). Thus, the fundamental group is acting as a permutation group of the set of vertices lying over \( v \). Such permutation representations of the fundamental groups of 2-complexes will play a key role in the proof of results like Miller’s theorem in Chapter 7.

To define the action, see that it makes sense, and is a homomorphism

\[
\pi_1(X, v) \to \text{Sym}(f^{-1}(v)),
\]

we require no more than the path and homotopy lifting of §3.1.2. So, the “path-lifting action” would probably be a sensible name: the action would then do exactly what it says on the box! However, it is traditional in topology to call this action *monodromy*, and so we will too.

The definition is illustrated in Figure 3.13: let \( \gamma \) be a closed path at \( v \) representing an element (which, by our custom, we also call \( \gamma \)) of the fundamental group \( \pi_1(X, v) \). To define the image of a vertex \( u \) in the fiber of \( v \), take the lift \( \tilde{\gamma} \) of \( \gamma \) at \( u \), and suppose that this lift has end vertex \( x \). Define \( \gamma(u) = x \).

We obviously have a well-defined issue to deal with, so that the action does not depend on our choice of path. If \( \gamma_1 \) is another closed path at \( v \) representing the same element of the fundamental group (so in the group we have \( \gamma = \gamma_1 \)), then the paths \( \gamma, \gamma_1 \) are homotopic. This homotopy can be lifted, via homotopy lifting, to a homotopy between the lifts \( \tilde{\gamma} \) and \( \tilde{\gamma}_1 \) at \( u \), and so the two lifts are homotopic in \( Y \).

But homotopic paths have the same endpoints! Thus \( \tilde{\gamma}_1 \) ends at \( x \) as well, and we get the same image vertex of \( u \) irrespective of the path chose to represent \( \gamma \).
Proposition 3.33. If \( Y \to X \) is a covering then monodromy gives a homomorphism
\[
\pi_1(X, v) \to \text{Sym}(f^{-1}(v)).
\]
In particular, a covering of finite degree gives a homomorphism from \( \pi_1(X, v) \) to a finite group.

Suppose that \( X \) has just the one vertex \( v \), so that the fiber of \( v \) consists of all the vertices of \( Y \). Monodromy thus gives an action on the whole 0-skeleton of \( Y \). The next exercise shows that in general this action cannot be extended any further than this, i.e.: there are examples where it cannot be extended from the 0-skeleton to the 1-skeleton.

Exercise 3.34. 1. Let \( X \) be the complex of Figure 3.14. Describe the universal cover \( \tilde{X} \to X \), showing that in particular that it is a covering of degree 6.

2. Show that in \( \tilde{X} \) there exists an edge that joins two vertices, but no edge joining the images of these two vertices under the monodromy action of \( \pi_1(X, v) \). Thus it is not possible to define an automorphism of \( \tilde{X} \) at this edge. Deduce that there can therefore be no homomorphism from \( \pi_1(X, v) \) to the automorphism group of the 1-skeleton extending the monodromy action on the 0-skeleton.

### 3.3 Operations on coverings

In the first chapter we had three constructions arising from a collection of complexes and maps between them: the pushout, pullback and Higman composition. In this section we show that all three are useful ways of creating new coverings from old. Throughout this section all complexes are connected.

#### 3.3.1 Pushouts of covers

In this subsection, let \( f : Y \to Z \) be a fixed covering of connected 2-complexes, and for \( i = 1, 2 \), let \( Y \overset{p_i}{\to} X_i \overset{q_i}{\to} Z \) be connected coverings intermediate to \( f \). Thus we have the commuting diagram of Figure 3.15 left, with all the maps in sight coverings.
As the $p_i$ are dimension preserving, we may, by Proposition 1.40, form the pushout $X_1 \coprod_Y X_2$, obtaining in the process maps $t_i : X_i \to X_1 \coprod Y X_2$ as the composition $X_i \leftarrow X_1 \coprod X_2 \to X_1 \coprod X_2/\sim$ of the inclusion of $X_i$ in the disjoint union and the quotient map. The universality of the pushout, Proposition 1.37, applied to the maps $q_i : X_i \to Z$, gives the commuting diagram on the right of Figure 3.15. If $[x]$ is a cell of the quotient $X_1 \coprod X_2/\sim$, then $x \in X_i$ for some $i$, and so the new map $h : X_1 \coprod Y X_2 \to Z$ sends $[x]$ to $q_i(x) \in Z$.

![Fig. 3.15.](image)

**Proposition 3.35 (pushouts of covers).** The maps $t_i : X_i \to X_1 \coprod Y X_2$, for $i = 1, 2$, and $h : X_1 \coprod Y X_2 \to Z$, are coverings. Thus, the pushout of two intermediate coverings is an intermediate covering.

**Proof.** We show that the $t_i$ are coverings, so part 2 of Proposition 3.22 gives us $h$ for free. For ease of expression we do $t_1$, with $t_2$ completely analogous. It is clearly dimension preserving, leaving us to show that the local continuity maps are bijections. This is similar for both edges and faces, so suppose that $[u]$ is a vertex of the pushout and $v$ a vertex of $X_1$ mapping to it via $t_1$. Thus $u$ and $v$ are equivalent under the quotient relation of the pushout, so we may write $[v]$ for the vertex instead. Suppose we have a cell of the pushout incident with the vertex $[v]$: an edge $[e']$ with start vertex $[v]$ or a face $[\sigma']$ with $[v]$ appearing in its boundary. Thus, $e'$ and $\sigma'$ are cells of the disjoint union $X_1 \coprod X_2$, and there is a vertex $v' \in X_1 \coprod X_2$, equivalent to $v$, that is the start vertex of $e'$, or appears in the boundary of $\sigma'$. The equivalence between $v$ and $v'$ is realized by a sequence of lifts and covers (of vertices) through the coverings $p_i : Y \to X_i$. The same sequence, but using path or face lifting instead, yields an edge $e \in X_1$ or a face $\sigma \in X_1$, equivalent to the $e'$ or $\sigma'$, and with $v$ the start vertex of $e$ or $v$ appearing in the boundary of $\sigma$.

Two objects at $v$ (edges starting at $v$ or appearances of $v$ in faces) mapping under $t_1$ to the same object at $[v]$ must then map under $h$ to the same object in $Z$. As $ht_1 = q_1$, these two objects at $v$ map via the covering $q_1$ to the same object at $q(v_1)$, a contradiction, and so the local continuity maps are injective. $\Box$

**Exercise 3.36.** Is the pushout of two immersions an immersion?

### 3.3.2 Pullbacks of covers

In this subsection, let $f : Y \to Z$ be a fixed covering of connected 2-complexes, and for $i = 1, 2$, let $Y \xrightarrow{q_i} X_i \xrightarrow{p_i} Z$ be connected coverings intermediate to $f$. Thus we have the commuting diagram on the left of Figure 3.16 with all the maps in sight coverings. As the coverings $q_i$ are dimension preserving, we may, via Definition 1.42, form the pullback $X_1 \coprod_Z X_2$, obtaining in the process maps $t_i : X_1 \coprod_Z X_2 \to X_i$ given by $t_i : x_1 \times x_2 \mapsto x_i$. The universality of the pullback, Proposition 1.43, applied to the maps $p_i : Y \to X_i$, gives the commuting diagram on the right of Figure 3.16. The new map $h : Y \to X_1 \coprod_Z X_2$ takes a cell $y \in Y$ and maps it to the cell $p_1(y) \times p_2(y) \in X_1 \coprod Z X_2$. 
Proposition 3.37 (pullbacks of covers). The maps

\[ t_i : X_1 \prod_Z X_2 \to X_i, \text{ for } i = 1, 2, \text{ and } h : Y \to X_1 \prod_Z X_2, \]

are coverings. Thus, the pullback of two intermediate coverings is an intermediate covering.

Proof. It suffices, by part 2 of Proposition 3.22 to show that \( h \) is a covering. It is dimension preserving as the \( p_i \) are, and so it remains to show that the various local continuity maps are bijections. This is similar for both edges and faces: suppose that \( v_1 \times v_2 \) is a vertex of the pullback and \( u \) a vertex of \( Y \) mapping to it via \( h \). If two objects at \( u \) (edges starting at \( u \) or appearances of \( u \) in faces) map under \( h \) to a single object at \( v_1 \times v_2 \), then these two map via the \( p_i \) to single objects at the \( v_i \) in the \( X_i \). The coverings \( p_i \) then ensure that the original two objects coincide, hence the injectivity of the local continuity maps.

Surjectivity requires a couple more steps: start with an object at the vertex \( v_1 \times v_2 \) of the pullback. It maps via the \( t_i \) to objects at the \( v_i \), and they in turn map via the \( q_i \) to the same object at \( v = q_i(v_i) \). The path and face lifting provided by the covers \( p_i \) give two objects at \( u \) mapping to this single object at \( u \), one via \( q_1 p_1 \) and the other via \( q_2 p_2 \). But then these two objects map via the covering \( f \) to this single object at \( u \), and so must be the same object. By definition, the image via \( h \) of this single object at \( u \) must be the original object at \( v_1 \times v_2 \) that we started with. \( \square \)

Exercise 3.38. Is the pullback of two immersions an immersion?

Exercise 3.39. Let \( Y \to X \leftarrow Z \) be coverings with \( Z \) a forest. Show that the pullback \( X \prod_Z Y \) is also a forest.

We pause to observe a slight asymmetry to the duality between pushouts and pullbacks: given coverings \( r_1, r_2 : Y_1, Y_2 \to X \), the \( t_1, t_2 : Y_1 \prod_Y Y_2 \to Y_1, Y_2 \) are coverings, whereas coverings \( q_1, q_2 : Z \to Y_1, Y_2 \) do not necessarily give coverings \( t_1, t_2 : Y_1, Y_2 \to Y_1 \prod_Z Y_2 \), unless the \( q_i \) are intermediate \( Z \to (Y_1 \text{ or } Y_2) \to X \). Indeed, taking the \( Y_1 = Y_2 \) to be two copies of the left hand graph,

\[ Y_1 = Y_2 = \]

and the coverings \( q_i : Z \to Y_i \) (described here by drawing the fibers of the vertices), then the \( t_i \) provided by the pushout construction are not coverings of the pushout.

3.3.3 Higman compositions of covers

3.4 Lattices of covers

3.4.1 Aside: posets and lattices

At this stage we pause and take a brief look at the general theory of posets and lattices. This will not be comprehensive: we will just familiarise ourselves with the basic terminology.
and the first of two important examples. There are many books on this subject. We have followed [14, Chapter 3].

Partially ordered sets (or posets) formalise the idea of ordering:

**Definition 3.40 (poset).** A poset is a set $P$ and a binary relation $\leq$ such that

1. $\leq$ is reflexive: $x \leq x$ for all $x \in P$.
2. $\leq$ is antisymmetric: if $x \leq y$ and $y \leq x$ then $x = y$.
3. $\leq$ is transitive: if $x \leq y$ and $y \leq z$ then $x \leq z$.

The motivating example is meant to be the integers $\mathbb{Z}$ with their usual ordering $\leq$, and the usual notational conventions from there are used in general: we write $x < y$ to mean $x \leq y$ but $x \neq y$. Elements $x, y$ with $x \leq y$ or $y \leq x$ are comparable, otherwise they are incomparable, a possibility that obviously doesn’t arise with the primordial example $\mathbb{Z}$. We say that $y$ covers $x$, written $x \prec y$, when $x < y$ and if $x \leq z \leq y$ then either $z = x$ or $z = y$.

A morphism (or just map) of posets $f : P \to Q$ is an order-preserving map of sets: if $x \leq y$ in $P$ then $f(x) \leq f(y)$ in $Q$. Notice that this is a one way business: comparable elements are sent to comparable elements, but incomparable elements are allowed to become comparable. Similarly an anti-morphism is an order-reversing map: if $x \leq y$ in $P$ then $f(y) \leq f(x)$ in $Q$.

Bijective morphisms have inverses, although they may not be morphisms. A bijective morphism with order-preserving inverse is an isomorphism: $x \leq y$ in $P$ if and only if $f(x) \leq f(y)$ in $Q$. Similarly a bijective anti-morphism with order-reversing inverse is an anti-isomorphism.

Posets are often illustrated using their Hasse diagram: a graph whose vertices are the elements of $P$ and whose edges are the covering relations. Thus, if $x \prec y$ then the vertex $y$ is drawn above the vertex $x$ with an edge connecting them. Two examples of Hasse diagrams (and posets) are given in Figure 3.17.

![Hasse diagram](image)

**Fig. 3.17.** Hasse diagram for the poset of subsets of the set $\{1, 2, 3\}$, ordered by inclusion (left) and for a poset with four elements (right) that is not a lattice.

A special place is reserved for those posets which have supremums and infimums. If $x, y \in P$ then $z$ is an upper bound for them when both $x \leq z$ and $y \leq z$. It is a least upper bound or supremum or join when it is an upper bound such that for any other upper bound $w$ we have $z \leq w$. Similarly, $z$ is a lower bound for $x$ and $y$ when both $z \leq x$ and $z \leq y$. It is a greatest lower bound or infimum or meet when it is an lower bound such that for any other lower bound $w$ we have $w \leq z$.

It is easy to show that if $x$ and $y$ have a join then it is unique (hint: any two joins must be $\leq$ each other) and similarly for the meet. Write $x \lor y$ for the join and $x \land y$ for the meet of $x$ and $y$.

**Definition 3.41 (lattice).** A lattice is a poset in which every pair of elements has a join and a meet.
The poset on the left of Figure 3.17 is a lattice, as can be checked directly from the Hasse diagram, but the example on the right is not: if \( x \) and \( y \) are the two minimal elements, then they have a join, but no meet.

**Exercise 3.42.** A \( \hat{1} \) in \( P \) is a unique maximal element: for all \( x \in P \) we have \( x \leq \hat{1} \). Similarly a \( \hat{0} \) in \( P \) is a unique minimal element: for all \( x \in P \) we have \( \hat{0} \leq x \). Show that a finite lattice has a \( \hat{0} \) and a \( \hat{1} \).

**Exercise 3.43.** A poset is a *meet-semilattice* if any two elements have a meet. Dually we have the notion of a *join-semilattice*. Show that if \( P \) is a finite meet-semilattice with a \( \hat{1} \) then \( P \) is a lattice (dually, if \( P \) is a finite join-semilattice with a \( \hat{0} \) then \( P \) is a lattice).

**Exercise 3.44.** Let \( P \) and \( Q \) be lattices and \( f : P \to Q \) a lattice isomorphism (respectively anti-isomorphism). Show that \( f \) sends joins to joins and meets to meets (resp. joins to meets and meets to joins), ie: \( f(x \lor y) = f(x) \lor f(y) \) and \( f(x \land y) = f(x) \land f(y) \).

The most commonly occurring lattice in nature is the *Boolean lattice* on a set \( X \): its elements are the subsets of \( X \) and \( A \leq B \) iff \( A \subset B \). Meets and joins are just intersections and unions: \( A \land B = A \cap B \) and \( A \lor B = A \cup B \).

**Exercise 3.45.** Let \( X \) be a finite set, \( P \) the Boolean lattice on \( X \) and \( V \) the real vector space with basis \( X \). If \( v = \sum_X \lambda_x x \in V \) define \( ||v||^2 = \sum_X \lambda_x^2 \), and let \( \square^n = \{ v \in V : ||v||^2 \leq 1 \} \), the \( n \)-dimensional *cube*. Embed the underlying set of \( P \) in \( V \) via the map sending \( A \subset X \) to \( \sum_{x \in A} x \), and show that the image of \( P \) is the set of vertices of \( \square^n \), while the vertices and edges of \( \square^n \) give the Hasse diagram for \( P \).

A slightly less trivial example is the lattice \( L_n(\mathbb{F}) \) of all *subspaces* of the \( n \)-dimensional vector space over the field \( \mathbb{F} \), with the ordering given by inclusion of one subspace in another. The meet of two subspaces is again their intersection, but this time the union is too small to be their join: the union of two subspaces is not a subspace! Instead we take \( U \lor V = U + V \), their sum, consisting of all vectors of the form \( u + v \) for \( u \in U \) and \( v \in V \). We leave it to the reader to verify that these are indeed infimums and supremums.

Here is one we are particularly interested in,

**Definition 3.46 (lattice of subgroups).** Let \( G \) be a group. The lattice of subgroups \( L(G) \) has as elements the subgroups of \( G \) ordered by inclusion, \( H \land K = H \cap K \) and \( H \lor K = \langle H, K \rangle \), the subgroup generated by \( H \) and \( K \).

![Fig. 3.18. Subgroup lattice for the symmetric group \( S_3 \): let \( \sigma = (1, 2, 3) \) and \( \tau = (2, 3) \) so that \( \sigma \tau = (1, 2) \) and \( \sigma^2 \tau = (1, 3) \).](image)

**Exercise 3.47.** Show that the set of *finite index* subgroups of a group \( G \) also forms a lattice, with the same meet and join as \( L(G) \).

**Exercise 3.48.** Show that the set of *finitely generated* subgroups of a group \( G \) also forms a lattice, with the same meet and join as \( L(G) \).
3.4.2 The poset of intermediate covers

In the previous section we said that a lattice of particular interest to us was the lattice of subgroups of a group. In this section and the next, we construct another lattice whose elements are, more or less, the coverings intermediate to a particular fixed covering $f$. In the next chapter we’ll see that if we look at this lattice sideways and squint our eyes a little, then it looks the same as the lattice of subgroups of the “group of automorphisms” of the covering $f$.

Throughout this section $f : Y \rightarrow Z$ is a fixed covering of connected 2-complexes. For $i = 1, 2$, let $Y \xrightarrow{p_i} (X_i) \xrightarrow{q_i} Z$ be connected coverings intermediate to $f$.

We call these two intermediate coverings equivalent if and only if there is an isomorphism $X_1 \rightarrow X_2$ making the diagram below right commute:

This is an equivalence relation on the set of coverings intermediate to $f$, and we write $L(Y \rightarrow^f Z)$ or just $L(Y, Z)$ for the set of equivalence classes of intermediate coverings. The notation here can become very cumbersome very quickly, so where possible we will write $X \in L(Y, Z)$ to mean that this equivalence class is represented by coverings $Y \rightarrow X \rightarrow Z$ intermediate to $f$.

It is possible to do everything in this section, and the next, in terms of intermediate coverings themselves, and not worry about equivalence at all. Nevertheless, it will be essential later to make sure all the accounting comes out in the wash. Figure 3.19 shows a pair of equivalent graph coverings. The graphs $X_i$ are necessarily the same, but the coverings are different.

![Fig. 3.19. Equivalent intermediate graph coverings: $Z$ is a bouquet of two loops and $Y$ is its universal cover, an infinite 4-valent tree. The intermediate $X_1 = X_2$, cover $Z$ (as it is so simple) by the same coverings $q_1 = q_2$. The coverings differ in how $Y$ covers the $X_i$ via the $p_i$: the first sends the green vertex of $Y$ to the red vertex of the $X_i$ and the second sends it to the blue vertex. There is an isomorphism $X_1 \rightarrow X_2$ interchanging the two vertices.](image)

We now turn $L(Y, Z)$ into a poset, for which the ordering is this: one (equivalence class of an) intermediate covering is “bigger” than another if the first covers the second. Specifically, if $X_1, X_2 \in L(Y, Z)$ then define $X_1 \leq X_2$ precisely when there is a covering $X_2 \rightarrow X_1$ making the diagram on the left of Figure 3.20 commute.

Presupposing for a minute that this definition makes sense and gives a partial order, we have
Definition 3.49 (poset of intermediate covers). For a fixed covering \( f : Y \to Z \) of conected complexes, the set \( L(Y, Z) \) of equivalence classes of connected intermediate coverings, together with the partial order \( \leq \) defined above is called the poset of intermediate coverings (to \( f \)).

It is not hard to check that this is well defined: suppose that for \( i = 1, 2 \), we have intermediate coverings \( X'_i \), equivalent to the \( X_i \) via isomorphisms \( X_i \leftrightarrow X'_i \). As isomorphisms are nothing other than degree one coverings, the red map across the middle of the diagram on the right of Figure 3.20 is a covering making the big outside square commute. Thus \( X'_1 \leq X'_2 \), and the order doesn’t depend on which representative for the equivalence class we choose.

Lemma 3.50. The set \( L(Y, Z) \) of equivalence classes of coverings intermediate to \( f : Y \to Z \), together with the \( \leq \) defined above, is a poset.

Proof. Reflexivity and transitivity are immediate, as the identity map is a covering and the composition of coverings is a covering. Anti-symmetry requires a moment’s thought: suppose we have intermediate coverings \( X_1, X_2 \in L(Y, Z) \) with \( X_1 \leq X_2 \) and \( X_2 \leq X_1 \), ie: there are coverings \( X_1 \Rightarrow X_2 \) making the appropriate diagrams commute. Let \( g \) be the composition \( X_1 \to X_2 \to X_1 \) of these two. Then consideration of these commuting diagrams gives that \( p_1 = gp_1 \), were \( p_1 : Y \to X_1 \) is the covering, and so by the surjectivity of \( p_1 \), \( g \) is the identity map on \( X_1 \). But then the covering \( X_1 \to X_2 \) must be injective, ie: of degree 1, and so an isomorphism. Thus \( X_1 = X_2 \) in \( L(Y, Z) \). \( \square \)

3.4.3 The lattice of intermediate covers

In the last section we introduced the poset \( L(Y, Z) \) of equivalence classes of coverings intermediate to a fixed covering \( Y \to Z \). In this section we show that we have in fact a lattice, with join a pullback and meet a pushout.

Throughout, \( f : Y_u \to Z_v \) is a fixed pointed covering of connected 2-complexes. All intermediate coverings \( Y \xrightarrow{p} X_s \xrightarrow{q} Z \) are connected and pointed, and \( L(Y_u, Z_v) \) is the poset of equivalence classes of pointed connected intermediate coverings.

We start by showing that we have a meet. Let \( X_{u, x_1} \xrightarrow{u} Y_u \xrightarrow{Y_{1, x_1}} Z_v \) and \( X_{u, x_2} \xrightarrow{u} Y_u \xrightarrow{Y_{2, x_2}} Z_v \) be intermediate to \( f \), and \( X_1 \coprod_Y X_2 \) the pushout of the coverings \( p_1 : Y_u \to Y_{1, x_1} \), and \( p_2 : Y_u \to (X_2)_{x_2} \). Let \( x = [x_1] = [x_2] \), where \( \cdot \mapsto [\cdot] \) is the quotient map arising from the construction of the pushout, and \( X_1 \coprod_Y X_2 \) the resulting pointed pushout.

By Proposition 3.35 we have a new element of \( L(Y_u, Z_v) \) given by the equivalence class of the intermediate covering,

\[ Y_u \to (X_1 \coprod_Y X_2)_x \to Z_v. \]

Our next result shows that if the \( X_i \) are replaced by equivalent coverings then the new pushout that results is equivalent to the old one:
Proposition 3.51. Let \((V_1)_{y_1}, (V_2)_{y_2} \in L(Y_u, Z_v)\) be equivalent to \((X_1)_{x_1}, (X_2)_{x_2}\) via the isomorphisms,
\[ g_1 : (X_1)_{x_1} \rightarrow (V_1)_{y_1} \text{ and } g_2 : (X_2)_{x_2} \rightarrow (V_2)_{y_2}. \]
Define a map \(g_1 \amalg g_2 : X_1 \amalg X_2 \rightarrow V_1 \amalg V_2\) between the disjoint unions by \(g_1 \amalg g_2 |_{X_i} = g_i\) and \(g_1 \amalg g_2 |_{X_3} = g_2\). Then the map
\[ g : (X_1 \coprod Y X_2)_x \rightarrow (V_1 \coprod Y V_2)_y, \quad (y = [y_1] = [y_2]), \]
defined by \(gr = r'(g_1 \amalg g_2)\), where \(r, r'\) are the quotient maps arising in the pushouts, is an isomorphism making these pointed pushouts equivalent.

Thus, the pushout can be extended in a well defined way to equivalence classes of intermediate coverings, and so for \((X_1)_{x_1}, (X_2)_{x_2} \in L(Y_u, Z_v)\) we write
\[ (X_1 \coprod Y X_2)_x \in L(Y_u, Z_v), \]
for the pushout of these two equivalence classes.

Proof (of Proposition 3.51), is a tedious but routine diagram chase. □

Now, Proposition 3.35 gives coverings \(t_i : X_i \rightarrow X_1 \coprod Y X_2\) so that \((X_1 \coprod Y X_2)_x \leq (X_i)_x\), is a lower bound in \(L(Y_u, Z_v)\) for each \(i\). If \(V_y\) is any other lower bound then we get coverings \(V_y \rightarrow (X_i)_x \rightarrow V_y\), and by the universality of the pushout, Proposition 1.37, we have a map \((X_1 \coprod Y X_2)_x \rightarrow (V)_y\), which by Proposition 3.22 is a covering. Thus \(V_y \leq (X_1 \coprod Y X_2)_x\), and the pushout is the meet of the two equivalence classes \((X_i)_x\).

Now to joins, which are similar. Let \(Y_0 \rightarrow (X_1)_{x_1} \rightarrow Z_v\) and \(Y_0 \rightarrow (X_2)_{x_2} \rightarrow Z_v\) be intermediate to \(f\), and \(X_1 \coprod Y Z X_2\) the pullback of the coverings \(g_1 : (X_1)_{x_1} \rightarrow Z_v\) and \(g_2 : (X_2)_{x_2} \rightarrow Z_v\). Let \(x = x_1 \times x_2\), a vertex of the pullback, and \((X_1 \coprod Z X_2)_x\) the pointed pullback consisting of the connected component containing \(x_1 \times x_2\).

We have a well-definedness result analogous to Proposition 3.51

Proposition 3.52. Let \((V_1)_{y_1}, (V_2)_{y_2} \in L(Y_u, Z_v)\) be equivalent to \((X_1)_{x_1}, (X_2)_{x_2}\) via the isomorphisms,
\[ g_1 : (X_1)_{x_1} \rightarrow (V_1)_{y_1} \text{ and } g_2 : (X_2)_{x_2} \rightarrow (V_2)_{y_2}. \]
Then the map
\[ g : (X_1 \coprod Z X_2)_x \rightarrow (V_1 \coprod Z V_2)_y, \quad (y = y_1 \times y_2), \]
defined by \(g(x_1 \times x_2) = g_1(x_1) \times g_2(x_2)\) is an isomorphism making these pointed pullbacks equivalent.

Proposition 3.37 gives coverings \(t_i : (X_1 \coprod Z X_2)_x \rightarrow (X_i)_x\) so that \((X_i)_x \leq (X_1 \coprod Y X_2)_x\) and the pullback is an upper bound in \(L(Y_u, Z_v)\) for each \(i\). If \(V_y\) is any other upper bound we get coverings \(V_y \rightarrow (X_i)_x \rightarrow Z_v\), and the by universality of the pullback, Proposition 1.43, we have a map \((V)_y \rightarrow (X_1 \coprod Z X_2)_x\). Proposition 3.22 again, this time applied to the commuting triangle formed by \((Y)_u \rightarrow (X_1 \coprod Y X_2)_x\) and \((Y)_u \rightarrow (V)_y \rightarrow (X_1 \coprod Z X_2)_x\), gives that the map \(V_y \rightarrow (X_1 \coprod Z X_2)_x\) is a covering. Thus \((X_1 \coprod Y X_2)_x \leq V_y\), and the pullback is the join of the two equivalence classes \((X_i)_x\).

Finally, recall from Exercise 3.42 that a \(\bar{1}\) in a poset is a unique maximal element, and dually, a \(\bar{0}\) is a unique minimal element. Here is the principle result of \S 3.4:

Theorem 3.53 (lattice of intermediate coverings). The poset \(L(Y_u, Z_v)\) of pointed connected covers intermediate to a fixed covering \(f : Y_u \rightarrow Z_v\) is a lattice with join \((X_1)_{x_1} \vee (X_2)_{x_2}\) the pullback \((X_1 \coprod Z X_2)_{x_1 \times x_2}\), meet \((X_1)_{x_1} \wedge (X_2)_{x_2}\) the pushout \((X_1 \coprod Y X_2)_{[x_i]}\), unique minimal element \(\bar{0} = Z_v\) and unique maximal element \(\bar{1} = Y_u\).
3 Coverings

The pointing of the covers in this section is essential if one wishes to work with connected intermediate coverings and also have a lattice structure (both of which we do). The problem is the pullback: because it is not in general connected, we need the pointing to tell us which component to choose.

3.5 Notes on Chapter 3
References

Books


Articles


[40] Stuart W. Margolis and John C. Meakin, Inverse monoids and rational Schreier subsets of the free group, Semigroup Forum 44 (1992), no. 2, 137–148.MR1141834 (93b:20110)


3.5 Notes on Chapter 3


