# The Geometry and Topology of Groups

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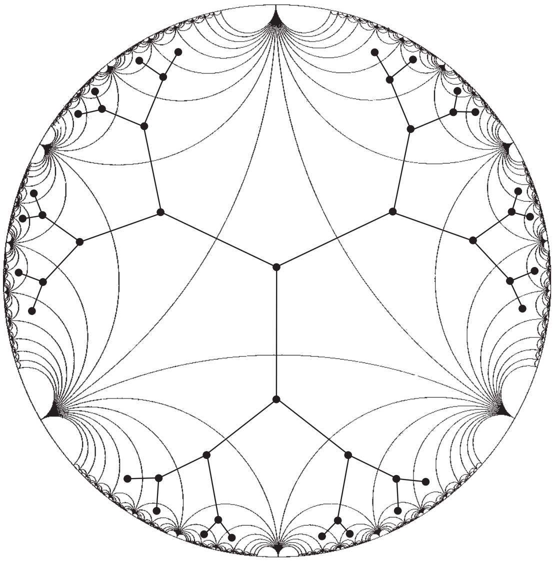

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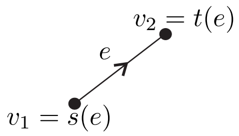
These notes are based on a series of lectures given at the Universidad Autonama Madrid in the Spring of 2003 and the University of York in the Autumn/Spring of 2001-2002.

# 1 Clockwork Topology

**(1.1)** A *combinatorial* $2$-*complex* $K$ is made up of three (countable) sets $V_K$, $E_K$ and $F_K$ (vertices, edges and faces), together with various incidence maps that describe how the pieces fit together. We have

$$s, t : E_K \longrightarrow V_K$$

$$^{-1} : E_K \longrightarrow E_K$$




so that $^{-1}$ assigns each edge to another, called its inverse, and $s, t$ assigns vertices to $e$, the start and terminal vertex. There are no restrictions on these maps except $e^{-1} \neq e$, $(e^{-1})^{-1} = e$, $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. Thus, we may draw pictures like the one on the right above, with $v_1 = s(e) = t(e^{-1})$ and $v_2 = t(e) = s(e^{-1})$, but it is important to keep in mind that such pictures are purely for illustrative purposes. One thinks of the inverse edge $e^{-1}$ as just $e$, but traversed in the reverse direction (or with the reverse orientation). The vertex and edge sets, together with these maps form a directed graph called the $1$-*skeleton* $K^1$ of $K$ (the vertices alone form the $0$-*skeleton* $K^0$).

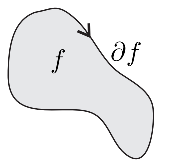
It is tempting to define faces of a 2-complex in terms of boundary paths, but we may want to have several faces with the same boundary, so we need to be a little more careful. A path in $K$ is a sequence of edges $e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k}$, $\varepsilon_i = \pm 1$, consecutively incident in the sense that $t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}})$, and closed if $t(e_k^{\varepsilon_k}) = s(e_1^{\varepsilon_1})$. The maps $s, t, ^{-1}$ can be extended to paths in the obvious way, with $(e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k})^{-1} = e_k^{-\varepsilon_k} \dots e_1^{-\varepsilon_1}$. We also allow a single vertex to be a path.

Two paths $w_1$ and $w_2$ are *cyclic permutations* of each other if $w_1 = e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k}$ then $w_2 = e_j^{\varepsilon_j} \dots e_k^{\varepsilon_k} e_1^{\varepsilon_1} \dots e_{j-1}^{\varepsilon_{j-1}}$ for some $k$. A *cycle* in the 1-skeleton is a set consisting of a path and all of its cyclic permutations.

The final ingredient then in the definition of a combinatorial $2$-complex is the incidence maps that say how the faces are glued onto the 1-skeleton,

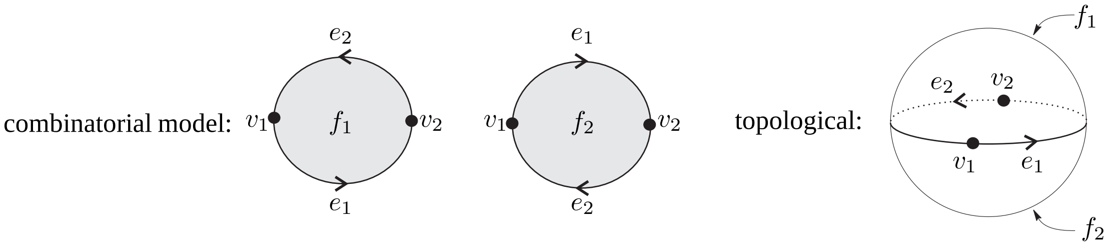
$$^{-1} : F_K \longrightarrow F_K$$

$$\partial : F_K \longrightarrow \text{cycles}$$

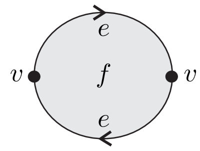



which must satisfy $f^{-1} \neq f$, $(f^{-1})^{-1} = f$, and $w \in \partial(f)$ iff $w^{-1} \in \partial(f^{-1})$. One thinks of the inverse face $f^{-1}$ as just $f$, but with the boundary traversed in the reverse direction. When there is no need to be more precise, we will often just write $\partial f$ to designate some boundary label of $f$. All our complexes will be *locally finite*, in that the image of any object under the incidence maps $s, t$ and $\partial$ is finite.

**(1.2)** Our first example is shown below left, where we have adopted the convention (which we will generally maintain) that parts of the complex with the same label give the same element of the set (so are to be thought of as being identified). This 2-complex has two vertices, two edges and two faces, drawn in a "face-centric" manner. Carrying out the identifications gives a 2-sphere as shown on the right.

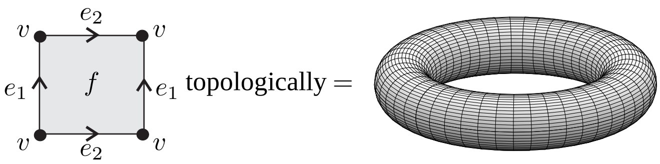


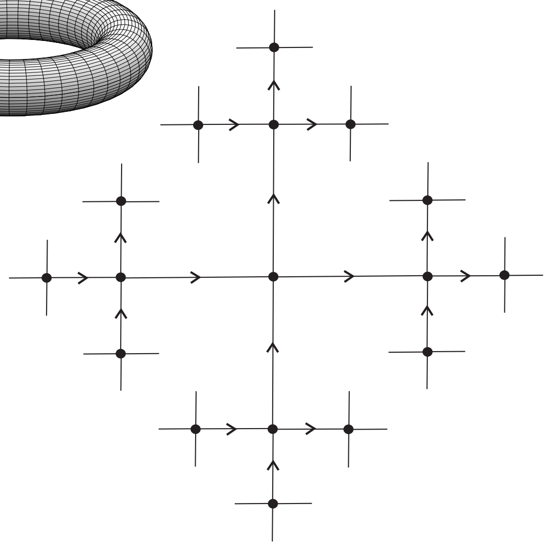

Neither the identifications nor the topological interpretation are an intrinsic part of the complex: they are purely to guide the intuition and to provide motivation. The complex is completely determined by the data: $V_K = \{v_1, v_2\}$, $E_K = \{e_1, e_2\}$ and $F_K = \{f_1, f_2\}$ with incidence maps $s(e_1) = v_1, t(e_1) = v_2$, and so on, which would suffice if necessary for a purely logical (ie: algebraic) development. We shall take advantage however of the nice topological analogy and not hesitate to draw topological pictures.

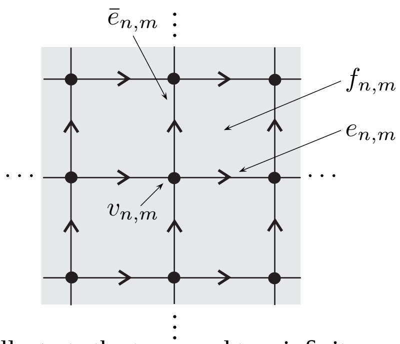



Similarly we call this complex the (real) projective plane $\mathbb{RP}^2$, a combinatorial model for the disc with antipodal points on the boundary identified. The justification for the name projective is the following: $\mathbb{RP}^2$ is the space of 1-dimensional linear subspaces of $\mathbb{R}^3$, the points of which are in one to one correspondence with pairs of antipodal points on the 2-sphere in $\mathbb{R}^3$. "Throwing away" the northern hemisphere gives a disc with antipodal points on the equator identified. We have again drawn the complex face-centrically at the expense of repeating the single vertex and edge twice in the picture. An alternative picture would have a single vertex and an edge loop with a face "sewn in" so that its boundary travels twice around the loop.

Similarly,

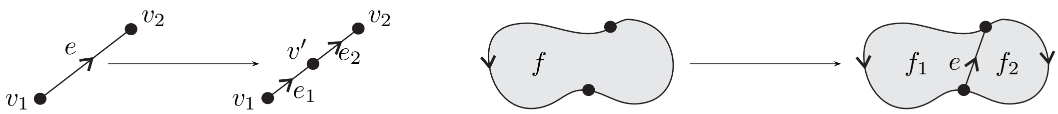





illustrate the torus and two infinite complexes: the plane and the infinite 4-valent tree.

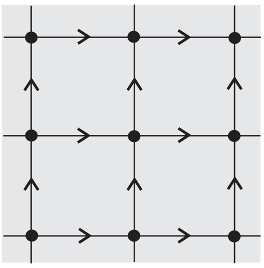
**(1.3)** Because of their very simple combinatorial nature, we often need to be able to make the structure of a 2-complex "finer" by chopping edges and faces into pieces. There are two fundamental moves:




namely subdividing an edge into two new ones (by adding a new vertex) or subdividing a face into two new ones (by adding a new edge). We leave it to the reader to verify the (obvious) fact that the new objects are 2-complexes. A *subdivision* of a complex is then the complex obtained by applying these two moves a finite number of times (in any order).

**(1.4)** A *subcomplex* $K'$ of $K$ is a collection of three subsets $V_{K'} \subseteq V_K, E_{K'} \subseteq E_K, F_{K'} \subseteq F_K$ such that the restriction to these subsets of the various incidence maps makes $K'$ into a 2-complex. Thus the equator of the 2-sphere complex is a subcomplex (as is indeed the 1-skeleton of any 2-complex) while the torus complex above is not a subcomplex of the plane complex.

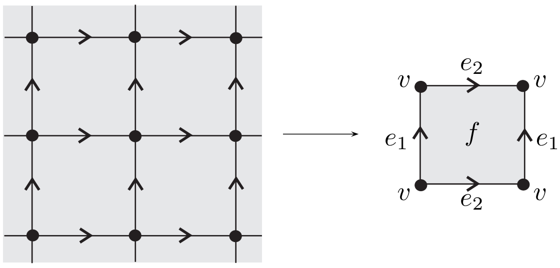
**(1.5)** We can glue complexes together (or even glue a complex to itself) to obtain new complexes. Suppose $\sim$ is an equivalence relation on the sets $V_K, E_K$ and $F_K$ (using the same symbol for the three different relations). The relation $\sim$ needs to be compatible with the incidence maps so that if edges $e_1 \sim e_2$ are equivalent then so are the start and terminal vertices $s(e_1) \sim s(e_2)$, $t(e_1) \sim t(e_2)$ as well as $e_1^{-1} \sim e_2^{-1}$; if faces $f_1 \sim f_2$, then for any $w_1 \in \partial f_1$ there is a unique $w_2 \in \partial f_2$ such that if $w_1 = e_{\alpha_1}^{\varepsilon_1} \dots e_{\alpha_n}^{\varepsilon_n}$ then $w_2 = e_{\beta_1}^{\nu_1} \dots e_{\beta_n}^{\nu_n}$ and $e_{\alpha_i} \sim e_{\beta_i}$; similarly $f_1 \sim f_2$ precisely when $f_1^{-1} \sim f_2^{-1}$. The final condition is that an object is never equivalent to its inverse: $e \not\sim e^{-1}$ and $f \not\sim f^{-1}$. This is so that when we glue equivalent things together, objects and their inverses remain distinct from each other, which is one of the fundamental properties of a 2-complex.



A trivial example is the "Euclidean plane" complex where $\sim$ makes all vertices and faces equivalent and two edges are equivalent iff they are both horizontal or both vertical. Given such a $\sim$, the *quotient complex* $K/\!\sim$ has vertices, edges and faces the sets $V_K/\!\sim, E_K/\!\sim$ and $F_K/\!\sim$; if $[\cdot]$ is some equivalence class, then the new incidence maps (for which we'll use the same letters as the old ones) are $s([e]) = [s(e)]$, $t([e]) = [t(e)]$, $[e]^{-1} = [e^{-1}]$, $\partial([f]) = [\partial(f)]$ and $[f]^{-1} = [f^{-1}]$. One can extend these maps to paths in an obvious way.

**Exercise 1** Check that these maps make sense and indeed give a 2-complex $K/\!\sim$. Show that the map $q$ associating to each vertex, edge and face, its equivalence class $q(\cdot) = [\cdot]$, is a mapping of 2-complexes $q : K \to K/\!\sim$ (see §2).

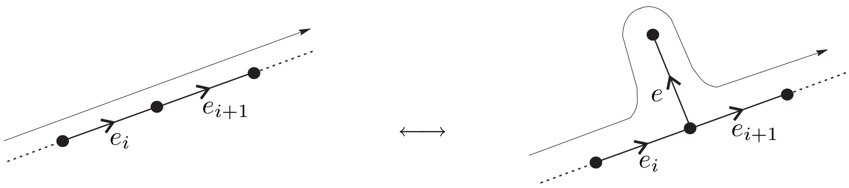
Thus the quotient in the example above is the torus complex.




If the relation is weakened a little so that two vertices, edges or faces are equivalent precisely when one can be shifted horizontally onto the other then $K/\!\sim$ is an infinite cylinder.

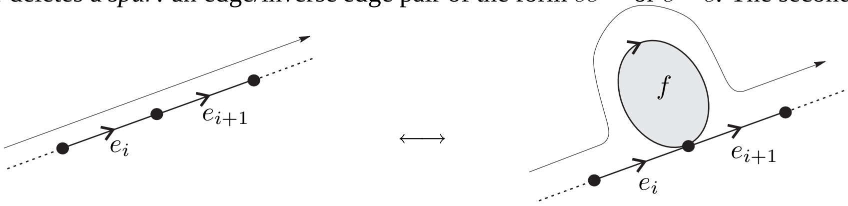
## 2 From 2-Complexes to Groups

**(2.1)** We now formulate a discrete version of homotopy–the deformation of paths in a 2-complex. Given a path in a 2-complex, we allow ourselves two fundamental moves; the first




$$e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k} \qquad\qquad e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} (ee^{-1}) e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k}$$

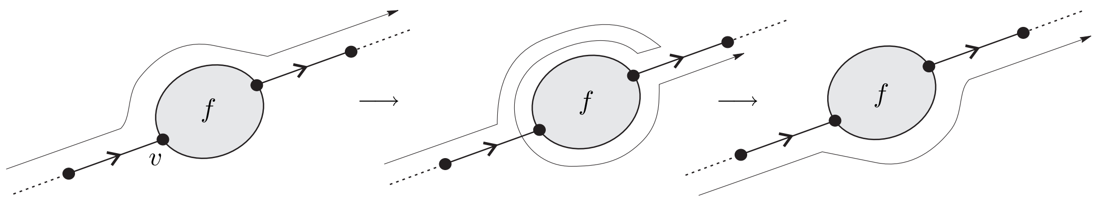
inserts or deletes a *spur*: an edge/inverse edge pair of the form $ee^{-1}$ or $e^{-1}e$. The second




$$e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k} \qquad\qquad e_1^{\varepsilon_1} \dots e_i^{\varepsilon_i} \partial f e_{i+1}^{\varepsilon_{i+1}} \dots e_k^{\varepsilon_k}$$

inserts or deletes the boundary of a face: a $w \in \partial(f)$ for $f$ some face of $K$ with $s(w) = t(w) = t(e_i^{\varepsilon_i}) = s(e_{i+1}^{\varepsilon_{i+1}})$.

Two paths $w_1, w_2$ are *homotopic* (written $w_1 \sim_h w_2$) iff there is a finite sequence of these two moves taking one path to the other. For example, two paths running different ways around a face are homotopic:

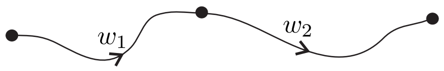



To get from the first picture to the second, insert the boundary of the face, using the $w \in \partial(f)$ with start vertex $v$; to get from the second to the third, remove the obvious spurs.

**Exercise 2** Show that homotopic paths must have the same start and terminal vertices and that homotopy is an equivalence relation on the set of paths in $K$.

Denote the homotopy equivalence class of a path $w$ by $[w]_h$.

**(2.2)** If $w_1, w_2$ are paths in a 2-complex $K$ with the terminal vertex of $w_1$ the start vertex of $w_2$, then let $w_1w_2$ be the path obtained by juxtaposing these two, ie: by traversing the edges of $w_1$ and then the edges of $w_2$:




We can think of this as a kind of "product" of paths, which in fact is defined upto homotopy in the sense that if,

$$w_1 \sim_h w_1' \text{ and } w_2 \sim_h w_2' \text{ then } w_1w_2 \sim_h w_1'w_2'.$$

(Exercise!) We can thus extend to a product on the homotopy classes of paths in $K$: if $[w_1]_h$ and $[w_2]_h$ are two such, where $t(w_1) = s(w_2)$, then

$$[w_1]_h[w_2]_h \stackrel{\text{def}}{=} [w_1w_2]_h.$$

This multiplication fails to give us a group as we can only multiply paths if the first finishes where the second starts.

**Aside 1** Assuming the reader knows the definition of a category, we have obvious "large" categories like the category of all groups, where the collection of objects (the groups) does not form a set. On the other hand there are the "small" categories where the objects *do* form a set. Indeed a single group can be turned into a category in the following way: there is just one object, $*$ say, and morphisms $* \to *$ in 1-1 correspondence with the elements of the group. The composition of morphisms is just the group operation, ie: if $f : * \to *$ and $g : * \to *$ are morphisms then $h : * \to *$ is their composition iff $h = fg$ in the group.

Generalising just a little, we can replace the single element object set by an arbitrary set $\mathsf{Ob}$. A *groupoid* is a small category such that every morphism is invertible: for each $x, y \in \mathsf{Ob}$ there are distinguished morphisms $\text{id}_x : x \to x$ and $\text{id}_y : y \to y$ such that for any $f : x \to y$ there is a $f^{-1} : y \to x$ with the compositions $ff^{-1}$ and $f^{-1}f$ equal to the respective identities.

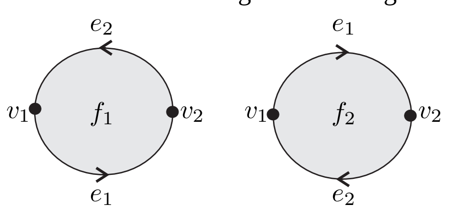
There are various interesting examples that arise in nature. The tiling of the plane by squares has a group of symmetries (isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$) whereas a Penrose tiling of the plane has a *groupoid* of "local symmetries". For another example let $X$ be a topological space so that a choice of basepoint gives the (topological) fundamental group of $X$. Instead we could construct a groupoid by taking as object set the space $X$ and for any two points in the space, the morphisms between them the homotopy classes of paths joining the points. One composes morphisms in the obvious way by considering the homotopy class of the juxtaposed path whenever the first finishes at the same point that the second starts. This is the *fundamental groupoid* of the space $X$.

**Exercise 3** Show that the collection of homotopy classes of paths in $K$, together with the multiplication above, forms a groupoid, *the fundamental groupoid of $K$*.

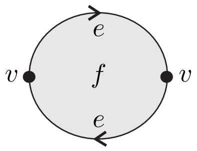
**(2.3)** To get a group from homotopy classes of paths we need to ensure we can always multiply paths. Let $K$ be a 2-complex and fix a vertex $v$. Let $\pi_1(K, v)$ be the set of all homotopy classes of closed paths with start (and hence terminal) vertex $v$.

**Theorem 2.1** *$\pi_1(K, v)$ together with the product $[w_1]_h[w_2]_h = [w_1w_2]_h$ forms a group with identity $[v]_h$ and $[w]_h^{-1} = [w^{-1}]_h$.*

If you are familiar with the corresponding result from topology (where $K$ is replaced by a topological space and homotopies are continuous) then you will know that the proof, while not difficult, is fiddly. For instance, associativity requires the construction of a homotopy from the path $(w_1w_2)w_3$ to the path $w_1(w_2w_3)$. In contrast, the proof of Theorem 2.1 is completely straightforward and left as an exercise (for example, associativity is immediate!). That the proofs are simpler, in the early stages at least, is one of the main advantages of working combinatorially rather than topologically.

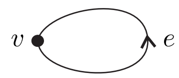



To compute $\pi_1(S^2, v_1)$, it is clear that any loop in the 1-skeleton based at $v_1$ must be of the form $w = (e_1e_2)^{\varepsilon_1} \dots (e_1e_2)^{\varepsilon_k}$ with the $\varepsilon_i = \pm 1$. Equally clearly, the loops $(e_1e_2)^{\pm 1}$ are homotopically trivial, bounding as they do the faces $f_1$ and $f_2$. Thus any loop is homotopically trivial and the group is the trivial group.



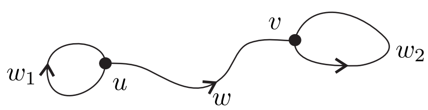

To compute the fundamental group of the projective plane, note that the edge $e$ is a loop based at $v$, homotopically non-trivial as it is neither the boundary of a face or a spur. On the other hand, $ee$ is a homotopically trivial loop, and in fact these are homotopically the only two possibilities. Indeed, any loop based at $v$ has the form $e^k$ for some $k \in \mathbb{Z}$, and is homotopically trivial if and only if $k \equiv 0 \text{ mod } 2$, otherwise it is homotopic to $e$. Thus, the fundamental group is isomorphic to the cyclic group $\mathbb{Z}/2$.

The single loop complex shown has a somewhat larger fundamental group. It is intuitively clear that any path has the form $e^k$ for some $k \in \mathbb{Z}$, where no two $e^k$ and $e^l$ are homotopic (unless $k = l$) making $\pi_1(K)$ in this case look like $\mathbb{Z}$ itself. Generalising, the "bouquet of circles" complex consists of a single vertex with a number of loops based at it. It has fundamental group consisting of all (homotopy classes of) paths $e_{i_1}^{\varepsilon_1} \dots e_{i_k}^{\varepsilon_k}$ with the $e_{i_j}$ edges and where two such are distinct if neither contains a spur and they are not identical (these intuitive statements will be justified later).




**(2.4)** A 2-complex is *connected* if there is a path between any two of its vertices. In this case there is an essentially unique fundamental group of $K$, for if $u, v$ are vertices and $w$ is a path in $K$ from $u$ to $v$, then define a map $\varphi : \pi_1(K, u) \longrightarrow \pi_1(K, v)$ by

$$[w_1]_h \overset{\varphi}{\longmapsto} [w^{-1}w_1w]_h.$$



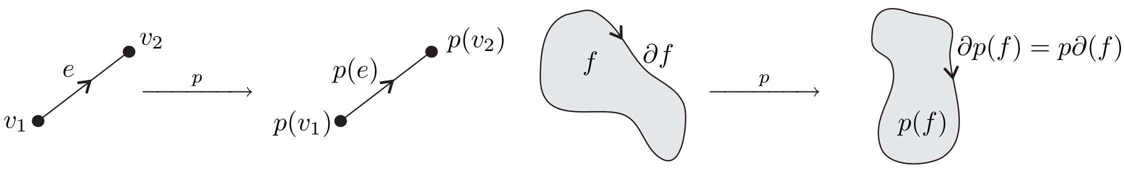

**Exercise 4**

1. Show that if $w_1 \sim_h w_1'$ then $ww_1w^{-1} \sim_h ww_1'w^{-1}$, thus $\varphi$ is well defined.
2. Show that $\varphi$ is a homomorphism.
3. Define $\psi : \pi_1(K, v) \longrightarrow \pi_1(K, u)$ by $[w_2]_h \longmapsto [ww_2w^{-1}]_h$. Show that $\psi$ is a well defined homomorphism and that $\varphi\psi$ and $\psi\varphi$ are the respective identity maps (this means that $ww^{-1}w_1ww^{-1} \sim_h w_1$ and $w^{-1}ww_2w^{-1}w \sim_h w_2$).

We will only ever deal with connected complexes, in which case the exercise means we get the same group (upto isomorphism) no matter which vertex we choose to base our closed paths at. We will often write just $\pi_1(K)$ for the fundamental group from now on.

**(2.5)** There are classes of complexes defined by their fundamental groups. For example, a complex $K$ is called a *tree* iff the face set of $K$ is empty and $\pi_1(K)$ is the trivial group. As there are no face boundaries to insert or delete from paths, this means that any loop in $K$ can be reduced to the empty loop by a finite sequence of insertions or deletions spurs. Alternatively, between any two vertices there is a unique path without spurs (as two paths differ by a loop, which must therefore be a collection of spurs).

**(2.6)** A *map* $K_1 \xrightarrow{p} K_2$ between 2-complexes assigns to each vertex of $K_1$ a vertex of $K_2$, each edge of $K_1$ an edge or vertex of $K_2$, and each face of $K_1$ a face, path or vertex $K_2$. All of this must be done in an incidence preserving manner:




and $p(e^{-1}) = p(e)^{-1}, p(f^{-1}) = p(f)^{-1}$. If $e$ is mapped to a vertex then it must be one of $p(v_1)$ or $p(v_2)$; if $f$ is mapped to a path then it must be homotopically trivial. The incidence preserving condition is meant to be a combinatorial version of continuity.

As it preserves the incidence of edges at a vertex, a map can be extended to paths in an obvious manner, and so in particular can be extended to closed paths. Thus, if $v$ is a vertex of $K_1$ we can define a map

$$\pi_1(K_1, v) \xrightarrow{p^*} \pi_1(K_2, p(v))$$

by taking $p^*[w]_h = [p(w)]_h$ for any closed path $w$ in $K_1$.

**Exercise 5**

1. Show that $p^*$ is well-defined (ie: sends homotopic paths to homotopic paths) and is a group homomorphism.
2. Let 2-Comp be the category with objects the 2-complexes and morphisms the maps between 2-complexes defined above. Show that $\pi_1$ is a (covariant) functor from 2-Comp to the category Groups (with morphisms the group homomorphisms).

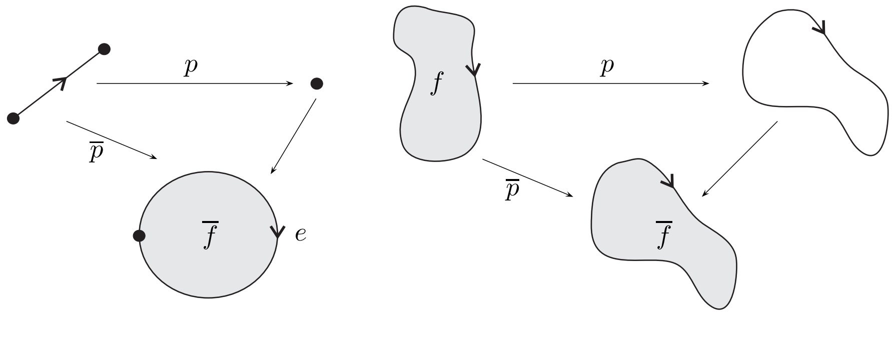
**(2.7)** A map is *dimension preserving* if $V_{K_1} \xrightarrow{p} V_{K_2}$, $E_{K_1} \xrightarrow{p} E_{K_2}$ and $F_{K_1} \xrightarrow{p} F_{K_2}$ (abusing notation by using the same letter for all three maps). One can always ensure that a map is dimension preserving by adjusting, if necessary, the image complex.

**Proposition 2.1** *If $K_1 \xrightarrow{p} K_2$ is a map of 2-complexes then there is a 2-complex $\overline{K}_2$ and a dimension preserving map $K_1 \xrightarrow{\overline{p}} \overline{K}_2$ such that $\pi_1(K_2, p(v)) \cong \pi_1(\overline{K}_2, \overline{p}(v))$, and*

$$\begin{array}{ccc} \pi_1(K_1, v) & \xrightarrow{p_*} & \pi_1(K_2, p(v)) \\ & \overline{p}_* \searrow \quad \nearrow \cong & \\ & \pi_1(\overline{K}_2, \overline{p}(v)) & \end{array}$$

*commutes.*

**Proof:** If the map is not dimension preserving, the two things that could happen is that



an edge is mapped to a vertex, or a face to a homotopically trivial path. In the second case, add a new face $\overline{f}$ to $K_2$ with boundary the image path. In the first, add a new edge-loop $e$ at the vertex and a new face $\overline{f}$ with boundary this loop. We leave it to the reader to show these modifications have the desired effect. □

**(2.8)** A map is an *isomorphism* if it preserves dimension and is a bijection on the vertex, edge and face sets. It is then easy to see that the inverse map $p^{-1}$ on each set preserves incidence and is also a 2-complex map $K_2 \to K_1$.

An *automorphism* of $K$ is an isomorphism $K \to K$, and as usual they form a group under composition Aut$(K)$. If $K$ is a finite complex then Aut$(K)$ is clearly a finite group.

**Exercise 6** What, if any, is the relationship between Aut$(K)$ and Aut$(\pi_1(K))$?

A map $K \xrightarrow{p} K$ of a complex to itself is an *inversion* iff $p(e) = e^{-1}$ for some edge $e$ or $p(f) = f^{-1}$ for some face $f$. Let Aut$^+(K)$ be the subset of Aut$(K)$ consisting of those automorhisms that are *not* inversions.

**Exercise 7** Is Aut$^+(K)$ a subgroup of Aut$(K)$?

**(2.9)** A group $G$ *acts* on a complex $K$ if there is a homomorphism of groups $G \to$ Aut$(K)$. The action is *orientation-preserving* when the image of the map $G \to$ Aut$(K)$ is contained in Aut$^+(K)$. One also says that $G$ acts *without inversions*.

Whenever we have an orientation-preserving action of a group on a complex we obtain a quotient complex. The orbits of the action form the equivalence classes of an equivalence relation on $K$, where two objects are equivalent iff the action takes one to the other. Let two vertices $v_1 \sim v_2$ precisely when they lie in the same orbit of the $G$-action on $V_K$; define $\sim$ in the same way on $E_K$ and $F_K$. It isn't hard, using the properties of maps, to see that this equivalence relation is compatible with the incidence maps for $K$. As the action is orientation-preserving, we never have an object $* \sim *^{-1}$, that is, equivalent to its inverse. From now on, *all group actions on complexes will be assumed to be orientation preserving*.

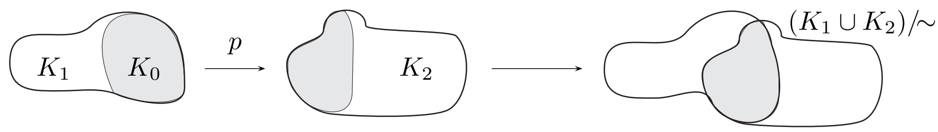
The quotient $K/\sim$, written $K/G$, is called the quotient of $K$ by the action of $G$. As an example, let $G = \mathbb{Z} \oplus \mathbb{Z}$ act on the Euclidean plane complex by

$$*_{n,m} \xrightarrow{(1,0)} *_{n+1,m} \text{ and } *_{n,m} \xrightarrow{(0,1)} *_{n,m+1},$$

where $*$ is some vertex, edge or face, and a general $(\alpha, \beta) \in \mathbb{Z} \oplus \mathbb{Z}$ acts as the composition of $\alpha$ times the first map followed by $\beta$ times the second. The quotient complex $K/G$ is the torus 2-complex.

As with any quotient of complexes we have a quotient map $q : K \to K/G$ sending an object of $K$ to its orbit. As this is a map of 2-complexes (see the exercise at the end of **(1.5)**) we have an induced (surjective) map $q_* : \pi_1(K) \to \pi_1(K/G)$. Hence *the fundamental group of a quotient is a quotient of the fundamental group*.

**(2.10)** We saw in §1 that an equivalence relation could be used to form the quotient of a 2-complex. We can also use a map between complexes to glue the two together. Suppose that $p : K_0 \to K_2$ is a map of 2-complexes where $K_0$ is a subcomplex of a larger $K_1$,




If $*$ is any vertex, edge or face of $K_0$, let $\sim$ be the equivalence relation on the disjoint union $K_1 \cup K_2$ *generated* by the relations $* \sim p(*)$. The quotient complex $(K_1 \cup K_2)/\sim$ is what is obtained by *gluing* $K_1$ *and* $K_2$ *together along* $K_0$, via the *attaching map* $p$.

**Aside 2** We say *generated* because $* \sim p(*)$ by itself does not give an equivalence relation. By generated we mean the following: any equivalence relation on a set $X$ corresponds to a subset $S \subseteq X \times X$ with $(x, y) \in S$ iff $x \sim y$. Given relations $\sim_1$ and $\sim_2$ on $X$ we can therefore say that $\sim_1$ is *weaker* than $\sim_2$ iff the corresponding $S_1 \subseteq S_2$. Given any subset $W \subseteq X \times X$, the equivalence relation *generated* by $W$ is then the weakest equivalence relation containing $W$ (weakest in the sense that no weaker relation contains $W$).

# 3 From 2-Complexes to Groups: more advanced features

**(3.1)** A group $G$ acts *freely* on a 2-complex $K$ if no non-trivial element of $G$ fixes a vertex. When we have a free action, we can relate the fundamental group of the quotient complex to $G$ in the following way: we have a map of 2-complexes $p : K \to K/G$ which induces a homomorphism $p^* : \pi_1(K) \to \pi_1(K/G)$.

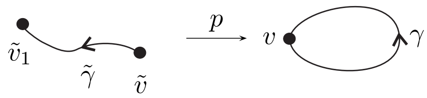
**Theorem 3.1** *Let $G$ act freely on the (connected) 2-complex $K$. Then,*

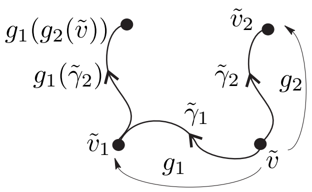
$$\pi_1(K/G)/p^*(\pi_1(K)) \cong G.$$

**Proof:** Given $v \in V_{K/G}$ and $e \in E_{K/G}$ with start vertex $v$, let $\tilde{v}$ be a vertex of $K$ such that $p(\tilde{v}) = v$ (ie: $v$ is really an equivalence class of vertices and $\tilde{v}$ a representative vertex of it). Now, there is certainly an edge $\tilde{e}$ of $K$ with start vertex $\tilde{v}$ that represents the equivalence class of edges $e$ (ie: $p(\tilde{e}) = e$). There cannot however be distinct $\tilde{e}_1, \tilde{e}_2 \in K$ starting at $\tilde{v}$ and representing $e$, for if so, there would be a non-trivial element of $G$ sending $\tilde{e}_1$ to $\tilde{e}_2$ and thus fixing $\tilde{v}$.

Thus for any representative vertex $\tilde{v}$ for $v$, there is a *unique* representative edge $\tilde{e}$ for $e$ starting at $\tilde{v}$. This can be extended (by induction) to paths: if $\gamma$ is a path in $K/G$ starting at $v$ there is a unique representative path $\tilde{\gamma}$ for $\gamma$ in $K$ and starting at the representative vertex $\tilde{v}$.

Fix a basepoint $v$ in $K/G$ and a representative vertex $\tilde{v}$ for it in $K$. If $\gamma$ is a closed path in $K/G$ at $v$ then let $\tilde{\gamma}$ be as above, hence a path in $K$ (now not necessarily closed) from $\tilde{v}$ to some other representative vertex for $v$, say $\tilde{v}_1$. In particular, there is a unique element $g \in G$ that takes $\tilde{v}$ to $\tilde{v}_1$ (unique, as if there are two such elements $g_1, g_2$, then $g_1 g_2^{-1}$ fixes $\tilde{v}$ hence is the identity, so $g_1 = g_2$). Define a map $\varphi : \pi_1(K/G) \to G$ by sending $[\gamma]_h$ to $g$. It is not hard to show that if $\gamma_1$ is homotopic to $\gamma$ then the unique representative $\tilde{\gamma}_1$ starting at $\tilde{v}$ also finishes at $\tilde{v}_1$. Thus $\varphi$ is a well-defined map.



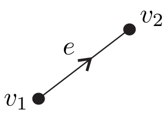

For $\gamma_1, \gamma_2$ loops at $v$ let $\tilde{\gamma}_i$ be the unique representative paths in $K$ starting at $\tilde{v}$ and $g_i$ the elements of $G$ taking $\tilde{v}$ to $\tilde{v}_i$. Then $\tilde{\gamma}_1 g_1(\tilde{\gamma}_2)$ is a path representing $\gamma_1\gamma_2$, starting at $\tilde{v}$ and finishing at $g_1(g_2(\tilde{v}))$, so by uniqueness, $\varphi[\gamma_1\gamma_2]_h = g_1 g_2 = \varphi[\gamma_1]_h \varphi[\gamma_2]_h$. For $g \in G$, let $\tilde{\gamma}$ be a path in $K$ from $\tilde{v}$ to $g(\tilde{v})$, a representative for a loop in $K/G$ (as $\tilde{v}$ and $g(\tilde{v})$ represent the same vertex of $K/G$). Thus $\varphi[\gamma]_h = g$ and so $\varphi$ in onto. Finally $[\gamma]_h$ is in the kernel precisely when the representative $\tilde{\gamma}$ is a loop (by the freeness of the action). It is easy to see that such $\gamma$ correspond precisely to $p^*(\pi_1(K))$. The result then follows by applying the first isomorphism theorem. $\square$

**Exercise 8** Show that the theorem is not true if we drop our standing insistence that group actions preserve orientation.

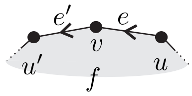
**(3.2)** Another algebraic invariant for topological spaces that is useful in the combinatorial setting is (integral) homology. For a 2-complex $K$, let $C_0(K)$ be the free Abelian group with generators the vertices $V_K$. Thus the elements of $C_0(K)$ are formal expressions of the form,

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k,$$

with the $a_i \in \mathbb{Z}$. Two such 0-*chains* are added component-wise by adding the coefficients of each vertex. Similarly we have the free Abelian group $C_1(K)$ of 1-chains on the generators $E_K$ and the 2-chains $C_2(K)$ on the faces. If the complex is finite, then these groups are nothing other than $C_0(K) \cong \mathbb{Z}^{|V_K|}, C_1(K) \cong \mathbb{Z}^{|E_K|}$ and $C_2(K) \cong \mathbb{Z}^{|F_K|}$.




The boundary of an edge $e$ is defined by $\partial_1(e) = v_2 - v_1 \in C_0(K)$. Given any 1-chain $\sum a_i e_i$, let $\partial_1(\sum a_i e_i) = \sum a_i \partial_1(e_i)$. It is straight forward to verify that $\partial_1 : C_1(K) \to C_0(K)$ is a group homomorphism. Similarly for a face $f$ with boundary label $e_1^{\alpha_1} \dots e_k^{\alpha_k}$ we have the boundary $\partial_2(f) = \sum \alpha_i e_i \in C_1(K)$, and extending this to 2-chains in exactly the same way as the above gives a homomorphism $\partial_2 : C_2(K) \to C_1(K)$.




If we now consider the image of a face $f$ under the composition map $\partial_1\partial_2$, we get for the section of boundary shown, that $\partial_2(f)$ is a sum whose terms include $e + e'$ in $C_1(K)$. In the image of this segment under $\partial_1$ the contribution of vertex $v$ is $v$ (from $\partial_1(e) = v - u$) and $-v$ (from $\partial_1(e') = u' - v$). Thus $\partial_1\partial_2$ maps this face to $0 \in C_0(K)$.

The sequence of Abelian groups and boundary maps,

$$C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K)$$

thus satisfies $\partial_1\partial_2 = 0$. Call an element of $C_i(K)$ that is in the kernel of the appropriate $\partial_i$ an *$i$-cycle*[1] (intuitively cycles have no boundary), while an element in the image is an *$i$-boundary*. By the previous paragraph, if $w$ is a 1-boundary then $w = \partial_2(\sum a_i f_i)$ for some 2-chain, hence $\partial_1(w) = \partial_1\partial_2(\sum a_i f_i) = 0$ as $\partial_1\partial_2$ is the zero map. In other words, *every 1-boundary is also a 1-cycle*.

Being the kernels and images of the groups $C_i(K)$ under the $\partial$ homomorphism, the $i$-cycles and boundaries are subgroups of $C_i(K)$, and as these groups are abelian, we may form quotients. The homology groups of $K$ are then,

$$H_0(K) = \frac{C_0(K)}{\text{0-boundaries}}, H_1(K) = \frac{\text{1-cycles}}{\text{1-boundaries}}, H_2(K) = \text{2-cycles}.$$

The middle group is the typical one; the outer two differ as there are no elements of $C_2(K)$ in an image and none in $C_0(K)$ in a kernel. The coset containing a particular object in any dimension is its *homology class*, denoted $[\cdot]_H$.

**Proposition 3.1** *1. If $K$ is connected then $H_0(K) \cong \mathbb{Z}$.*

*2. If $K$ is a graph then $H_2(K)$ is trivial.*

*3. If $w$ is a loop at the basepoint $v$ of $K$ with label $e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k}$, then the map given by*

$$[e_1^{\varepsilon_1} \dots e_k^{\varepsilon_k}]_h \mapsto \left[\sum \varepsilon_i e_i\right]_H$$

*is a homomorphism $\pi_1(K, v) \to H_1(K)$.*

The proof is left as an exercise. The homomorphism from the third part is called the *Hurewicz map*.

**Exercise 9** Show that the Hurewicz map is surjective with kernel the commutator subgroup of $\pi(K, v)$, ie: the subgroup whose elements consist of all products of commutators $[a, b] = aba^{-1}b^{-1}$ and their inverses.

**Exercise 10** Call an abelian group *finitely generated* if it is isomorphic to $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \times \cdots \times \mathbb{Z}/n_k \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, a finite product of finite cyclic group $\mathbb{Z}/n$'s and $\mathbb{Z}$'s. Find examples of graphs $K$ where $H_1(K)$ is not finitely generated.

**(3.3)** If the homologies of a complex $K$ are finitely generated, then call the number of $\mathbb{Z}$'s in $H_i(K)$ its (torsion free) *rank* and denote it $\text{rank}_{\mathbb{Z}} H_i(K)$. The *Euler characteristic* of $K$ is then defined as

$$\chi(K) = \sum (-1)^i \text{rank}_{\mathbb{Z}} H_i(K).$$

**Proposition 3.2** *If $K$ is a finite complex, show that the Euler characteristic is given by*

$$\chi(K) = |V_K| - |E_K| + |F_K|.$$

**Proof:** Is left as an exercise.

[1] Don't confuse these with the cycles of §1!

# 4 Presentations

**(4.1)** Let $X = \{x_\alpha \,|\, \alpha \in A\}$ be a *countable* set, and $X^{-1} = \{x_\alpha^{-1} \,|\, \alpha \in A\}$ another of the same cardinality disjoint from $X$. A *word* in the symbols $X \cup X^{-1}$ is an expression of the form

$$x_{\alpha_1}^{\varepsilon_1} x_{\alpha_2}^{\varepsilon_2} \dots x_{\alpha_k}^{\varepsilon_k},$$

where $\varepsilon_i = \pm 1$. We also have the empty word, which is denoted by a complete absence of symbols! For words $w_1, w_2$, let $w_1 w_2$ be their juxtaposition (so that if either is empty then $w_1 w_2$ is just the other) and if $w = x_{\alpha_1}^{\varepsilon_1} \dots x_{\alpha_k}^{\varepsilon_k}$ then the inverse word is $w^{-1} = x_{\alpha_k}^{-\varepsilon_k} \dots x_{\alpha_1}^{-\varepsilon_1}$. A cyclic permutation of $w$ is a word of the form $x_{\alpha_i}^{\varepsilon_i} \dots x_{\alpha_k}^{\varepsilon_k} x_{\alpha_1}^{\varepsilon_1} \dots x_{\alpha_{i-1}}^{\varepsilon_{i-1}}$.

Let $R = \{w_\beta \,|\, \beta \in B\}$ be a fixed set of words in $X \cup X^{-1}$. We can use the data $X$ and $R$ to construct a group. Let $W$ be the set of all words in $X \cup X^{-1}$, and define an equivalence relation $\sim_p$ on $W$ by $w_1 \sim_p w_2$ iff $w_2$ can be obtained from $w_1$ by a finite sequence consisting of the following two moves:

1. insert or delete at any place an expression of the form $x_\alpha x_\alpha^{-1}$ or $x_\alpha^{-1} x_\alpha$;
2. insert or delete at any place a $w_\beta, w_\beta^{-1}$ or any cyclic permutation of these.

It is easy to show that this gives an equivalence relation. Let $[w]_p$ be the equivalence class of the word $w$. Define a product on these equivalence classes by

$$[w_1]_p [w_2]_p \stackrel{\text{def}}{=} [w_1 w_2]_p.$$

It is not hard to show that the definition is independent of the choice of representatives for the equivalence class, and that in fact,

**Theorem 4.1** *The set $W/\!\sim_p$ of equivalence classes forms a group under this multiplication, with identity $[$empty word$]_p$ and $[w]_p^{-1} = [w^{-1}]_p$. Denote this group by $\langle X; R\rangle$.*

If $G$ is a group, then we say that $\langle X; R\rangle$ is a *presentation* for $G$ iff $G \cong \langle X; R\rangle$. Intuitively one thinks of a presentation as a means of expressing the elements of $G$ in terms of the $x_\alpha$ (the *generators*), with the $w_\beta$ (the *relations*) giving rules for the manipulation of these expressions. In practice the $[w]_p$ notation is too unwieldy, and so we write just $w$, bearing in mind that there may be (many) other ways of expressing the same element of the group. If $w'$ is another (ie: $w' \sim_p w$) then write $w =_G w'$, or even $w = w'$, and say these two *represent the same element* of the group $G$. Such ambiguity is part and parcel of dealing with group presentations. Write $1_G$ or just 1 for the identity.

**(4.2)** Some examples of presentations for common garden variety groups are given (without justification) below. Not all of them are obvious. A sample isomorphism is given, but note that this is not in general unique.

1. $\mathbb{Z} \cong \langle x\,;\, –\rangle$ where – indicates an empty set of relators. An isomorphism is uniquely defined by $x \mapsto 1$ ($x \mapsto -1$ also works, and these are the only two).
2. $\mathbb{Z}/n \cong \langle x\,;\, x^n\rangle$ with $x \mapsto 1$ mod $n$ (in general, $x \mapsto k$ mod $n$ works for any $k$ relatively prime to $n$);
3. $\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ ($n$ times) $\cong \langle x_1, \dots, x_n\,;\, x_i x_j x_i^{-1} x_j^{-1}$ for all $i \neq j\rangle$ with

   $$x_i \mapsto (0, \dots, 0, 1, 0, \dots, 0),$$

   the 1 in the $i$-th coordinate.
4. The symmetric group

   $$S_{n+1} \cong \langle x_1, \dots, x_n\,;\, x_i^2 \text{ for all } i; (x_i x_{i+1})^3 \text{ for } 1 \leq i \leq n-1; (x_i x_j)^2 \text{ for } j \neq i \pm 1\rangle.$$

   where an isomorphism is given by $x_i \mapsto$ the transposition $(i, i+1)$. For those in the know, this means that $S_{n+1}$ is a Coxeter group.

5. $\mathrm{PSL}_2(\mathbb{Z}) \stackrel{\text{def}}{=} \mathrm{SL}_2\mathbb{Z}/\{\pm I_2\} \cong \langle x, y\,;\, x^2, y^3\rangle$ where

$$x \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   Here we are using the (standard) abuse of denoting the elements of $\mathrm{PSL}_2(\mathbb{Z})$ as matrices: the matrix $A$ is written when we really mean the coset $\{A, -A\}$.

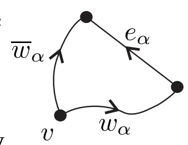
6. Any countable group $G$ has a presentation

$$\langle g \in G\,;\, g_i g_j g_k^{-1} \text{ whenever } g_k = g_i g_j\rangle.$$

7. Lie groups like $\mathbb{R}, \mathrm{GL}_2(\mathbb{R}), \mathrm{SU}_2(\mathbb{C})$ and so on, do not have presentations in the sense that we have defined them, for the simple reason that these groups are uncountable, while the set of words on $X \cup X^{-1}$ is countable. To at least generate such groups, one approach involves the use of, necessarily uncountably many, "infinitesimal" generators. Although this point of view is important to the Lie theory, it is not one that benefits from the use of techniques in geometric group theory, and so will not be discussed further in these lectures.

**(4.3)** How does one find a presentation for the fundamental group $\pi_1(K)$ of a 2-complex $K$? Intuitively, the generators will be certain homotopy classes of loops at the basepoint and the relations should arise from the faces. In other words, the generators arise from 1-dimensional information and the relations from 2-dimensional. If we take a spanning tree for the 1-skeleton, then loops contained entirely in the tree play no role, as the fundamental group of a tree is trivial. Thus, the generators should come from edges not in the tree.

Let $K$ be a connected 2-complex and $v$ a vertex of $K$. Let $T$ be a connected tree that contains all the vertices of $K$ (it is an elementary result from graph theory that such trees always exist). Choose an edge $e_\alpha$ from each edge/inverse-edge pair in $K^1 \setminus T$. Then there are unique paths $w_\alpha, \overline{w}_\alpha$ without spurs in $T$, such that $w_\alpha$ connects $v$ to the start vertex of $e_\alpha$ and $\overline{w}_\alpha$ connects $v$ to the terminal vertex. Let $x_\alpha = w_\alpha e_\alpha \overline{w}_\alpha^{-1}$, a loop based at $v$, and $X = \{x_\alpha \mid \alpha \in K^1 \setminus T\}$, the set of loops arising in this way. Choose $f_\beta$ from each face/inverse-face pair in $K$ and $\partial f_\beta$ a boundary label for $f_\beta$. Let $\partial f_\beta = e_{\alpha_1}^{\varepsilon_1} e_{\alpha_2}^{\varepsilon_2} \ldots e_{\alpha_k}^{\varepsilon_k}$ be the boundary label *after* the edges that are contained in the tree $T$ have been removed. Take $w_\beta = x_{\alpha_1}^{\varepsilon_1} x_{\alpha_2}^{\varepsilon_2} \ldots x_{\alpha_k}^{\varepsilon_k}$, a word in $X \cup X^{-1}$, and $R = \{w_\beta \mid f_\beta \text{ a face of } K\}$.




**Exercise 11**

1. Show that if $e_{\alpha_1}^{\varepsilon_1} e_{\alpha_2}^{\varepsilon_2} \ldots e_{\alpha_k}^{\varepsilon_k}$ is the boundary label of a face (with the edges in the spanning tree removed) then the loop $w_\beta = x_{\alpha_1}^{\varepsilon_1} x_{\alpha_2}^{\varepsilon_2} \ldots x_{\alpha_k}^{\varepsilon_k}$ is homotopically trivial.
2. If $w$ is a loop in $K$ at the basepoint $v$, and $e_{\alpha_1}, \ldots e_{\alpha_k}$ are the edges (in the order that $w$ is traversed) not contained in the tree $T$, then show that $w$ is homotopic to the loop corresponding to $x_{\alpha_1} \ldots x_{\alpha_k}$.

**Theorem 4.2** *$\langle X; R\rangle$ is a presentation for the fundamental group of $K$.*

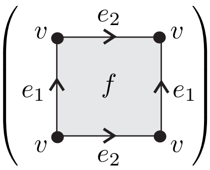
**Proof:** Any word $w$ in the generators $X$ corresponds in the obvious way to a loop in $K$ based at $v$ (as each generator itself does), so define a map $\varphi : \langle X; R\rangle \to \pi_1(K, v)$ by letting $\varphi[w]_p = [w]_h$. To see that $\varphi$ is well-defined suppose that $w, w'$ are words in the $x_\alpha$ and $w \sim_p w'$ by the insertion or deletion of an $x_\alpha x_\alpha^{-1}$ or $x_\alpha^{-1} x_\alpha$. Then the path in $K$ corresponding to $w'$ traverses the loop $w_\alpha e_\alpha \overline{w}_\alpha^{-1}$ and then the loop $w_\alpha^{-1} e_\alpha \overline{w}_\alpha$. Removing the obvious collection of spurs, gives a homotopy to the path corresponding to $w$, ie: $w \sim_h w'$. If $w \sim_p w'$ via the insertion or deletion of a word $w_\beta$ in the $x_{\alpha_i}$, then the corresponding paths are homotopic by part (1) of the exercise above. Thus, in any case, if $w \sim_p w'$ then $w \sim_h w'$, giving that $\varphi$ is well-defined.

It is trivial that $\varphi$ is a homomorphism which is onto by part (2) of the exercise above. Suppose now that $w$ and $\overline{w}$ are words in the $x_\alpha$ with $\varphi[w]_p = \varphi[\overline{w}]_p$, hence $w$ and $\overline{w}$ are homotopic in $K$. Each of these is a loop in $K$ composed of sub-loops of the form shown in the figure above. Thus, if the homotopy between them is achieved by the insertion/deletion of spurs, these must be composed of paths of the form $w_\alpha e_\alpha \overline{w}_\alpha^{-1} w_\alpha^{-1} e_\alpha^{-1} \overline{w}_\alpha$ or their inverses. This corresponds to an insertion/deletion of a $x_\alpha x_\alpha^{-1}$ or its inverse. If the homotopy involves the insertion/deletion of the boundary label $e_{\alpha_1}, \ldots, e_{\alpha_k}$ of a face, the edges not contained in the tree $T$, then this corresponds to the insertion/deletion of $x_{\alpha_1} \ldots x_{\alpha_k}$. In any case, $w \sim_h \overline{w}$ gives $w \sim_p \overline{w}$, ie: $\varphi[w]_p = \varphi[\overline{w}]_p \Rightarrow [w]_p = [\overline{w}]_p$ and the map is injective. $\square$

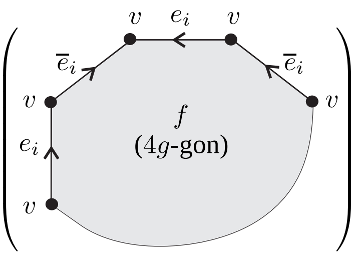
The elements of $X$ are called *Schreier generators* for the fundamental group.

**(4.4)** Suppose that $K$ is a graph, ie: its face set $F_K$ is empty. Then rather trivially there are no relations obtained as there are no faces, so we have a presentation of the form $\langle x_1, \ldots, x_k; - \rangle$. In fact, since a tree has one fewer edges than it does vertices, we get that if $K$ has $|V_K|$ vertices and $|E_K|$ edges (counting edge/inverse-edge pairs as one) then we can find such a presentation for $\pi_1(K)$ with $|E_K| - |V_K| + 1 = 1 - \chi(K)$ generators.

**(4.5)** The construction gives the expected answers when we look at the combinatorial versions of well known topological 2-manifolds:

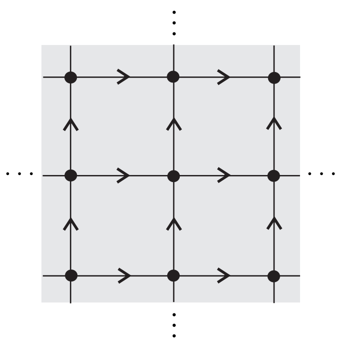
$\pi_1$  $\cong \langle x, y; [x, y] \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.

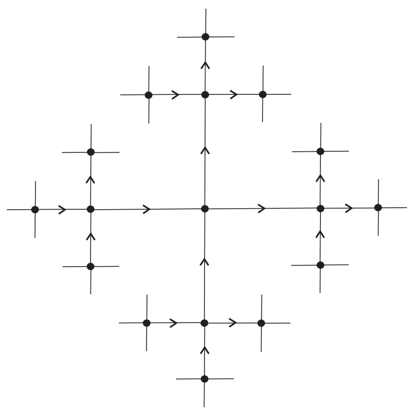

**(4.6)** Similarly,

$\pi_1$  $\cong \langle x_1, y_1, \ldots, x_g, y_g \,;\, \prod_i [x_i, y_i] \rangle$,


a so-called *surface group* of genus $g$.

**(4.7)** The complexes below are *simply-connected*, which is to say, their fundamental groups are trivial. You can either take this to be obvious, or for the infinite tree, a justification is given in the section on coverings.





**(4.8)** Subdividing a 2-complex gives one with the same fundamental group:

**Proposition 4.1** *If $\overline{K}$ is a 2-complex resulting from a subdivision of $K$ then $\pi_1(\overline{K}, v) \cong \pi_1(K, v)$.*

**Proof:** We need only verify the two cases where $\overline{K}$ results from the single subdivision of an edge $e$ or a face $f$. In the first, let $T$ be a spanning tree for the 1-skeleton of $K$. If $T$ includes $e$ then the applications of Theorem 4.2 are identical for both $K$ and $\overline{K}$, yielding the same presentations. If $e \notin T$ then adding edge $e_1$ (see the figures in **(1.3)**) to $T$ gives a spanning tree $T'$ for the 1-skeleton of $\overline{K}$. As $e$ is not in $T$ it contributes a Schreier generator to $K$, and similarly $e_2$ does to $\overline{K}$. The resulting presentations are identical except $e$ in one is replaced by $e_2$ in the other.

If $\overline{K}$ arises by subdividing the face $f$ into $f_1, f_2$ by a new edge $e$, then a spanning tree $T$ for the 1-skeleton of $K$ suffices for the 1-skeleton of $\overline{K}$. We have a new Schreier generator $x_\alpha$ corresponding to this edge. The only differences in the applications of Theorem 4.2 is the relations arising from the subdivided face for which we get in the two cases,


the presentations $\langle X; R, w\rangle$ and $\langle X, x_\alpha; R, w_1 x_\alpha, x_\alpha^{-1} w_2\rangle$. There are no occurrences of $x_\alpha$ in the $w_i$ or any of the relations $R$. As $x_\alpha = w_2$ in the second presentation it is clearly superfluous, so we may remove it, the relation $x_\alpha = w_2$ and replace all other occurrences of $x_\alpha$ by $w_2$. This gives us the first presentation. □

**(4.9)** Now that we can obtain a group presentation from a 2-complex, the question arises as to whether we can obtain a 2-complex from a group presentation? We would like the construction to be "natural", in the sense that the resulting complex should have fundamental group with presentation the one we started with. In general there may be many ways to do this, but the following is the most standard.

Let $\langle X; R\rangle$ be a presentation for a group $G$, and define a 2-complex $K = K(X; R)$ with a single vertex $v$. For each $x \in X$ take a $e_x^{\pm 1} \in E_K$ and for each $w \in R$ a $f_w^{\pm 1} \in F_K$. The incidence maps are given by

$$s, t(e_x^{\pm 1}) = v, \partial(f_w) = \text{ the cyclic permutations of } e_{x_{\alpha_1}}^{\varepsilon_1} \dots e_{x_{\alpha_k}}^{\varepsilon_k},$$

if $w = x_{\alpha_1}^{\varepsilon_1} \dots x_{\alpha_k}^{\varepsilon_k}$. Intuitively, $K$ has edges that are loops based at the vertex that are in one-one correspondence with the generators; the faces are "sewn" onto the loops so that their boundaries are the relators.

**Theorem 4.3** *$\langle X; R\rangle$ is a presentation for $\pi_1(K(X; R), v)$.*

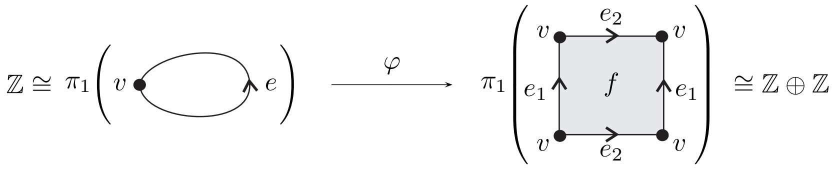
The proof is a straight forward application of Theorem 4.2, although notice that initially we get a presentation $\langle X; \overline{R}\rangle$ where the $\overline{R}$ are the $R$, but possibly cyclically permuted. The Theorem then follows from the fact that $\langle X; R\rangle$ and $\langle X; \overline{R}\rangle$ are obviously isomorphic.

Call $K(X; R)$ a *presentation 2-complex* for $G$.

**(4.10)** We have already seen that maps between complexes induce homomorphism between the fundamental groups. In certain situations the converse is (almost) true. What we want is that if

$$\varphi : \pi_1(K_1, v_1) \to \pi_1(K_2, v_2)$$

is a homomorphism then there is a map $p : K_1 \to K_2$ with $p_*$ being the homomorphism $\varphi$. To be so, the following in particular must happen: if the homotopy class of a loop $w_1 \in K_1$ is sent by $\varphi$ to the homotopy class of a loop $w_2 \in K_2$, then $p$ would need to send the *path* $w_1$ to the *path* $w_2$. The problem is that at a combinatorial level, the paths $w_1$ and $w_2$ may be quite different. For example, take the homomorphism $\varphi : \mathbb{Z} = \langle t \rangle \to \mathbb{Z} \oplus \mathbb{Z} = \langle a, b \rangle$ given by $t \mapsto ab$ interpreted as a map between fundamental groups as follows,




where loop $e$ corresponds to the generator $t$; $e_1$ and $e_2$ to $a$ and $b$. There is no map from the first complex to the second sending the generating loop on the left to the loop $e_1 e_2$ on the right as 2-complex maps must send edges to edges (rather than to paths). The solution is to subdivide the complex on the left.

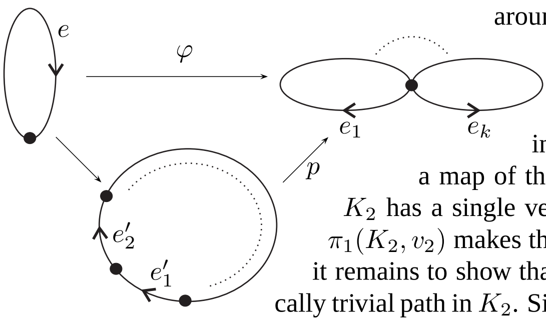
**Proposition 4.2** *Let $K_1, K_2$ be 2-complexes, each with a single vertex $v_i$, and $\varphi : \pi_1(K_1, v_1) \to \pi_1(K_2, v_2)$ a homomorphism of groups. Then there is a subdivision $\overline{K}_1$ of $K_1$ and a 2-complex map $p : \overline{K}_1 \to K_2$ such that*

$$\begin{array}{ccc} \pi_1(K_1, v_1) & \xrightarrow{\varphi} & \pi_1(K_2, v_2) \\ & \cong \searrow \quad \nearrow p_* & \\ & \pi_1(\overline{K}_1, v_1) & \end{array}$$

*commutes.*

Thus we may replace $K_1$ by a 2-complex with isomorphic fundamental group so that the map $p_*$ is the same, upto this isomorphism, as the original map $\varphi$.

**Proof:** Let $e$ be an edge of $K_1$ and suppose that $e_1^{\alpha_1} \dots e_k^{\alpha_k}, \alpha_i = \pm 1$ is a representative for the homotopy class of the image of $[e]_h$. Subdivide $e$ into new edges $e'_1, \dots, e'_k$ oriented so that running around them following the direction of $e$ gives the label $(e'_1)^{\alpha_1} \dots (e'_k)^{\alpha_k}$. Perform this procedure for all the edges of $K_1$ and call the resulting subdivision $\overline{K}_1$. Define $p : (\overline{K}_1)^1 \to K_2^1$ by $p(e'_j) = e_j$, sending the new vertices to $v_2$. it is easy to see that this is a map of the 1-skeletons (although beware: it uses the fact that $K_2$ has a single vertex) and that the induced map $p_* : \pi_1(\overline{K}_1, v_1) \to \pi_1(K_2, v_2)$ makes the diagram commute. To define $p$ on the faces of $\overline{K}_1$, it remains to show that for every face $f'$ in $\overline{K}_1$ that $p\partial(f')$ is a homotopically trivial path in $K_2$. Since $\partial f'$ is just the subdivided version of the boundary $\partial f$ of a face $f$ of $K_1$, and $\varphi$ is a homomorphism we have that $[\partial f]_h$, hence $\varphi[\partial f]_h$ is the trivial element (homomorphisms sending the trivial element to the trivial element). But $\varphi[\partial f]_h$ is by definition the homotopy class of $p(\partial f')$, which is thus homotopically trivial as required. □

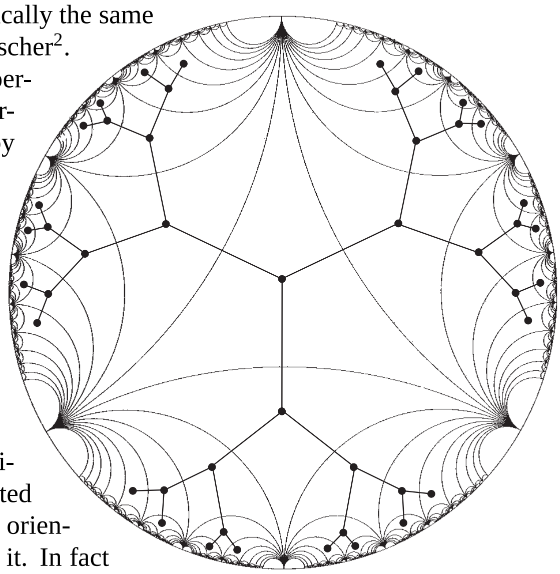



The problem that this proposition gets us out of does not arise with *topological* (ie: $CW$-) complexes as maps between them can legitimately send edges to paths.

# 5 Fundamental groups of graphs

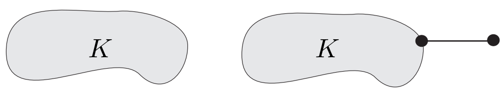
**(5.1)** A group is *free* if and only if it is the fundamental group of a graph. We will eventually give four definitions for free groups, with this the only one that does not immediately explain the use of the word "free". Nevertheless, the topological definition has a nice feel.

**(5.2)** There are many examples of free groups occurring in nature. If $K$ is the graph consisting of one vertex and a sinlge loop, then we saw that the group is nothing other than the integers $\mathbb{Z}$. For non-Abelian examples, consider the picture at left, which is topologically the same as a well known picture by the graphic artist M C Escher[2]. Let the disc represent the Poincaré model for the hyperbolic plane $\mathbb{H}^2$ (see Section 10). The arcs are hyperbolic lines (or *geodesics*), and the regions bounded by them are hyperbolic triangles. It is one of the intriguing properties of hyperbolic geometry that one can find regular triangles, squares, and so on, with sides of infinite length, but bounding areas that are finite in size. In fact all the triangles in the picture are congruent and have area $\pi$ by the Gauss-Bonnet theorem. An *automorphism* of this picture is an isometry $\mathbb{H}^2 \to \mathbb{H}^2$ taking the tessellation to itself. The group of automorphisms is generated by the three reflections $a, b, c$ in the sides of the large triangle in the middle (where reflection can be interpreted suitably). Reflections are obviously maps that reverse orientation, while the product of two reflections preserves it. In fact it turns out that the orientation preserving automorphisms form a free group.



Now place a vertex at the center of each triangle, and join two vertices by an edge iff they are in the interior of adjacent triangles (ie: triangles sharing an edge). The graph obtained is an infinite 3-valent tree. Moreover, the free group consisitng of the orientation preserving automorphisms acts on this tree, with no non-trivial element fixing a vertex, so is a free action. It will turn out to be a *characterising* property of free groups that they act freely on trees.

**(5.3)** If a group is the fundamental group of a graph, there will be many different graphs that can play this role. Indeed, if $K$ is any 2-complex then the following




have the same fundamental group by Theorem 4.2. Thus the graph plays no intrinsic role in the definition of free group.

Nevertheless, the *connected* graphs that realise a given free group do have one thing in common, namely, they all have the same homology. Let $\mathscr{C}_G$ be the collection of connected graphs $K$ with $\pi_1(K) \cong G$. We have $H_0(K) \cong \mathbb{Z}$ and $H_2(K)$ trivial by Proposition 3.1. Now $H_1(K)$ is the image of $\pi_1(K) \cong G$ under the Hurewicz map, thus by Exercise 9, $H_1(K) \cong G/[G, G]$ where $[G, G]$ is the commutator subgroup of $G$.

**Exercise 12** Show that for a graph $K$, having finitely generated homology groups reduces to the condition that $K$ has finitely many distinct 1-cycles.

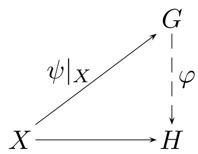
For a given $G$ the graphs in $\mathscr{C}_G$ may or may not have finitely generated homology. If they do then the Euler characteristic is defined and constant across $\mathscr{C}_G$ and so the quantity $1 - \chi(K), K \in \mathscr{C}_G$ is an invariant of $G$. Call it the *rank* of the free group $G$. If the homologies are not finitely generated, say that $G$ has (countably) *infinite rank*.

[2] Actually, this is no coincidence. Escher's pictures were inspired by a meeting with Coxeter at the ICM in Vancouver in 1954.

**(5.4)** We saw in **(4.4)** that if $K$ is a graph then $G = \pi_1(K)$ has a presentation of the form $\langle x_1, \ldots, x_n; - \rangle$, where $n = |E_k| - |V_K| + 1 = 1 - \chi(K)$ is the rank just defined. Call this a *free presentation* for $G$ (it is free of relations). On the other hand, if $G$ is the group with free presentation $\langle x_1, \ldots, x_n; - \rangle$, then the presentation 2-complex $K(X; R)$ has fundamental group $G$, and is a graph, because of the complete absence of relators. Moreover, it has Euler characteristic $1 - n$. Thus, a group is free of rank $n$ if and only if it has a free presentation on $n$ generators.

**(5.5)** The importance of free groups to group theory stems from the,

**Substitution Theorem.** *Let $G$ have the presentation $\langle X; R \rangle$ via the isomorphism $\psi : \langle X; R \rangle \longrightarrow G$. Let $H$ be any other group and $X \longrightarrow H$ a map with $x_\alpha \mapsto h_\alpha$. Then there is a unique homomorphism $\varphi$ such that*

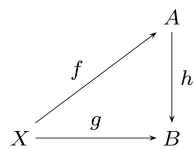



*commutes if and only if for each $w_\beta = x_{\alpha_1}^{\varepsilon_1} \ldots x_{\alpha_k}^{\varepsilon_k} \in R$ we have $h_{\alpha_1}^{\varepsilon_1} \ldots h_{\alpha_k}^{\varepsilon_k} = 1$ in $H$.*

Once decoded, the usefulness of this result becomes apparent. It says that to find a homomorphism from $G$ to $H$, choose images for the generators of $G$ (this is the map $X \to H$) in such a way that for each relation in $G$, the corresponding relation obtained by substituting the $h_\alpha$ for the $x_\alpha$, also holds in $H$.

**Proof:** We give the proof for $G = \langle X; R \rangle$, identifying $x_\alpha \in X$ with $[x_\alpha]_p \in G$, and leave the straightforward extension to $G \cong \langle X; R \rangle$ to the reader. The only if part is trivial. For the if part, any $\varphi$ making the diagram commute must send $[x_\alpha]_p \mapsto h_\alpha$, and to be a homomorphism, must extend linearly to $[x_{\alpha_1}^{\varepsilon_1} \ldots x_{\alpha_k}^{\varepsilon_k}]_p \mapsto h_{\alpha_1}^{\varepsilon_1} \ldots h_{\alpha_k}^{\varepsilon_k}$. Thus $\varphi$ is uniquely defined. If $w_\beta = x_{\beta_1}^{\varepsilon_1} \ldots x_{\beta_l}^{\varepsilon_l} \in R$ (or a cyclic permutation), then $[x_{\alpha_1}^{\varepsilon_1} \ldots w_\beta \ldots x_{\alpha_k}^{\varepsilon_k}]_p \mapsto h_{\alpha_1}^{\varepsilon_1} \ldots h_{\beta_1}^{\varepsilon_1} \ldots h_{\beta_l}^{\varepsilon_l} \ldots h_{\alpha_k}^{\varepsilon_k} = h_{\alpha_1}^{\varepsilon_1} \ldots h_{\alpha_k}^{\varepsilon_k}$ by the condition given in the theorem. Trivially, $[x_{\alpha_1}^{\varepsilon_1} \ldots x_\alpha x_\alpha^{-1} \ldots x_{\alpha_k}^{\varepsilon_k}]_p \mapsto h_{\alpha_1}^{\varepsilon_1} \ldots h_{\alpha_k}^{\varepsilon_k}$, thus $\varphi$ is well defined. □

**Aside 3** In many categories there is the notion of a *free object*. If $\mathsf{C}$ is a category and $X$ is a set, construct a new category $\mathsf{K}$ as follows: the objects of $\mathsf{K}$ are pairs $(A, f)$ where $A$ is an object of $\mathsf{C}$ and $f$ is a mapping $f : X \to A$. Given two such, a $\mathsf{K}$-morphism $h : (A, f) \to (B, g)$ is a $\mathsf{C}$-morphism $h : A \to B$ such that the diagram,




commutes. A *free* $\mathsf{C}$*-object on the set* $X$ is an object $(F, \varphi)$ of $\mathsf{K}$ such that there is precisely *one* morphism in $\mathsf{K}$ from $(F, \varphi)$ to any other object. By abuse we call $F$ *free* in $\mathsf{C}$ when this happens for some $X$.

Once the formal nonsense has been decoded the reader may be able to see that free objects exist in a number of familiar categories: in the category of $R$-modules for $R$ some (commutative) ring, they are the free modules (ie: modules with a basis), so in particular vector spaces are examples of free objects, with $X$ a basis.

**Corollary 5.1** *1. $G$ is a free group iff $G$ is a free object in the category of groups.*

*2. If $G$ is free on the set $X$ and $H$ is a group with presentation $\langle X; R \rangle$, then $H$ is a homomorphic image of $G$.*

**Proof:** The first part is a special case of the substitution theorem. For the second, apply the theorem with $G \cong \langle X; - \rangle$, $H \cong \langle X; R \rangle$ and $h_\alpha = x_\alpha$.

**(5.6)** Here is an example of the Corollary. A group is said to be *simple* iff it has no non-trivial normal subgroups except for the trivial subgroup and the whole group. Much like integers and primes, any group can be decomposed, in an essentially unique way, into simple pieces. Examples of finite simple groups are easy to find; for instance a cyclic group $\mathbb{Z}/p$ for $p$ a prime (use Lagrange's theorem). Infinite simple groups are a little harder, but nevertheless can be found in the "back garden", for example $\text{PSL}_2\mathbb{C}$, or indeed, $\text{PSL}_2 k$ for any infinite field $k$. Infinite simple groups that are finitely generated are quite hard to construct and generally don't seem to occur in nature (ie: all the examples are quite artificial).

There is a periodic table for the finite simple groups. It contains the cyclic groups of prime order; the alternating groups $A_n$ for $n \neq 1, 2$ or $4$; sixteen infinite families of matrix groups over finite fields (the groups of *Lie type*, so-called as they arise as groups of automorphisms of simple Lie algebras over finite fields); and twenty six exceptional examples that don't seem to fit into any of the previous categories.

One consequence of the classification is that every finite simple group can be generated by just two of its elements (is 2-*generated*). I am not aware of a general conceptual proof; one can analyze the situation case by case, as was done by Steinberg for the groups of Lie type and Aschbacher for the sporadic groups (it is trivial for the cyclics and an easy exercise for the alternating groups).

Thus, every finite simple group is an image of the free group on two generators.

**Exercise 13** Show that if $p$ is a prime, then the product of cyclic groups $\mathbb{Z}/p \times \mathbb{Z}/p^2 \times \cdots \times \mathbb{Z}/p^k$ cannot be generated by fewer than $k$ elements (hint: use the fundamental theorem for Abelian groups).

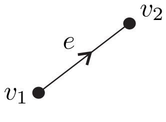
**(5.7)** Tying together the different definitions of free group, we are not yet in a position to prove all of the following, but this seems an appropriate place to state the result:

**Theorem 5.1** *The following are equivalent for a group $G$:*

1. *is the fundamental group of a graph;*
2. *has a free presentation $\langle X; - \rangle$;*
3. *is a free object in the category of groups;*
4. *acts freely on a tree.*

**Proof:** The equivalence of the first three has already been established, so we prove that $(1)$ and $(4)$ are equivalent. To prove $(1) \Rightarrow (4)$ requires the covering space theory of the next section, so we'll postpone the proof until later. To see $(4) \Rightarrow (1)$, let $T$ be a tree and $G$ a group acting freely on it. Since trees are simply connected we have $\pi_1(T)$ is trivial, hence Theorem 3.1 becomes $\pi_1(T/G) \cong G$. But $T/G$ is clearly a graph (the face set of $T$ is empty, hence so must be the face set of $T/G$). □

If we have a group $G$ acting on a tree but now the stabilisers of the vertices are not trivial, then something can still be said about the $G$. See §8 for the general story.




As an illustrative example of an easy trap to fall into, let $\mathbb{Z}/2$ act on the complex shown, with the non trivial element of the group swapping the two vertices. Then this is a free action of $\mathbb{Z}/2$ on a tree. What has gone wrong is that while we have a $\mathbb{Z}/2$ action here, it is not orientation preserving.

Another point to recall is that when we say a group $G$ acts freely on some complex, we mean acts freely on the vertices only, not the whole complex. In the Theorem we have the free group $G$ acting freely on some tree: the action cannot fix the vertices incident with some edge $e$, nor can it interchange them, as all our actions preserve orientation. Thus it acts freely on the edges as well.

**(5.8)** The last part of the Theorem immediately gives,

**Nielsen-Schreier Theorem (version 1).** *A subgroup of a free group is free.*

For the free group acts freely on some tree $T$, hence so does any subgroup, which is thus free as well!

**Exercise 14** Show that if $G$ is any finitely generated group then there is a graph on which $G$ acts freely (hint: see §7).

**(5.9)** We finish with a somewhat more profound application of Theorem 5.1 due to Serre [8].

**Aside 4** The rationals $\mathbb{Q}$ are an incomplete field which can be completed by considering equivalence classes of Cauchy sequences. As usual a Cauchy sequence is one that eventually has its terms arbitrarily close together. What though do we mean by close together, ie: what is the metric?

The obvious metric gives the reals $\mathbb{R}$ as our completion, but a more sensible metric from the point of view of number theory is the $p$-*adic*. Fix a prime $p$ and for any $x \in \mathbb{Q}$ let $|x| = p^{-v(x)}$ where $v(x)$ is the largest power of $p$ dividing $x$ in the sense that $x$ can be written as $p^{v(x)}a/b$ with $a, b$ relatively prime to $p$. Then the distance from $x$ to $y$ in the $p$-adic metric is $|x - y|$. The qualitative effect of the $p$-adic distance is to make integers highly divisible by $p$ very close to 0. This has important ramifications to the solution of Diophantine equations.

Call the completion of $\mathbb{Q}$ with respect to this metric the $p$-adic numbers $\mathbb{Q}_p$. The $p$-adics have a distinguished subring, playing the same role that $\mathbb{Z}$ does in $\mathbb{Q}$, called the *$p$-adic integers* $\mathbb{Z}_p$ and defined as those $x \in \mathbb{Q}_p$ with $v(x) \geq 0$. Like $\mathbb{Z}$, the $p$-adic integers are a principal ideal domain, and have distinguished elements, called *uniformizers* and traditionally denoted $\pi$, such that any ideal has the form $\langle \pi^m \rangle$ for some $m \in \mathbb{Z}$.

**Aside 5** Let $k = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$ for some $p$. A $k$-Lie group $G$ is a $k$-analytic manifold together with an $k$-analytic map $G \times G \to G$ making it into a group. Thus, the multiplication and inverse maps are analytic. A subgroup $\Gamma$ is called a *lattice* iff it is discrete (in the manifold topology) and the set of cosets $G/\Gamma$ can be endowed with a $G$-invariant Borel measure $\mu$ such that $\mu(G/\Gamma) < \infty$.

Lattices can be thought of as discrete approximations to the Lie group, with $\mu(G/\Gamma)$ a measure of the accuracy of the approximation. So they are as good an approximation as we could hope to find given that we are approximating something continuous by something discrete.

A typical example of a lattice is $SL_2\mathbb{Z}$ in the real Lie group $SL_2\mathbb{R}$, and a typical example of a non-lattice is $GL_2\mathbb{Z}$ in $GL_2\mathbb{R}$ (it is discrete, but the coset space $GL_2\mathbb{R}/GL_2\mathbb{Z}$ is non-compact of infinite volume.)

Lattices in the real Lie group $SL_2\mathbb{R}$, called *Fuchsian groups*[3], have a classical importance in mathematics. In particular, one has

**Theorem 5.2 (Fricke, Klein)** *A torsion free lattice in $SL_2\mathbb{R}$ is either a surface group or a free group.*

What about torsion free lattices in $SL_2\mathbb{Q}_p$? We construct a complex on which $SL_2\mathbb{Q}_2$ acts in a natural way. Let $V$ be a 2-dimensional vector space over $\mathbb{Q}_p$ and $\Lambda$ a free $\mathbb{Z}_p$-submodule of the form $\mathbb{Z}_p\boldsymbol{u} \oplus \mathbb{Z}_p\boldsymbol{v}$ (it turns out that they all have this form for linearly independent $\boldsymbol{u}$ and $\boldsymbol{v}$). Given two such, $\Lambda$ and $\Lambda'$, call them equivalent iff $\Lambda = a\Lambda'$ for some $a \in \mathbb{Q}_p$. The vertices of our complex will be the equivalence classes of the $\Lambda$ (which we will denote using the same symbol).

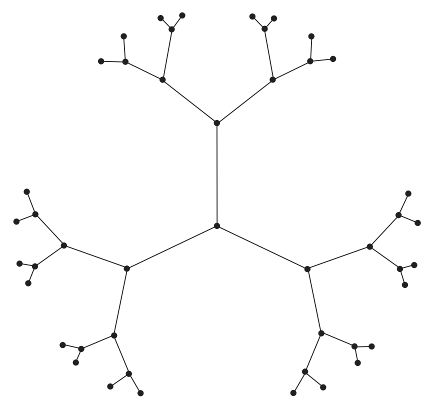
To construct the edges, consider $(\Lambda + \Lambda')/\Lambda'$, which is a finitely generated $\mathbb{Z}_p$-module, hence by the general theory of modules over a PID, a direct sum of cyclic $\mathbb{Z}_p$-modules,

$$\frac{\Lambda + \Lambda'}{\Lambda'} \cong \frac{\mathbb{Z}_p}{\langle \pi^m \rangle} \oplus \frac{\mathbb{Z}_p}{\langle \pi^n \rangle},$$

for $m, n \in \mathbb{Z}$. Join the vertices corresponding to $\Lambda, \Lambda'$ by an edge iff $|m - n| = 1$.

Here is the deep bit: this graph turns out to be the infinite $(p+1)$-valent tree[4]. The result for $\mathbb{Q}_2$ is shown below:

tree for $SL_2\mathbb{Q}_2$



[3] Many authors use the word "Fuchsian" for discrete subgroups of $SL_2\mathbb{R}$, without imposing the lattice condition.

[4] It is an example of a *Bruhat-Tits building*. These play the same role for Lie groups over local fields like $\mathbb{Q}_p$ that symmetric spaces play for real and complex (semi-simple) Lie groups.

$GL_2\mathbb{Q}_p$ acts on the vector space $V$ by the usual $\boldsymbol{u} \mapsto A\boldsymbol{u}$, hence on the equivalence classes $\Lambda$ of lattices, and so on the vertices of the tree. Indeed, the action can be extended to the 1-skeleton, although it does not preserve orientation. Passing to $SL_2\mathbb{Q}_p$ gives an orientation preserving action. The "Arboreal dictionary" of §8 will then give us some information about the structure of $SL_2\mathbb{Q}_p$. For now, suppose that $\Gamma$ is a torsion free lattice in $SL_2\mathbb{Q}_p$ and restrict the $SL_2\mathbb{Q}_p$-action on the tree to a $\Gamma$-action. It turns out that the torsion free property means that no non-trivial element of $\Gamma$ fixes a vertex, ie: the $\Gamma$-action is free. Thus,

**Theorem 5.3 (Ihara, Serre)** *A torsion free lattice in $SL_2\mathbb{Q}_p$ is a free group.*

# 6 Coverings and subgroups

We have seen how to think of groups as topological objects, so how does one model the subgroups of a group using this topological picture?

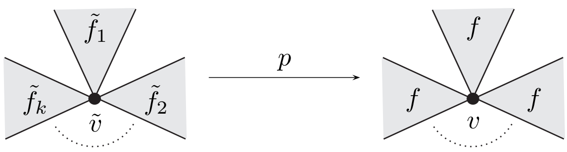
**(6.1)** A map $\widetilde{K} \xrightarrow{p} K$ of 2-complexes is a *covering* iff

1. $p$ preserves dimension;
2. if $\tilde{v} \xrightarrow{p} v$ then $p$ is a bijection from the set of edges in $\widetilde{K}$ with initial vertex $\tilde{v}$ to the set of edges in $K$ with initial vertex $v$;
3. if $f$ is a face and $v$ a vertex of $K$, let $m(f, v)$ be the number of times that $v$ appears in the boundary of $f$. Then for any $\tilde{v} \xrightarrow{p} v$,
$$\sum_{\tilde{f}\to f} m(\tilde{f}, \tilde{v}) = m(f, v),$$
the sum being over all faces $\tilde{f}$ covering $f$.

As is the case with any definition, there is the worry that it is "not the right one". As much as one may want to preserve the intuition of a topological covering, from a formal point of view the definition is not important as long as the resulting coverings have the properties of path-lifting (§6.5) and homotopy lifting (§6.6).

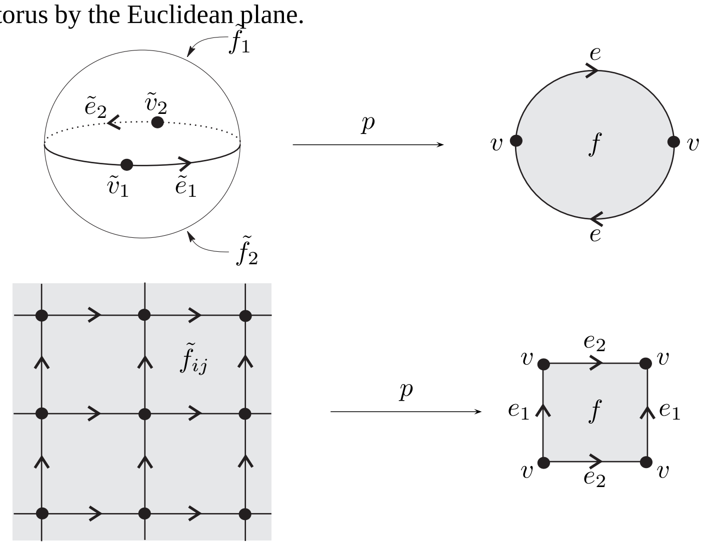
**(6.2)** We'll only cover *connected* $K$. The terminology cover/lift is used for images/pre-images of the covering map $p$: if $p(\tilde{*}) = *$, then one says that $\tilde{*}$ *covers* $*$, or that $*$ *lifts* to $\tilde{*}$. The set of all lifts of $*$ is its *fiber*.

The last two parts in the definition express local properties of coverings: if $\tilde{v}$ covers $v$, then $\widetilde{K}$ looks the same near $\tilde{v}$ as $K$ does near $v$. Specifically, the configuration of edges around a vertex looks the same both upstairs and downstairs. Given a vertex $v$ and incident face $f$ downstairs, this face looks the same near $v$ as its lifts do near any vertex $\tilde{v}$ covering $v$: if $f$ contains $v$ in its boundary $k$ times, so there are $k$ wedge-shaped pieces of $f$ fitting together around $v$, then there are $k$ wedge-shaped pieces fitting together around $\tilde{v}$, where these belong to the faces $\tilde{f}_i$ that cover $f$ and contain $\tilde{v}$ in their boundary:




Note that there is nothing to say that the faces $\tilde{f}_i$ are distinct, and in general, they won't be.

**(6.3)** The following two illustrate, in a combinatorial fashion, the coverings of the projective plane by the 2-sphere and the torus by the Euclidean plane.

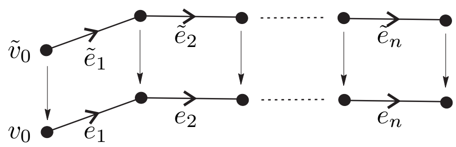



**(6.4)** As with any map of 2-complexes there is an induced homomorphism of groups

$$\pi_1(\widetilde{K}, \tilde{v}) \xrightarrow{p^*} \pi_1(K, v).$$

If $w$ is a loop at $\tilde{v}$ then $p : [w]_h \to [p(w)]_h$ so that $p_*(\pi_1(\widetilde{K}, \tilde{v}))$ may be identified with the homotopy classes of loops at $v$ that lift to loops at $\tilde{v}$.

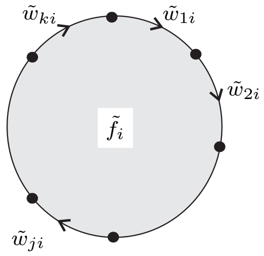
**(6.5)** There are two really crucial properties coverings need. The first is *path lifting*: given a path $w = e_1 e_2 \ldots e_n$ starting at $v_0$ and any $\tilde{v}_0$ covering $v_0$, there is a *unique* path $\tilde{w} = \tilde{e}_1 \ldots \tilde{e}_n$ starting at $\tilde{v}_0$ covering $w$ in the sense that $\tilde{e}_i$ covers $e_i$. This is easily seen, as in the picture,




since there is a unique edge $\tilde{e}_1$ corresponding to $e_1$ under the bijection between the edges starting at $\tilde{v}_0$ and those starting at $v_0$. This edge $\tilde{e}_1$ must end at a vertex that covers the end vertex of edge $e_1$, as coverings (being maps of complexes) preserve vertex-edge incidences. The process can be repeated starting at this new vertex.

Call $\tilde{w}$ the *lift* of $w$ at $\tilde{v}$.

**Exercise 15** Let $\widetilde{K} \xrightarrow{p} K$ be a covering. Show that all the fibers have the same cardinality, whether they be fibers of vertices, edges, faces, paths . . . This is called the *sheet number* of the covering.




*Hint*: one can show in fact that there are *incidence preserving* bijections between these fibers, in the following sense. If $e$ is an edge, show that there is a bijection between the fiber of $e$ and the fiber of the initial vertex of $e$. Let $f$ be a face of $K$ containing the vertex $v$ in its boundary $k$ times, ie: $f$ has a boundary label $w_1 w_2 \ldots w_k$ where each $w_i$ is a closed loop at $v$ that otherwise does not contain $v$. Then show that for any fixed $j$, there is a bijection between the fiber of $f$ and the fiber of $w_j = \{\tilde{w}_{j1}, \ldots, \tilde{w}_{ji}, \ldots\}$, with $\tilde{f}_i$ corresponding under this bijection to $\tilde{w}_{ji}$ where $\tilde{f}_i$ has boundary label $\tilde{w}_{1i}\tilde{w}_{2i} \ldots \tilde{w}_{ki}$ as shown in the figure to the left.

**(6.6)** The other crucial property coverings have is *homotopy lifting*: let $\widetilde{K} \xrightarrow{p} K$ be a covering and $w_1, w_2$ homotopic paths in $K$. If $v$ is the vertex at the start of $w_1$ and $w_2$, and $\tilde{v}$ covers $v$, then the lifts $\tilde{w}_1$ and $\tilde{w}_2$ at $\tilde{v}$ are homotopic. Thus, homotopies can be lifted upstairs.

Since a homotopy is a finite sequence of insertion/deletion of spurs or boundaries of faces, we need show that spurs lift to spurs and boundaries of faces lift to boundaries of faces. Homotopy lifting then follows by induction.

That spurs lift to spurs follows straight from path-lifting. Suppose $f$ is a face downstairs with boundary as in the exercise above. Let $w_j \dots w_k \, w_1 \dots w_{j-1}$ be a boundary path of $f$ and $\tilde{v}$ a vertex in the fiber of $v$. We lift this boundary path to $\tilde{v}$: lifting $w_j$ to $\tilde{v}$ we get one of the paths $\tilde{w}_{ji}$ in the fiber of $w_j$, and this lies, by the exercise, in the boundary of a face $\tilde{f}_i$ in the fiber of $f$. As we lift each successive piece of the boundary of $f$, we move around the boundary of $\tilde{f}_i$.

**(6.7)** The key result on coverings is the,

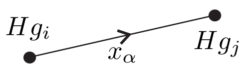
**Subgroup Theorem.**

1. *Let $\widetilde{K} \xrightarrow{p} K$ be a covering. Then the induced map $\pi_1(\widetilde{K}, \tilde{v}) \xrightarrow{p^*} \pi_1(K, v)$ is injective.*
2. *Let $K$ be a 2-complex and $H$ a subgroup of $\pi_1(K, v)$. Then there is a connected $\widetilde{K}$ and a covering of 2-complexes $\widetilde{K} \xrightarrow{p} K$ with $H \cong \pi_1(\widetilde{K}, \tilde{v})$, where $\tilde{v} \xrightarrow{p} v$.*

The first bit says that coverings give subgroups; the second that subgroups give coverings. Questions about subgroups can thus be turned into questions about coverings, and vice-versa.

**Proof:** The first part follows immediately from homotopy lifting, since if $w_1$ and $w_2$ are non-homotopic paths upstairs, then they must map under $p$ to non-homotopic paths downstairs, ie: $[w_1] \neq [w_2] \in \widetilde{K} \Rightarrow p^*[w_1] \neq p^*[w_2] \in K$.

We will only need the result from the second part in the special case where $K = K\langle X; R\rangle$, the presentation 2-complex for some (finite) presentation $\langle X; R\rangle$. The interested reader can fill in the details for the case of an arbitrary complex $K$.



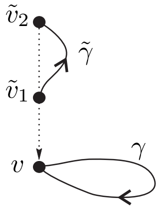

Let $\{Hg_i\}$ be the right cosets of $H$ in $\pi_1(K, v)$. Define $\widetilde{K}$ by taking as vertex set these cosets. Define an edge/inverse edge pair as shown in the picture precisely when $Hg_i x_\alpha = Hg_j$ where $x_\alpha \in X$. It's standard practice to abuse notation and label such an edge by $x_\alpha$. For each $r \in R$ a relator word, and each vertex $Hg_i$ of $\widetilde{K}$, consider the path starting at $Hg_i$ with label $r$. It must be a closed path, as $Hg_i r = Hg_i$ since $r = 1$ in $\langle X; R\rangle \cong \pi_1(K, v)$. Attach a face to $\widetilde{K}$ with boundary label this path. Repeat this procedure for each pair consisting of a relator word/vertex of $\widetilde{K}$, obtaining a *distinct* face each time. Note that you may get the same closed path from different vertices of $\widetilde{K}$ via this process (this will happen when the relator word is a proper power $r = w^n$). Attach distinct faces with these same boundary labels anyway.

Define a map $\widetilde{K} \xrightarrow{p} K$ by sending every vertex of $\widetilde{K}$ to the single vertex of $K\langle X; R\rangle$; edges labelled $x_\alpha$ to the edge labelled $x_\alpha$ in $K\langle X; R\rangle$, and faces with boundary label $r$ to the face of $K\langle X; R\rangle$ with boundary label $r$. Then this is a covering of 2-complexes (Exercise!)

Let $\tilde{v}$ be the vertex of $\widetilde{K}$ corresponding to the subgroup $H$ itself. As the induced map $\pi_1(\widetilde{K}, \tilde{v}) \xrightarrow{p^*} \pi_1(K, v)$ is injective, $\pi_1(\widetilde{K}, \tilde{v})$ is isomorphic to its image in $\pi_1(K, v)$. But this image is obtained by taking the images of closed paths in $\widetilde{K}$ based at $\tilde{v}$; such a path has label $w$ and is closed $\Leftrightarrow Hw = H \Leftrightarrow w \in H$. Thus $\pi_1(\widetilde{K}, \tilde{v}) \cong H$. $\square$

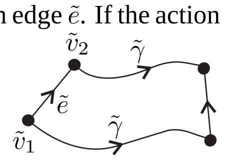
The coverings so constructed are called *Schreier coset diagrams* for reasons that the proof makes clear!

**(6.8)** Whenever we have a covering $\widetilde{K} \to K$ we get actions of two groups on the covering complex $\widetilde{K}$, both related to $\pi_1(K)$,: the *path-lifting* and *deck transformation* actions.

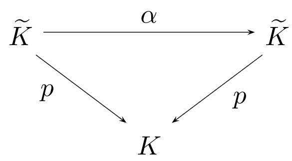



Path-lifting gives an action of the fundamental group $\pi_1(K, v)$ on the fiber of $v$: let $v$ be the basepoint for $K$ and $\gamma$ a loop in $K$ at $v$. For any vertex $\tilde{v}_1$ in the fiber of $v$, let $(\tilde{v}_1)\gamma = \tilde{v}_2$ be the terminal vertex of the lifted path $\tilde{\gamma}$ at $\tilde{v}_1$. If $\gamma'$ is another loop at $v$ homotopic to $\gamma$ then the lifts of each loop at $\tilde{v}_1$ are homotopic as well by homotopy lifting, so in particular must finish at the same vertex. Thus in a well defined way, we can define the image of the vertex $\tilde{v}_1$ under the action of the *homotopy class* of $\gamma$ to be $\tilde{v}_2$. If $\gamma_1, \gamma_2$ are loops at $v$, with $\tilde{\gamma}_1$ the lift of $\gamma_1$ at $\tilde{v}_1$ and $\tilde{\gamma}_2$ the lift of $\gamma_2$ at the terminal vertex of $\tilde{\gamma}_1$, then the path $\tilde{\gamma}_1\tilde{\gamma}_2$ covers $\gamma_1\gamma_2$ and starts at $\tilde{v}_1$. Hence it is *the* lift of $\gamma_1\gamma_2$ at $\tilde{v}_1$. Thus $\tilde{v}_1(\gamma_1\gamma_2) = ((\tilde{v}_1)\gamma_1)\gamma_2$, giving us a homomorphism $\pi_1(K) \rightarrow \text{Sym}(\widetilde{K}^0)$. Notice in particular that if $K$ has just one vertex then $\pi_1(K)$ acts on *all* the vertices of $\widetilde{K}$.

In general this action cannot be extended beyond the vertices of $\widetilde{K}$, not even to the edges. As an example, suppose that $\gamma \in \pi_1(K)$ and $\tilde{v}_1, \tilde{v}_2$ are vertices in the cover connected by an edge $\tilde{e}$. If the action extends to a the edges of $\widetilde{K}$, then the images of the vertices under $\gamma$ must be joined by the image of the edge $\tilde{e}$ under $\gamma$ (remember that an action is a homomorphism $\pi_1(K) \rightarrow \text{Aut}(\widetilde{K})$ meaning that $\gamma$ must act in a way that preserves incidence). In the particular case that $\widetilde{K}$ is the Cayley complex of $\pi_1(K)$ (see §7), this amounts to saying that the generator of $\pi_1(K)$ corresponding to the edge $\tilde{e}$ commutes with $\gamma$. This clearly is not always going to happen.



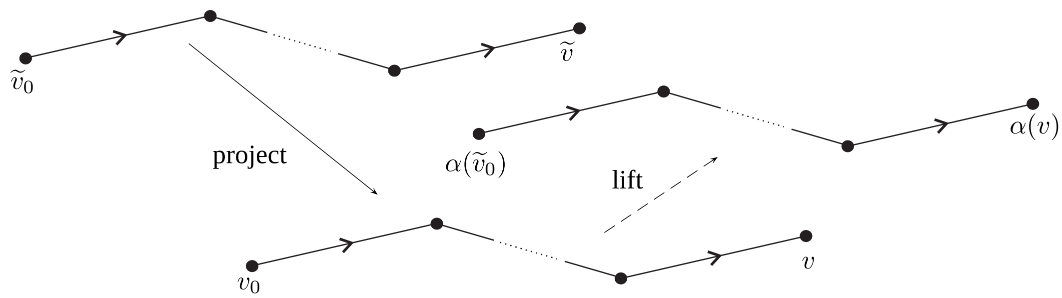

**(6.9)** A *deck transformation* of a covering $\widetilde{K} \xrightarrow{p} K$ is an automorphism $\widetilde{K} \xrightarrow{\alpha} \widetilde{K}$ such that the diagram




commutes, ie: it is a permutation of the fibers (this is the commuting diagram) that rearranges the complex with a result that appears the same to the naked eye (this is the automorphism) [5].

**Proposition 6.1** *If $\widetilde{K}$ is connected then the effect of a deck transformation on the vertices is completely determined its effect on a single vertex. In particular, a non-trivial deck transformation acts freely on $\widetilde{K}$.*

**Proof:** Suppose we know $\alpha(\widetilde{v}_0)$ for some vertex $\widetilde{v}_0$ in $\widetilde{K}$ and for any other $\widetilde{v}$, take a path in the 1-skeleton of $\widetilde{K}$ from $\widetilde{v}_0$ to $\widetilde{v}$:




The path projects to a path in $K$, which can then be lifted to a unique path at $\alpha(\widetilde{v}_0)$. The commutativity of the diagram means this lifted path is the image of the original one under the deck transformation, so in particular, $\alpha(\widetilde{v})$ is the vertex at the end of it. Thus the transformation is uniquely determined by its effect on a single vertex, and in particular, only the identity map can fix a vertex. □

[5] I'm not sure about the origins of the phrase "deck", but one imagines a fiber to be like a deck of cards stacked over the vertex, edge or face being covered, and the deck transformation shuffles the deck.

The deck transformations form a group under composition which we'll denote by $\mathscr{D}(\widetilde{K} \to K)$ or just $\mathscr{D}(\widetilde{K})$. We will see a little later on that $\mathscr{D}$ is closely related to a *subgroup* of the fundamental group $\pi_1(K)$ of the complex being covered.

**(6.10)** In summary then, we have two actions on $\widetilde{K}$, both deficient in some way. The Path-lifting action is of the *whole* group $\pi_1(K)$, but only on *part* of the complex $\widetilde{K}$, namely the vertices. The deck transformation action is the other way around: it has the virtue of being on the whole complex, vertices, edges and faces, but one has to sacrifice some of $\pi_1(K)$.

**(6.11)** Let $\widetilde{K} \xrightarrow{p} K$ be a covering and $g \in \pi_1(K, v)$. Any loop representing $g$ lifts to a path $\widetilde{g}$ at $\widetilde{v}$ which finishes at the vertex $\widetilde{v}_1$ say. We say that $g$ *normalises the covering* iff for a loop $w$ at $v$, either

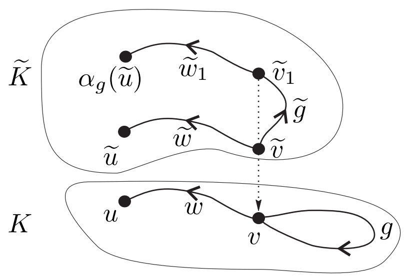
1. $w$ lifts to a loop at $\widetilde{v}$ and a loop at $\widetilde{v}_1$, or
2. $w$ lifts to a non-loop at $\widetilde{v}$ and a non-loop at $\widetilde{v}_1$.

**Lemma 6.1** *$g$ normalises the covering iff $g$ normalises $p^*(\pi_1(\widetilde{K}, \widetilde{v}))$ in $\pi_1(K, v)$, ie: if $w \in p^*(\pi_1(\widetilde{K}, \widetilde{v}))$ then $gwg^{-1} \in p^*(\pi_1(\widetilde{K}, \widetilde{v}))$.*

**Proof:** Is an easy exercise. □

**Aside 6** If $G$ is a group and $H$ a subgroup, the *normaliser* $N_G(H)$ of $H$ in $G$ is the set $N_G(H) = \{g \in G : gHg^{-1} = H\}$. One can show quite easily that the normaliser is a subgroup of $G$, and indeed that it is the largest subgroup of $G$ in which $H$ is normal in the following sense: if $H \lhd N \leq G$, then $N \leq N_G(H)$.

**(6.12)** For a covering $\widetilde{K} \xrightarrow{p} K$, let $G = \pi_1(K, v)$ and $H = p^*(\pi_1(\widetilde{K}, \widetilde{v}))$. If $\widetilde{K}$ is connected and $g \in G$ define a map $\alpha_g : \widetilde{K}_0 \to \widetilde{K}_0$ on the vertices of $\widetilde{K}$ as follows:



Here we have taken a representative loop for $g$, and lifted to a path $\widetilde{g}$ at $\widetilde{v}$ that finishes at the vertex $\widetilde{v}_1$. If $\widetilde{u}$ is a vertex of $\widetilde{K}$, take a path $\widetilde{w}$ from $\widetilde{v}$ to $\widetilde{u}$. This covers a path $w$ in $K$ (ie: $p(\widetilde{w}) = w$) which can then be lifted to a path $\widetilde{w}_1$ at $\widetilde{v}_1$. Define $\alpha_g(\widetilde{u})$ to be the terminal vertex of this lifted path.

The following is an easy exercise:

**Lemma 6.2** *The map $\alpha_g$ is well defined iff $g$ normalises the covering, ie: iff $g \in N_G(H)$.*

Indeed, $\alpha_g$ acts not only on the vertices, but on the whole of the complex $\widetilde{K}$:

**Theorem 6.1** *$\alpha_g$ extends to a deck transformation $\in \mathscr{D}(\widetilde{K} \to K)$.*

**Proof:** We sketch the proof and leave the details to the reader. To extend $\alpha_g$ to a deck transformation we need to extend it to the edges and faces of $\widetilde{K}$. If $\widetilde{e}$ is an edge of $\widetilde{K}$ with vertices $\widetilde{v}_0$ and $\widetilde{v}_1$, then it is not hard to show that their images $\alpha_g(\widetilde{v}_0), \alpha_g(\widetilde{v}_1)$ form the endpoints of an edge in $\widetilde{K}$. Define $\alpha_g(\widetilde{e})$ to be this edge. Similarly, if $\widetilde{f}$ is a face, one can show that $\alpha_g(\partial \widetilde{f})$ is a closed path forming the boundary of a face, and define $\alpha_g(\widetilde{f})$ to be this face. It follows pretty much by definition that $\alpha_g$ is an automorphism $\alpha_g : \widetilde{K} \to \widetilde{K}$, and it can be checked that it permutes the fibers of the covering, giving a deck transformation. □

**Theorem 6.2** *The map $g \mapsto \alpha_g$ induces an isomorphism $N_G(H)/H \cong \mathscr{D}(\widetilde{K} \to K)$.*

**Proof:** It is not hard to check that $g \mapsto \alpha_g$ is a homomorphism $N_G(H) \to \mathscr{D}(\widetilde{K} \to K)$ with kernel $H$ and image $\mathscr{D}(\widetilde{K} \to K)$. $\square$

**Corollary 6.1** *If $\widetilde{K} \xrightarrow{p} K$ is a covering with $\widetilde{K}$ simply connected then $\mathscr{D}(\widetilde{K} \to K) \cong \pi_1(K, v)$. In particular, $\pi_1(K, v)$ acts freely on $\widetilde{K}$.*

**Proof:** By the Subgroup Theorem, $H$ is the trivial subgroup of $G$, hence $N_G(H)/H \cong G = \pi_1(K, v)$, and the result follows immediately from Theorem 6.2. The second part then follows from Proposition 6.1.

**Exercise 16** Let $\widetilde{K} \xrightarrow{p} K\langle X; R\rangle$ be a covering of the presentation 2-complex for some group $\langle X; R\rangle$. The proof of the subgroup theorem showed that the vertices of $\widetilde{K}$ can be identified with the right cosets $Hg$ of $H = \pi_1(\widetilde{K}, \tilde{v})$ in $G = \pi_1(K, v)$.

1. If $g \in \pi_1(K, v)$, show that the action of $g$ by path lifting is given by $Hw \mapsto H(wg)$.
2. Show that the action of $g$ by deck transformations (if $g \in N_G(H)$) is given by $Hw \mapsto H(g^{-1}w)$.

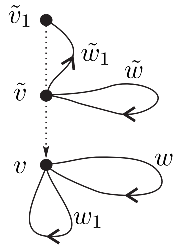
**(6.13)** A covering $\widetilde{K} \xrightarrow{p} K$ is *regular* if it looks the same at every vertex in the following sense: if $w$ is a loop at the basepoint $v$ of $K$, then the lifts of $w$ at each vertex in the fiber of $v$ are either *all loops* or *all non-loops*.

**(6.14)** As regular coverings are particularly nice they should correspond to particularly nice subgroups. This gives a normal subgroup version of the Subgroup Theorem:

**Theorem 6.3**
1. *Let $\widetilde{K} \xrightarrow{p} K$ be a regular covering. Then the image of the map $\pi_1(\widetilde{K}, \tilde{v}) \xrightarrow{p^*} \pi_1(K, v)$ is a normal subgroup.*
2. *Let $K$ be a 2-complex and $H$ a normal subgroup of $\pi_1(K, v)$. Then there is a connected $\widetilde{K}$ and a regular covering $\widetilde{K} \xrightarrow{p} K$ with $\tilde{v} \xrightarrow{p} v$ and $H \cong \pi_1(\widetilde{K}, \tilde{v})$.*

**Proof:** If $w$ is a loop at $\tilde{v}$ then $p : [w]_h \to [p(w)]_h$ so that $p_*(\pi_1(\widetilde{K}, \tilde{v}))$ may be identified with the homotopy classes of loops at $v$ that lift to loops at $\tilde{v}$.

Let $w_1$ be another loop at $v$ and consider the loop $w_1 w w_1^{-1}$, which we want to lift to a loop at $\tilde{v}$. Lifting first $w_1$ to $\tilde{v}$ may or may not give a loop. If it does, then lifting all of $w_1 w w_1^{-1}$ clearly gives a loop too. If it doesn't (as in the picture) then the lift of $w_1 w w_1^{-1}$ is obtained by lifting $w_1$, lifting $w$ at $\tilde{v}_1$ and lifting $w_1^{-1}$ at the terminal vertex of this. But by regularity, the lift of $w$ at $\tilde{v}_1$ is a loop too, so $w_1 w w_1^{-1}$ does indeed lift to a loop at $\tilde{v}$. Thus if the homotopy class of $w$ is in $p_*(\pi_1(\widetilde{K}, \tilde{v}))$ so is the homotopy class of $w_1 w w_1^{-1}$.




For the second part, construct $\widetilde{K}$ as in the proof of the Subgroup Theorem, so we need only show that the corresponding covering is regular. The elements of $H$ correspond to the loops at $v$ lifting to loops at $\tilde{v}$. Let $w$ be any loop at $v$ and $\tilde{w}_1 \xrightarrow{p} w_1$ a path in $\widetilde{K}$ connecting $\tilde{v}$ to some other vertex $\tilde{v}_1$ in the fiber. We want that $w$ lifts to a loop at $\tilde{v}_1$ if and only if it lifts to a loop at $\tilde{v}$. But this second one happens if and only if $w \in H$, and by normality, this is the case if and only if $w_1 w w_1^{-1}$ is in $H$ and thus $w_1 w w_1^{-1}$ lifts to a loop at $\tilde{v}$. Thus we need that $w$ lifts to a loop at $\tilde{v}_1$ if and only if $w_1 w w_1^{-1}$ lifts to a loop at $\tilde{v}$, and this is clearly the case. $\square$

In particular, we have the

**Corollary 6.2** *If $\widetilde{K} \to K$ is a regular covering with $G = \pi_1(K, v)$ and $H = p^*(\pi_1(\widetilde{K}, \tilde{v}))$, then the quotient group $G/H$ acts freely on the cover $\widetilde{K}$.*

**Proof:** Apply Theorems 6.2 and 6.3 using the fact that $H$ normal in $G$ gives $N_G(H) = G$. $\square$

**(6.15)** Coverings that are the antithesis of regular should correspond to subgroups that are the antithesis of normal. With this in mind, a covering $\widetilde{K} \xrightarrow{p} K$ is *completely irregular at* $v \in K$ iff there is a vertex $\tilde{v} \in \widetilde{K}$ in the fiber of $v$ such that any homotopically non-trivial loop at $v$ that lifts to a loop at $\tilde{v}$, lifts to a non-loop at every other vertex of the fiber.

**Aside 7** A subgroup $H$ of a group $G$ is said to be *malnormal* when $g \in G \setminus H$ gives that $gHg^{-1} \cap H = \{1\}$. Thus, malnormal subgroups are the antithesis of normal ones. Nevertheless, examples will arise quite naturally in §8.

**Theorem 6.4** *1. Let $\widetilde{K} \xrightarrow{p} K$ be a covering, completely irregular at $v$ and suppose $\tilde{v} \to v$. Then the image of the map $\pi_1(\widetilde{K}, \tilde{v}) \xrightarrow{p^*} \pi_1(K, v)$ is a malnormal subgroup.*

*2. Let $K$ be a $2$-complex and $H$ a malnormal subgroup of $\pi_1(K, v)$. Then there is a connected $\widetilde{K}$ and a covering $\widetilde{K} \xrightarrow{p} K$ with $\tilde{v} \longrightarrow v$, completely irregular at $v$, and $H \cong \pi_1(\widetilde{K}, \tilde{v})$.*

The proof is sufficiently analagous to that of the previous Theorem for it to be safely left to the reader.

# 7 Applications of Coverings

**(7.1)** First we have a piece of unfinished business to clear up, namely the proof of Theorem 5.1 that if a group is the fundamental group of a graph, then it acts freely on a tree. If the graph is $K$, then the covering $\widetilde{K}$ corresponding to the trivial subgroup of $G$ is a simply connected graph, hence a tree. $G$ then acts freely on this tree by Corollary 6.1.

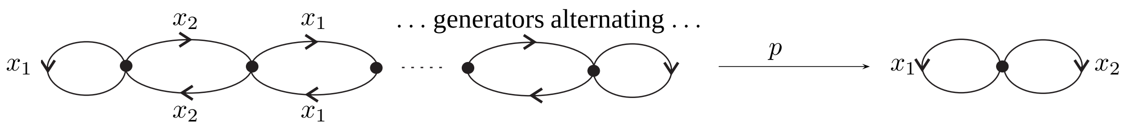
**(7.2)** Here is another proof of the,

**Nielsen-Schreier Theorem (version 2).** *A subgroup of a free group is free.*

The following is a common incorrect "proof": if $H$ is a subgroup of free group $F$, and $r = 1$ is a relation in $H$, then this must also be a relation in $F$, and as $F$ is free this cannot be. The point is that the relation in $H$ is amongst a given set of generators for $H$–there is no reason to suspect that we still can't find a set of generators for $F$ which have no relations amongst them. Instead we can deduce the theorem easily from coverings:

**Proof:** If $F$ is free then $F \cong \langle X; - \rangle$, hence $K\langle X; - \rangle$ is a bouquet of circles. For a subgroup $H$ of $F$ there is a covering $\widetilde{K} \xrightarrow{p} K\langle X; - \rangle$, where the complex $\widetilde{K}$ must be a graph (ie: have empty set of faces) and $H \cong \pi_1(\widetilde{K})$. But the fundamental group of a graph is also free, hence $H$ is free. □

**(7.3)** The proof of the Nielsen-Schreier theorem requires only the second part of the subgroup theorem. The first part can be used to construct subgroups:



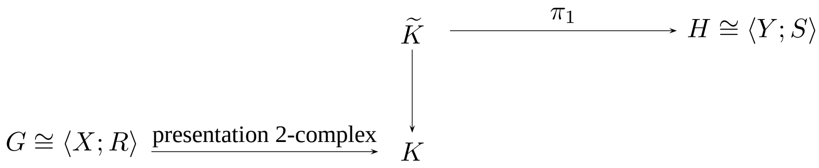

Here we have a covering of the presentation 2-complex for the free group $\langle x_1, x_2; - \rangle$. The corresponding subgroup has index $n$ (the covering complex has $n$ vertices) and is free of rank $n + 1$ (use Theorem 4.2). Thus,

**Proposition 7.1** *For any $n \geq 2$ the free group of rank two contains a free subgroup of rank $n + 1$ and index $n$.*

**Exercise 17** Formulate a similar proposition for free subgroups of the free group of rank $m$.

**Exercise 18** Show (using coverings) that if $F$ is the free group of rank two then the commutator subgroup $[F, F]$ has infinite index in $F$. Show that $[F, F]$ is free of countably infinite rank.

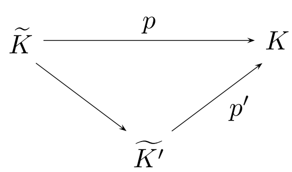
**(7.4) The Reidemeister-Schreier algorithm**. Let $G \cong \langle X; R\rangle$ and $H$ a subgroup. We can obtain a presentation for $H$ if we have the right cosets for $H$ in $G$:




The algorithm works by following the diagram from left to right: form the presentation 2-complex $K$ for $\langle X; R\rangle$; let $\widetilde{K}$ be the covering corresponding to $H$, constructed as in the proof of the subgroup theorem using the cosets for $H$ in $G$; find the presentation for $\pi_1(\widetilde{K}) \cong H$ using Theorem 4.2.

**Exercise 19** If $G \cong \langle X; R\rangle$ with $|X| = n, |R| = m$ and $H$ a subgroup of $G$ of index $k$, then show that the presentation for $H$ obtained from the Reidemeister-Schreier algorithm has $kn - k + 1$ generators and $mk$ relations.

**(7.5) The Cayley complex**. For a complex $K$, a covering $\widetilde{K} \overset{p}{\longrightarrow} K$ is *universal* iff for any other covering $\widetilde{K'} \overset{p'}{\longrightarrow} K$ there is a covering $\widetilde{K} \longrightarrow \widetilde{K'}$ with




commuting. Call such a $\widetilde{K}$ the[6] *universal cover* of $K$. Thus, not only is the universal cover a covering of $K$, but also of any other covering of $K$. In particular, given $G \cong \langle X; R\rangle$, the universal cover of the presentation 2-complex $K(X; R)$ is called the *Cayley complex* of $G$ with respect to $\langle X; R\rangle$. It stores a great deal of useful information about the group.

In view of the subgroup theorem, $\widetilde{K}$ covers all the covers of $K$ precisely when $\pi_1(\widetilde{K})$ is a subgroup of all the subgroups of $\pi_1(K)$, ie: $\pi_1(\widetilde{K})$ is the trivial subgroup so that $\widetilde{K}$ is simply connected. This gives a slightly simpler way of showing that a covering $\widetilde{K}$ is the Cayley complex.

From the construction given in the proof of the Subgroup Theorem, the vertices of the Cayley complex are the cosets in $G$ of the trivial subgroup, so of course can be identified with the elements of $G$ itself. The edges depict the multiplicative action of the generators $X$ on the elements of $G$: in particular, if a path labelled $w$ starts at (the vertex corresponding to) 1 and finishes at (the vertex corresponding to) $g$, then $w = g$ in $G$. Thus, two words $w_1, w_2$ in the generators represent the same element of $G$, ie: $w_1 =_G w_2$, exactly when paths labelled with these words and starting at 1 finish at the same vertex. In particular, a word $w$ represents the identity in $G$ exactly when a path labelled $w$ and starting at 1 is a loop.

**(7.6)** The 1-skeleton of the Cayley complex is the *Cayley graph* for $G$ with respect to the generators $X$ (the relations play no role as there are no faces). This object is very commonly found throughout combinatorial group theory, but it suffers from the serious drawback that it has no convenient *topological* description, whereas the Cayley complex has the virtue of being characterised by its simply connectedness. Nevertheless it is convenient to have a simple description of it:

**Proposition 7.2** *A 1-complex $K$ is the Cayley graph of $G$ with respect to $\langle X\rangle$ if and only if there is a covering $K \to K(X)$ and a bijection $f : K_0 \to G$ such that if $e \in C$ is an edge with initial vertex $v$ and terminal vertex $u$, and $p(e) = x_i \in X$, then $f(u) = f(v)x_i$ in $G$.*

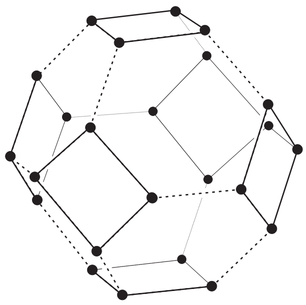
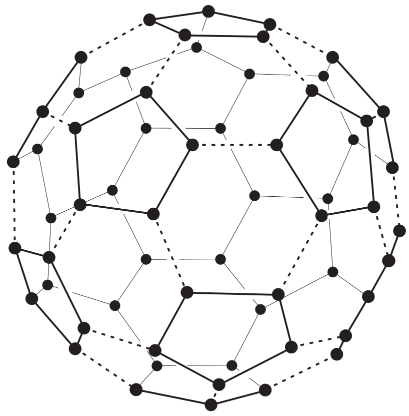
(Here, $K(X)$ is the presentation 2-complex of the free group $\langle X; –\rangle$, ie: a bouquet of loops).

**Exercise 20** Prove Proposition 7.2.

---

[6] The uniqueness of $\widetilde{K}$, upto a 2-complex isomorphism, follows from the definition.

**(7.7)** The symmetric group $S_4$ has a presentation $\langle x, y; x^2, y^4, (xy)^3\rangle$ where $x = (1, 2)$ and $y = (1, 2, 3, 4)$. The Cayley complex with respect to this presentation is shown below left, where a number of details have been suppressed for clarity. The solid edges correspond to the generator $y$ with the orientations running anti clockwise around each square face with respect to the outward pointing normal (and the square faces the lifts of the face corresponding to the relation $y^4 = 1$). There should also be 2-gonal faces with boundary label $x^2$, but these have been squashed out and replaced by a single dotted edge, which thus represents both $x$ and $x^{-1}$ edges. Similarly for the Cayley complex on the right for the alternating group $A_5$ with respect to the presentation $\langle x, y; x^2, y^5, (xy)^3\rangle$ where $x = (1, 2)(3, 4)$ and $y = (1, 2, 3, 4, 5)$.

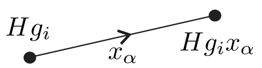
**Aside 8** Let $G$ be a group, $\Omega$ a set and $G \to \text{Sym}(\Omega)$ given by $\omega \mapsto (\omega)g$ an action of $G$ on $\Omega$. For any $\omega \in \Omega$ write $\omega^g$ for its image under the action of the element $g \in G$. The action is *transitive* iff for any $\omega_1, \omega_2 \in \Omega$ there is a $g \in G$ with $\omega_1^g = \omega_2$. It is *regular* iff for every choice of $\omega_1$ and $\omega_2$ this element $g$ is unique.

**Exercise 21** Let $G = \langle X\rangle$ act regularly on $\Omega$ and define a 1-complex $C$ as follows: the vertices $C_0$ are the set $\Omega$, and for every $\omega \in \Omega$ and $x_i \in X$ there is an edge from $\omega$ to $\omega^{x_i}$. Show that the resulting 1-complex is the Cayley graph of $G$ with respect to $X$.

**(7.8)** Cayley complexes give us a little bit more information about the normal subgroups of a group:

**Theorem 7.1** *Let $\widetilde{K} \xrightarrow{p} K(X; R)$ be a regular covering of the presentation 2-complex for $\langle X; R\rangle$, $G = \langle X; R\rangle$ and $H = p_*(\pi_1(\widetilde{K}, \tilde{v}))$. Then the 1-skeleton of $\widetilde{K}$ is the Cayley graph for $G/H$ with respect to the generators $\langle Hx\rangle_{x\in X}$.*

**Proof:** The group $G$ acts on the vertices of $\widetilde{K}$ by path-lifting, indeed transitively, but not regularly. The regularity of the covering however gives that loops in $K$ lift to loops at every vertex in $\widetilde{K}$ if and only if these loops represent an element of $H$. Thus the stabiliser of every vertex of $\widetilde{K}$ under the path-lifting action of $G$ is $H$, and this induces a regular action of $G/H$ on the vertices of $\widetilde{K}$. Applying Exercise 21 with $\Omega = \widetilde{K}_0$ gives, by Exercise 16 part 1, the 1-skeleton of $\widetilde{K}$. Finally, the proof of the Subgroup Theorem gives the following form




for the edges of $\widetilde{K}$. Interpreting $Hg_ix_\alpha$ as an element of the quotient group $G/H$, we can replace it by $Hg_iHx_\alpha$ and interpret the edge label as $Hx_\alpha \in \langle Hx\rangle_{x\in X}$. $\square$

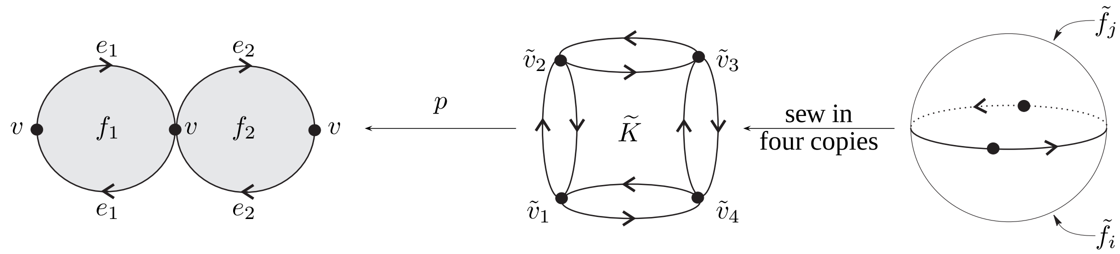
Theorem 7.1 cannot be extended to the 2-skeletons as is easily seen: the full 2-complex $\widetilde{K}$ (ie: faces included) has fundamental group $p_*(\pi_1(\widetilde{K}, \tilde{v}))$ by the Subgroup Theorem, but the full Cayley complex has trivial fundamental group. While sharing the same 1-skeleton, the Cayley complex will have many more faces than $\widetilde{K}$.

**Aside 9** Recall that a group $G$ is *soluble* if and only if there is a sequence of subgroups

$$\{1\} = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G,$$

with the successive quotients $N_i/N_{i-1}$ Abelian. Thus a soluble group is the result of taking successive extensions of the trivial group by Abelian groups. By a celebrated theorem of Galois (and indeed this is the source of the concept in group theory), a polynomial with $\mathbb{Q}$-coefficients has roots that can be expressed in terms of $\mathbb{Q}, +, -, \times, \div$ and $\sqrt[n]{\ }$ if and only if its Galois group is soluble.

**(7.9)** As an application of Theorem 7.1, we show that the free product $G = \mathbb{Z}/2 * \mathbb{Z}/2$ (see §8) is soluble. First, realise $\mathbb{Z}/2 * \mathbb{Z}/2$ as the fundamental group of a 2-complex. This can be done either using the construction described in §8, or by using Theorem 8.1, from which we get a presentation $\langle x, y; x^2, y^2\rangle$, and hence a presentation 2-complex as shown below on the left:




Now consider the commutator subgroup $[G, G]$ generated by all commutators $[a, b], a, b \in G$, which is well known to be a normal subgroup. Moreover, the quotient $G/[G, G]$ is the abelianisation of $\mathbb{Z}/2*\mathbb{Z}/2$, which is Klein's 4-group, $\mathbb{Z}/2 \times \mathbb{Z}/2$.

By Theorem 7.1, the covering complex $\widetilde{K}$ corresponding to $[G, G]$ has 1-skeleton that of the Cayley complex for the 4-group with presentation $\langle x, y; x^2, y^2, [x, y]\rangle$. Thus the 1-skeleton is as shown above in the middle. As the covering has degree four, the fiber of each face must contain four faces. On the other hand, for each face $f_i$ in the presentation 2-complex there must be a lift $\tilde{f}_i$ with boundary any one of the four pairs of edges shown in $\widetilde{K}$. Thus the faces of $\widetilde{K}$ are obtained by sewing in four copies of the 2-sphere complex, one into each of the edge pairs.

Now that we have the $\widetilde{K}$ corresponding to $[G, G]$ we can apply Theorem 4.2 and obtain the presentation

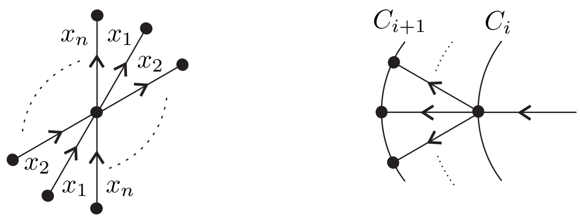
$$\pi_1(\widetilde{K}) \cong [G, G] = \langle a, b; ab = 1\rangle = \langle b; -\rangle \cong \mathbb{Z},$$

and so we have the sequence $\{1\} \lhd \mathbb{Z} \lhd \mathbb{Z}/2 * \mathbb{Z}/2$ with successive quotients $\mathbb{Z}$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

**Exercise 22** Show that the commutator subgroup of $\mathbb{Z}/m * \mathbb{Z}/n$ is free for all $m, n \geq 2$. Find the rank of this free group and deduce that the only case in which the free group is Abelian is when $m = n = 2$.

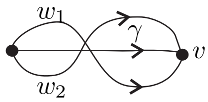
**Aside 10** Actually, for any non-trivial finite groups $A$ and $B$, the free product $A * B$ is soluble if and only if $A \cong B \cong \mathbb{Z}/2$.

**(7.10)** Another important example is the Cayley complex for a finitely generated free group $\langle x_1, \ldots, x_n ; -\rangle$. Take a family $C_i$ of concentric circles with radii $i \in \mathbb{Z}^+$. At the center of the innermost circle place the configuration of edges shown on the left,




with the outer vertices sitting on $C_1$. Construct the rest of the complex inductively: for each vertex on the circle $C_i$, place $2n - 1$ *distinct* new vertices on $C_{i+1}$ as shown on the right above, labelling the $2n$ mutually incident edges in the same way as the first (it doesn't matter what order this is done in). That ends the construction, so in particular there are no faces.

To see that the complex so constructed is the Cayley complex for $\langle x_1, \ldots, x_n ; -\rangle$, we need to show that it is a covering of $K\langle x_1, \ldots, x_n ; -\rangle$ and that it is simply-connected, ie: $\pi_1$ is trivial. That it covers (by the obvious covering map) is trivial. If $w$ is a closed path starting at the central vertex, then there must be a maximal $n$ such that $w$ contains a vertex lying on $C_n$. This vertex is thus connected to $C_{n-1}$ by two edges from the path: one that arrives and one that leaves. But these two edges must be a spur, for otherwise there would be distinct vertices on $C_{n-1}$ connected to the vertex on $C_n$, contradicting the construction of the complex. Remove this spur from $w$ to get a path with two fewer edges and repeat the argument, until $w$ is seen to be (freely) homotopic to the central vertex. Thus the complex is simply connected. Alternatively, you can just say that the complex is "obviously" a tree!




From the general comments about Cayley complexes two words $w_1, w_2$ in the generators are equal in $\langle x_1, \dots, x_n\,; - \rangle$ precisely when paths starting at the central vertex and labelled $w_i$ end at the same vertex $v$ of the Cayley complex. Since we've already seen that this complex is a tree, there is a *unique path* $\gamma$ without spurs starting at the central vertex and ending at $v$. Thus removing spurs from the paths $w_i$ must give $\gamma$, and so we have the

**Normal Form for Free Groups.** *Two words represent the same element of a free group precisely when the removal of all occurrences of* $x_\alpha x_\alpha^{-1}$ *or* $x_\alpha^{-1} x_\alpha$ *results in* identical *words.*

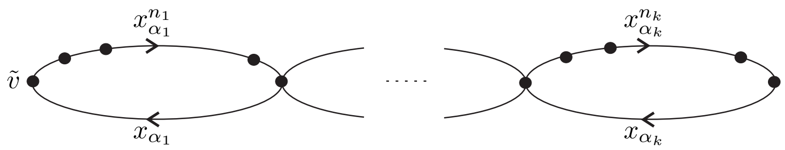
Words containing no occurrences of $x_\alpha x_\alpha^{-1}$ or $x_\alpha^{-1} x_\alpha$ are called *reduced*, so the theorem says that any element of a free group is uniquely represented by a reduced word.

In general, a normal form for $G \cong \langle X; R \rangle$ is a canonical choice of path to each vertex in the Cayley complex from the base vertex. Equivalently, it is a list of words such that every element of $G$ is represented by a unique word in the list. In the example above this is particularly easy as the complex is a tree, so there is an obvious choice of path, namely a *geodesic*–a path containing as few edges as possible.

Clearly any group will have a normal form with respect to any presentation $\langle X; R \rangle$–just choose a path to each vertex in the Cayley graph–but the resulting list of words may not be easy to describe[7].

**(7.11)** A group $G$ is *residually finite* iff for any $g \in G$ not equal to the identity, there is a finite group $K$ and a homomorphism $\varphi : G \to K$ such that $\varphi(g) \neq 1$. It is a *local* property of the group $G$ in the sense that standing at any $g \neq 1$ we may pretend, by suitable squinting of the eyes (this is $\varphi$) that we are in a finite group.

We show that free groups are residually finite. Let $F$ be free with free generators $x_i$. By collecting together consecutive occurrences of the same generator, any word can be written as $w = x_{\alpha_1}^{n_1} \dots x_{\alpha_k}^{n_k}$ with $x_{\alpha_1} \neq x_{\alpha_{i+1}}$. On the other hand, such a word is clearly reduced. Thus, any non-trivial element of the free group can be expressed in this form with the $n_i \neq 0$. Consider,




a 2-complex $\widetilde{K}$ with a loop labelled by $x_i$ at every vertex that is not already incident with any $x_i$ edges as shown in the picture (for convenience the picture has been drawn for the case that all the $n_i$ are positive; similar pictures apply whenever any are negative). It is easy to see that this is a covering of the presentation 2-complex for $F$. The free group $F$ thus acts by path lifting on $\widetilde{K}$ giving a homomorphism $F \to \text{Aut}(\widetilde{K})$ to a finite group. The element $w$ acts on the vertex $\tilde{v}$ as given in the picture, so in particular, not fixing it. Hence $w$ maps to a non trivial element of $\text{Aut}(\widetilde{K})$.

**Exercise 23** Show that the modular group $\text{PSL}_2\mathbb{Z}$ is residually finite (*hint*: use the presentation found in §4).

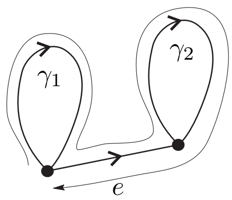
**(7.12)** There is a celebrated conjecture due to Hanna Nuemann that is more or less (more rather than less) open.

[7] When it *is* easy to describe the normal form can be used to solve the word problem for the group (see §9). Be warned though, there are groups for which no nicely described normal form exists.

# 8 Fundamental groups of graphs of groups

The two most important constructions in combinatorial group theory arise from what happens when you glue complexes together: we can glue two complexes together along a common sub-complex, or glue a complex to itself along two different copies of a complex inside of it. In any case, taking the fundamental groups of everything in sight (including the maps, ie: considering the homomorphisms induced by the maps) we can interpret this set-up as "gluings" of two different groups along common subgroups, or of a group to itself along two isomorphic subgroups. Because of the symmetry of these ideas, the two constructions are often thought of as dual to each other.

**(8.1)** Given a pair of complexes with the same fundamental groups, we construct a "tube-like" complex having these two at the ends. It is meant to mimic $K \times [0,1]$.




Let $K_1, K_2$ be 2-complexes and

$$\varphi : \pi_1(K_1, v_1) \to \pi_1(K_2, v_2)$$

an isomorphism. A $(K_1, K_2)$-*handle* is constructed as follows: join the base points by a new edge $e$. Let $\gamma_1$ be a loop representing a generator for $\pi_1(K_1, v_1)$ arising via Theorem 4.2 and $\gamma_2$ a representative loop for its image under $\varphi$. Sew in a new face with boundary label the path $\gamma_1 e \gamma_2 e^{-1}$. Perform this process for each generator of $\pi_1(K_1, v_1)$.

**(8.2)** Let $G_1, G_2$ and $H$ be groups and $\varphi_1, \varphi_2$ homomorphisms,

$$G_1 \xleftarrow{\varphi_1} H \xrightarrow{\varphi_2} G_2$$

Let $K_1, K_2$ and $K_0$ be single-vertexed 2-complexes with $\pi_1(K_i) \cong G_i$ and $\pi_1(K_0) \cong H$ (for example they could be presentation 2-complexes with respect to some presentations for the three groups). Apply Proposition 4.2 to get subdivisions $\overline{K}, \underline{K}$ of $K_0$ and 2-complex maps $p_i$ such that the diagrams

$$\begin{array}{ccccc} \pi_1(K_0, v) & & \xrightarrow{\varphi_1} & & \pi_1(K_1, v_1) \\ & \cong \searrow & & \nearrow (p_1)_* & \\ & & \pi_1(\overline{K}, v) & & \end{array} \qquad \begin{array}{ccccc} \pi_1(K_0, v) & & \xrightarrow{\varphi_2} & & \pi_1(K_2, v_2) \\ & \cong \searrow & & \nearrow (p_2)_* & \\ & & \pi_1(\underline{K}, v) & & \end{array}$$

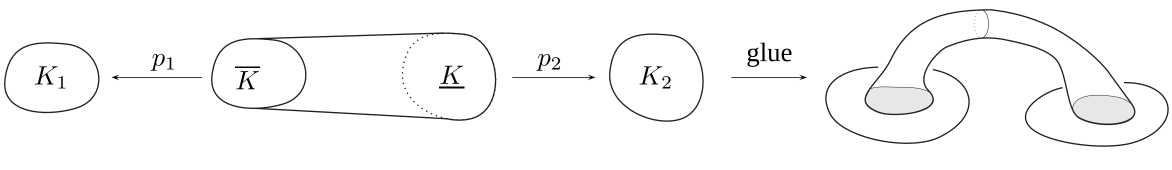
commute.

We now have 2-complexes and 2-complex maps,

$$K_1 \xleftarrow{p_1} \overline{K},\ \underline{K} \xrightarrow{p_2} K_2 \xrightarrow{\text{fundamental groups}} G_1 \xleftarrow{\varphi_1} H \xrightarrow{\varphi_2} G_2$$

that completely models the group situation we started with: the complexes and maps on the left induce the groups and homomorphisms on the right. By Proposition 2.1 we can ensure, by modifying the $K_i$ if necessary, that the $p_i$ are dimension preserving.

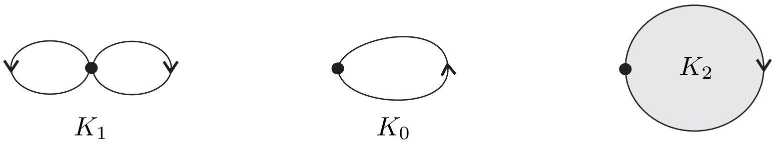
Form the $(\overline{K}, \underline{K})$-handle using the isomorphism $\pi_1(\overline{K}) \cong \pi_1(K_0) \cong \pi_1(\underline{K})$, so that we have the following picture:



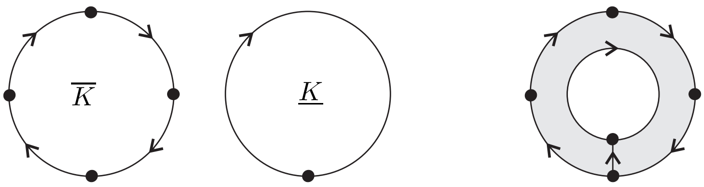

Glue the three complexes together using the $p_i$ as attaching maps. If $K$ is the resulting 2-complex, call its fundamental group the *type I amalgam of $G_1$ and $G_2$ along $H$ (with respect to $\varphi_1$ and $\varphi_2$)*.

**(8.3)** Mainly for historical reasons, a special place is reserved for those amalgams that arise via maps $\varphi_i$ that are *injections*. In this case we may think of the situation as having two groups $G_1, G_2$ each with a subgroup $H_i$ which are isomorphic $H_1 \cong H \cong H_2$. The resulting amalgam is called the *free product of $G_1$ and $G_2$ amalgamated over $H$*, and normally written without reference to the injective maps as $G_1 *_H G_2$.

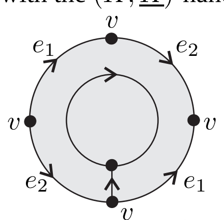
**(8.4)** Here is an example nevertheless where the homomorphisms $\varphi_i$ are not injective. Let $G_1 = \langle a, b; - \rangle$, $G_2$ be the trivial group and $H = \langle c; - \rangle$, the integers. The map $\varphi_1$ sends $c$ to the commutator $[a, b] = aba^{-1}b^{-1}$, and $\varphi_2$ must send $c$ to the identity in $G_2$. We realise each of the groups as fundamental groups of the complexes:




To realise the map $\varphi_1$ by a 2-complex map we need to subdivide the complex $K_0$ as described in Proposition 4.2. The map $\varphi_2$ sends the single generating loop of $K_0$ to the single generating loop of $K_2$ so no subdivision is needed there. Thus the complexes $\overline{K}$ and $\underline{K}$ are as on the left,




with the $(\overline{K}, \underline{K})$-handle on the right a cylinder.




The effect of carrying out the gluings is on the one hand to glue the complex $K_2$ into the interior "hole" of the cylinder, while on the other to identify the edges around the outside according to the scheme given by the commutator. Applying Theorem 4.2 to this complex gives $\pi_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}$. Notice that we have simplified the drawing of the complex. Strictly speaking there should be edges running from the single vertex on the inside circle to all four vertices on the outside one as the gluing has identified these four vertices into a single vertex.

**(8.5)** Finding a presentation for a type I amalgam,

**Theorem 8.1** *If*

$$G_1 \xleftarrow{\varphi_1} H \xrightarrow{\varphi_2} G_2$$

*where the $G_i \cong \langle X_i; R_i \rangle$ and $H = \langle Y \rangle$, then the type I amalgam has presentation,*

$$\langle X_1, X_2; R_1, R_2, \varphi_1(y) = \varphi_2(y) \text{ for all } y \in Y \rangle$$

Notice that a consequence of the Theorem is that the amalgam depends only on the $H, G_i$ and the $\varphi_i$, and not on the choice of complexes used in the construction.

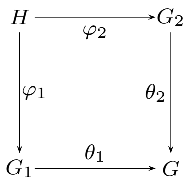
**Proof:** We may assume that the presentations given arise by applying Theorem 4.2 to the complexes $K_i$. Thus we have spanning trees $T_i$ for the $K_i$ with the $X_i$ Schreier generators corresponding to the edges of $K_i$ not in the tree $T_i$ and the $R_i$ the boundaries of faces. Let $K$ be the glued up complex described in the construction, letting the basepoint $v$ of $K_1$ be the basepoint for $K$. Let $T$ be the spanning tree obtained by taking the union of $T_1$ and $T_2$ together with the edge $e$ of the handle. Schreier generators for $\pi_1(K)$ are then the $X_1$ together with the homotopy classes of the loops $eX_2e^{-1} = \{ex_\alpha e^{-1} \,|\, x_\alpha$ represents a generator in $X_2\}$. The relations are the $R_1, R_2'$ together with those arising from the faces of the handle (here $R_2'$ are the relations $R_2$ with the $x_\alpha \in X_2$ replaced by the $ex_\alpha e^{-1}$). The handle relations are clearly of the form $\varphi_1(y) = \varphi_2(y)$ for all $y \in Y$. □

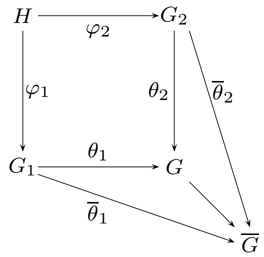
**(8.6)** If $H$ is the trivial group then the homomorphisms $\varphi_i$ are automatically injective, sending the single element of $H$ to the identities of the $G_i$. Theorem 8.1 gives the presentation

$$\langle X_1, X_2; R_1, R_2\rangle$$

for the type I amalgam, called the *free product* of $G_1$ and $G_2$.

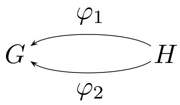
**Exercise 24** If the group $G$ is the type I amalgam of $G_1, G_2$ along $H$, show that there are maps $\theta_1, \theta_2$ making the diagram on the left commute,



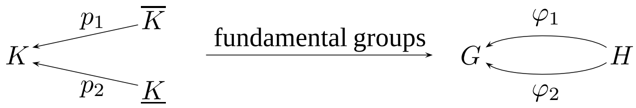




and moreover, if $\overline{G}$ also makes this diagram commute (through the use of maps $\overline{\theta}_1, \overline{\theta}_2$) then there is a homomorphism $G \to \overline{G}$ with the diagram above right commuting. Thus the type I amalgam is a *push-out* in the category of groups.

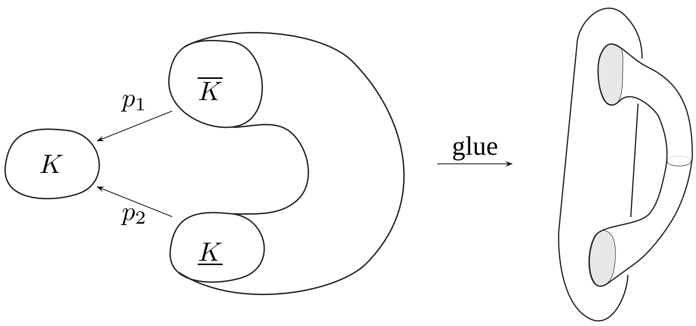
**(8.7)** If the type I amalgam arose from gluing different complexes together along a "common" subcomplex then the type II arises by gluing a complex to itself along two "copies" of a sub-complex. Let $G$ and $H$ be groups and $\varphi_1, \varphi_2$ homomorphisms,




Let $K$ and $K_0$ be single-vertexed 2-complexes with fundamental groups $G$ and $H$. Apply Proposition 4.2 to get subdivisions $\overline{K}, \underline{K}$ of $K_0$ and maps $p_i$ so that as before, the $p_i$ induce homomorphisms that are the same as the $\varphi_i$ (upto an automorphism of $H$). Thus, we now have 2-complexes and 2-complex maps that induce the original group theoretic picture:



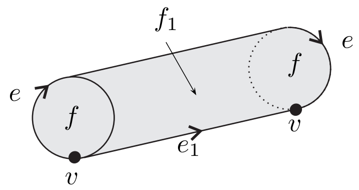

Form the $(\overline{K}, \underline{K})$-handle using the isomorphism $\pi_1(\overline{K}) \cong \pi_1(K_0) \cong \pi_1(\underline{K})$, so that we have the following picture:




and glue the two complexes together using the $p_i$ as attaching maps. Call the fundamental group of the resulting complex the *type II amalgam of* $G$ *and* $H$ *(with respect to* $\varphi_1$ *and* $\varphi_2$*)*.

**(8.8)** History again singles out the special case where the maps $\varphi_i$ are injective, calling the amalgam the *HNN-extension of G by H*. The HNN are the initials of Graham Higman, Bernard Neumann and Hanna Neumann who first concocted (an algebraic version of) this construction. We will justify the use of the word "extension" later on.

**(8.9)** As an example let $G$ and $H$ both be the trivial groups, $G = \langle a; a = 1\rangle$ and $H = \langle b; b = 1\rangle$ with $\varphi_1 = \varphi_2$ being the obvious maps. Realising all these groups using the same complex as in the example above we get as our final complex,

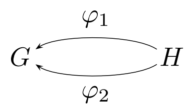



a cylinder with its ends identified. Applying Theorem 4.2, we get that the HNN-extension of the trivial group by the trivial group is $\mathbb{Z}$.

**Exercise 25** Show that the HNN-extension of $G$ by the trivial group is the free product $G * \mathbb{Z}$.

**(8.10)** As with the type I amalgams we have,

**Theorem 8.2** *If*




*where* $G \cong \langle X; R\rangle$ *and* $H = \langle Y\rangle$, *then the type II amalgam has presentation,*

$$\langle X, t; R, \varphi_1(y) = t^{-1}\varphi_2(y)t \text{ for all } y \in Y\rangle$$

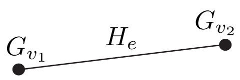
**Proof:** The proof is much as for Theorem 8.1. We have a spanning tree $T$ for the 1-skeleton of $K_1$ so that the generators $X$ correspond to the edges not in this tree. To get a spanning tree for $K$ we take $T$ together with the edge $e$ from the handle (which by the gluings has been turned into a loop). The relators are those from $G$ (the faces of $K_1$ are still there in $K$) together with those arising from the faces of the handle. These are of the form given by definition. □

**(8.11)** In §7 we had the normal form for free groups, which was a list of words in the generators for the group such that any element was represented by precisely one word on the list. There is a similar procedure for the two types of amalgams, the details of which we won't go into here (see for instance [6]). Nevertheless, a useful consequence is the,

**Corollary 8.1** *The vertex groups in a free product with amalgamation or HNN-extension inject.*

In particular, $G$ injects into the HNN-extension of $G$ by $H$, so it is indeed an extension.

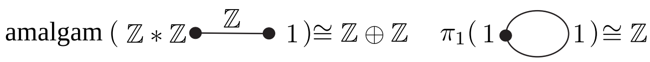
**(8.12)** The type I and II amalgams (and in particular the free product with amalgamation and HNN-extension) are special cases of a more general construction.



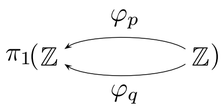

Suppose we have a graph $\Gamma$ with each vertex incident with finitely many edges. We allow for the possibility that $\Gamma$ has multiple edges between vertices and loops at a vertex. For each vertex $v$ of $\Gamma$ we have a *vertex group* $G_v$, and for each edge $e$ an *edge group* $H_e$. If the edge connects the vertices $v_1$ and $v_2$ there are homomorphisms $\varphi_i : H_e \to G_{v_i}$; if the edge is a loop at the vertex $v$ we thus have two homomorphisms of $H_e$ into $G_v$. The graph, together with the groups and homomorphisms is called a *graph of groups*.

Form complexes $K_{v_1}, K_{v_2}, K_0$ for all the vertices and edges realising the respective groups. For each edge, form the handle complex as we did above and carry out all the gluings. The fundamental group of the resulting complex is called the *amalgam of the graph of groups*. If all the homomorphisms are injections we have a *fundamental group of graph of groups*.

The type I and II amalgams are clearly amalgams in this sense where the graph has either two vertices joined by a single edge, or a single vertex incident with a loop. The examples above are,



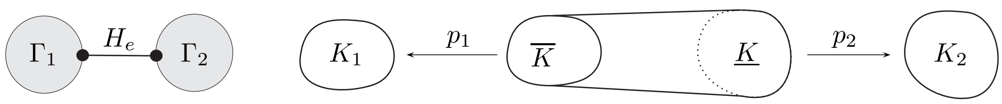

**(8.13)** The map $\varphi_t : \mathbb{Z} = \langle z \rangle \to \mathbb{Z}$ given by $\varphi_t(z) = z^t$ is an injective homomorphism, so that the data




gives the HNN-extension with presentation $\langle z, t \,:\, z^q = t^{-1} z^p t \rangle$, a so-called *Baumslag-Solitar group*.

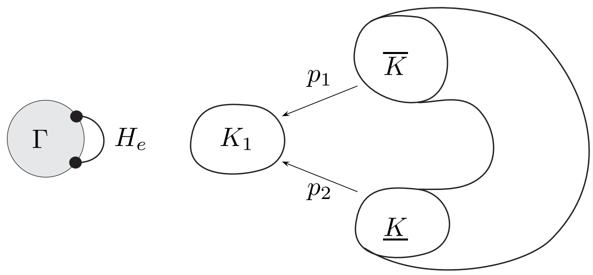
**(8.14)** In finding a presentation for an amalgam, the reader may have sensed that we have already done all the hard work. Indeed, an amalgam as defined above is just a sequence of type I and II amalgams (so in particular a fundamental group of a graph of groups is just a sequence of free products with amalgamation and HNN-extensions).

If the graph $\Gamma$ has the form shown below left, then to find the amalgam we need to construct a complex of the form on the right,




where the subcomplexes $K_1, K_2$ are the complexes needed for the amalgams arising from the graphs of groups $\Gamma_1, \Gamma_2$. Thus the amalgam is a type I amalgam of groups $G_1$ and $G_2$ over $H_e$, where the $G_i$ are the amalgams coming from the $\Gamma_i$.

Similarly if $\Gamma$ has the form below left




the amalgam is a type II amalgam of $G_1$ and $H_e$ with $G_1$ the amalgam coming from the graph $\Gamma_1$.

Thus, to obtain a presentation for an amalgam, take a spanning tree $T$ for the graph $\Gamma$, and perform type I amalgams along the edges of the tree and type II amalgams along the edges not in the tree.

**(8.15)** Recall that a *tree* is a simply connected 1-dimensional complex $T$, and that a group $G$ acts on $T$ orientation preservingly, which is to say, without inversions.

We saw group actions on trees in §5, where a group that does so without fixing a vertex is a free group. In general, if the stabilisers of the vertices is non-trivial we have,

**The Aboreal Dictionary (Serre)** *A group acts on a tree if and only if it is the fundamental group of a graph of groups.*

If $G$ acts on $T$ then the graph is $T/G$ with the vertex/edge groups stabilizers in the action; if $G = \pi_1(\text{graph of groups } \Gamma)$ then the tree $T$ is the universal cover of $\Gamma$ (as a 1-complex).

**(8.16)** We saw in §7 that the free product

$$\pi_1(\mathbb{Z}/m \bullet\!\!\xrightarrow{\;\;1\;\;}\!\!\bullet \mathbb{Z}/n)$$

contained a free subgroup as a subgroup of index $mn$. In general, a group is *virtually free* if and only if it contains a free subgroup of *finite* index. It turns out that examples such as the one above account for all the virtually free groups.

**Theorem 8.3 (Karrass-Pietrowski-Solitar)** *A group is virtually free if and only if it it the fundamental group of a graph of groups with all the vertex groups finite.*

(Notice that the edge groups, injecting as they do the vertex groups, will also be finite).

**(8.17)** A consequence of the theorem above is the following

**Corollary 8.2 (Stallings-Swan)** *A virtually free torsion free group is free.*

**Proof:** The group is virtually free hence of the form given in the theorem. As the group is torsion free and the vertex groups inject into it, these vertex groups must all be trivial (and hence the edge groups are too). Thus we have $\pi_1(\text{graph of groups})$ with a trivial group at each vertex and edge. Taking a spanning tree for the graph, we perform free products with amalgamation of trivial groups over trivial groups (the result is the trivial group) followed by a series of HNN-extensions by the trivial group (each of which gives a free factor of $\mathbb{Z}$). Thus the group is $\mathbb{Z} * \cdots * \mathbb{Z}$, ie: free. $\square$

# 9 Word Problems

In dealing with a presentation for a group there is a great deal of ambiguity in how one can express individual elements. Sometimes it's obvious when two different expressions in fact represent the same element, as we saw with free groups in §7. In general though, life can be made arbitrarily hard. The analysis of this, the *word problem*, is the subject of this section, and indirectly, the rest of these lectures. It is single handedly responsible for most of the major developments in the subject, in the sense that they arose initially through attempts to understand better this situation.

**(9.1)** Consider the group $\langle X; R \rangle$ with $X$ finite. The *word problem* for $\langle X; R \rangle$ asks for the existence of an algorithm which takes as input

$$w \xrightarrow{\text{input}} \boxed{\text{algorithm A}} \xrightarrow{\text{output}} \begin{cases} \text{yes}, & \text{if } w =_{\langle X;R \rangle} 1 \\ \text{no}, & \text{if } w \neq_{\langle X;R \rangle} 1 \end{cases}$$

any word $w$ in the generators $X$ and in a finite number of steps produces the output "yes" if $w =_{\langle X;R \rangle} 1$ and "no" otherwise. We say that $\langle X; R \rangle$ has solvable word problem in this case.

**Aside 11** By algorithm we mean Turing machine. There is of course a formal definition, but here is the guts of it: the machine has a control unit which can be in any one of a finite number of states. The unit controls a tape head positioned above a square on a tape of arbitrary length, which is divided into these squares. Each square has a symbol printed in it from a finite alphabet. The machine works in discrete time, with each new step either changing the state of the control unit, overwriting the symbol in the square currently being scanned by the tape head, or moving the tape head one square to the left or right on the tape.

The *Church-Turing thesis* is that this definition captures the intuitive notion of an algorithm. In practical situations the thesis is used in the following way: a process that "feels" algorithmic is described, and the existence of a Turing machine performing the algorithm is deduced. It is all very convenient, as constructing Turing machines to do even the most basic tasks is horrendously complicated.

**(9.2)** For example, $\langle x; x^n \rangle$ could use an algorithm that counts in $w$ the exponent sum $e(w)$ of $x$, with output "yes" iff $e(w) \equiv 0 \bmod n$.

**(9.3)** Given two presentations $\langle X; R \rangle$ and $\langle Y; S \rangle$ for the same group $G$ (with $|X|, |Y| < \infty$), then any $y_i \in Y$ can be expressed as a word[8] $x_{i_1} \dots x_{i_k(i)}$ in the $X$'s. If $\langle X; R \rangle$ has solvable word problem and $w$ a word in the generators $Y$, then replacing each $y_i$ in $w$ with $x_{i_1} \dots x_{i_k(i)}$ gives a word in the generators $X$ that can be fed through the Turing machine solving the word problem for $\langle X; R \rangle$. This gives a solution for the word problem in $\langle Y; S \rangle$.

Thus, having solvable word problem is a property of groups, not presentations, and in future we will just say that $G$ has solvable word problem whenever some presentation for it does. *A priori* the relations in a presentation play no role in the formulation of the word problem, so in determining if a group has solvable word problem, we may do so with reference just to a set of generators.

**Exercise 26** Show that if two groups have solvable word problem using the same Turing machine then they are isomorphic.

**(9.4)** So now for the bad news:

**Theorem 9.1 (Boone-Novikov, Higman)** *There exists a finitely presented group with unsolvable word problem.*

The theorem can be made quite concrete in that there is a certain presentation on 10 generators and 29 relations that can be shown to have unsolvable word problem.

**(9.5)** Perhaps the most striking result on algorithmic unsolvability is the following theorem of S. Adian. Let $\mathscr{P}$ be a collection of groups such that

1. If $G$ and $H$ are isomorphic then $G \in \mathscr{P}$ if and only if $H \in \mathscr{P}$;
2. there is a finitely presented group $G \in \mathscr{P}$;
3. there is finitely presented group $G$ that cannot be embedded in any group in $\mathscr{P}$.

Call $\mathscr{P}$ a *Markov property* of groups. Examples of Markov properties are: being the trivial group; being a finite group; being Abelian; being simple and being free.

**Theorem 9.2 (Adian)** *For any Markov property $\mathscr{P}$ there is no algorithm to determine if an arbitrarily finitely presented group is in $\mathscr{P}$.*

**(9.6)** So much for the bad news. The good news is that many classes of important groups occurring in nature are *known* to have solvable word problem.

The first obvious examples are the finite groups: if $G = \{g_0 = 1, \dots, g_{n-1}\}$ then create the multiplication table for $G$, ie: the $n \times n$ array with $(i, j)$-th entry the product $g_i g_j$. Using the elements of the group itself as generators, any word $g_{i_1} \dots g_{i_k}$ can be evaluated using the table, representing the identity if this evaluation yields $g_0 = 1$.

**(9.7)** A group $G$ is a *one relator* group iff $G \cong \langle X; R \rangle$ with $|R| = 1$.

**Theorem 9.3 (Magnus)** *One relator groups have solvable word problem.*

An important example is given at the end of this section.

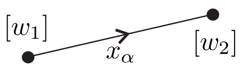
**(9.8)** Many groups arising in geometry and topology have solvable word problem: Coxeter groups, Braid and Artin groups, the fundamental groups of closed 2-manifolds, the fundamental groups of closed orientable 3-manifolds, the fundamental groups of any geometric 3-manifold satisfying Thurstons geometrisation conjecture (except those containing a Sol-manifold component), mapping class groups of surfaces, and linear groups!

[8] Strictly speaking, the groups $\langle X; R \rangle$ and $\langle Y; S \rangle$ are isomorphic, with $x_{i_1} \dots x_{i_k(i)}$ the image under this isomorphism of $y_i$.

**(9.9)** Returning to our topological viewpoint, the word problem for $G = \langle X; R\rangle$ can be phrased in terms of the Cayley complex of the presentation $\langle X; R\rangle$. In fact, only the 1-dimensional information is needed, so we consider the *Cayley graph*, the 1-skeleton of this complex. If $u$ and $v$ are vertices, let $d(u, v)$ be the minimum number of edges in a path between them.

**Proposition 9.1** *$G \cong \langle X; R\rangle$ has solvable word problem precisely when there is an algorithm that constructs the ball $B(n) = \{$all vertices $u$ with $d(u, 1) \leq n\} \cup \{$incident edges$\}$ in the Cayley graph for $\langle X; R\rangle$.*

**Proof:** Given such an algorithm for constructing a ball in the Cayley graph and $w$ a word in $\langle X; R\rangle$, construct a ball of sufficient size to accommodate the path labelled $w$ starting at the vertex corresponding to the identity. The word represents the identity element precisely when this path closes to form a loop.

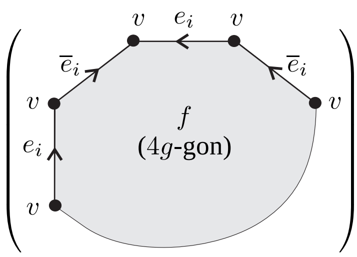



On the other hand, let $A$ be an algorithm for solving the word problem in $\langle X; R\rangle$ and $\mathscr{B}$ the set of all words in the generators $X$ involving at most $n$ of the $X$'s. Define an equivalence relation on this set of words by $w_1 \sim w_2$ iff $w_1$ and $w_2$ represent the same elements of the group $G$, which you determine by feeding $w_1 w_2^{-1}$ into the algorithm $A$. Call an equivalence class in $\mathscr{B}/\!\sim$ a vertex, and join two vertices $[w_1]$ and $[w_2]$ by an edge labelled $x_\alpha \in X$ precisely when $w_1 x_\alpha = w_2$, determined once again by feeding $w_1 x_\alpha w_2^{-1}$ into algorithm $A$. The result is clearly the ball of radius $n$ in the Cayley graph. $\square$

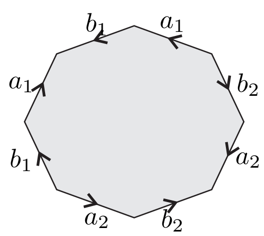
**(9.10)** Use the previous and the constructions of Cayley complexes to show that the word problem is solvable for free groups; fundamental groups of graphs of groups if the vertex groups have solvable word problem; eg: finite groups do so $\mathbb{Z}/2 * \mathbb{Z}/3 \cong \text{PSL}_2\mathbb{Z}$ has solvable word problem.

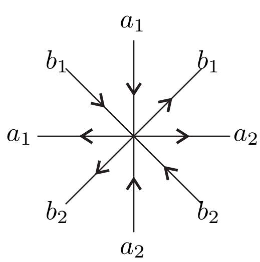
**(9.11)** An important example is the solution to the word problem in a surface group.We have,

$$\Sigma_g \cong \langle x_1, y_1, \dots, x_g, y_g\, ;\, \textstyle\prod_i [x_i, y_i]\rangle = \pi_1$$

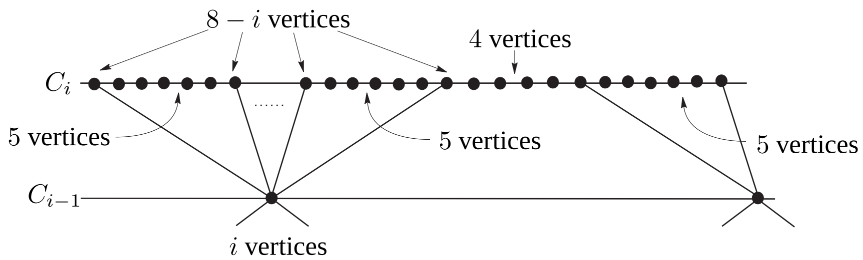



the presentation 2-complex having $2g$ edges $e_i, \overline{e}_i$ each occurring twice in the boundary of a single face $f$. The Cayley complex is then constructed inductively, much as for free groups. We will do the case $g = 2$. Take a family of concentric circles $C_i$ with radii $i \in \mathbb{Z}^{>0}$, and place an octagonal face as shown below left,




with its vertices sitting on the first circle $C_1$. Unlike the octagon in the presentation 2-complex, every vertex and edge on the octagon is *distinct*. We are therefore breaking our usual convention for drawing complexes whereby similarly labelled objects are identified. At the $i$-th step in the construction we have vertices on $C_{i-1}$, each already incident with $i = 2$ or 3 edges. For each such vertex, place $8 - i$ new vertices on $C_i$, joined by new edges to the vertex on $C_{i-1}$. Each consecutive pair of new edges is meant to demarcate an octagonal region, so new vertices need to be placed on $C_i$ as shown below:

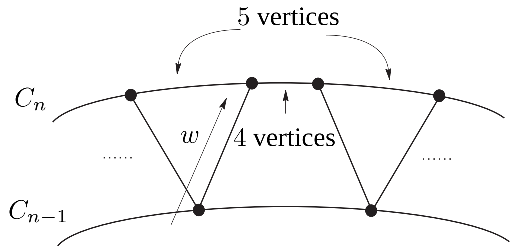



Finally, the edges are labelled according to the scheme illustrated above right (the scheme shows how the edges are arranged around the single vertex in the presentation 2-complex; just carry out the gluings of the edges!)

**Exercise 27**

1. Show that the edges can be labelled in a consistent manner.
2. Show that the mapping sending all the vertices of the resulting complex to the single vertex, edges similarly labelled to those with the same labels and all faces to $f$ is a covering of the presentation 2-complex.
3. Show that the complex constructed has trivial fundamental group, hence is the Cayley complex for the surface group $\Sigma_2$.

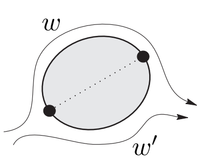
**(9.12)** Putting the Cayley complex to work, suppose that $w$ is a word in the $x_i, y_i$ representing the identity element of $\Sigma_2$, so in particular, a path labelled $w$ starting at the identity vertex closes. Removing all the spurs from $w$ there is a maximal $n$ such that the path contains vertices on $C_n$. Moreover, the path must traverse an edge from $C_{n-1}$ to $C_n$ in order to reach $C_n$, and cannot return immediately to $C_{n-1}$ by going back down this same edge (as this would be a spur). The path thus has no choice but to return to $C_{n-1}$ via one of the other edges, but not before it has passed through a complete set of 4 or 5 vertices on $C_n$:




The conclusion is that $w$ must travel along at least 5 edges that are consecutive on the same face of the Cayley complex (there are 5 if the path turns right in the picture above, otherwise there are 6). Consequently, the word $w$ contains as a subword at least 5 consecutive letters in some cyclic permutation of the relator word $x_1y_1x_1^{-1}y_1^{-1}x_2y_2x_2^{-1}y_2^{-1}$.

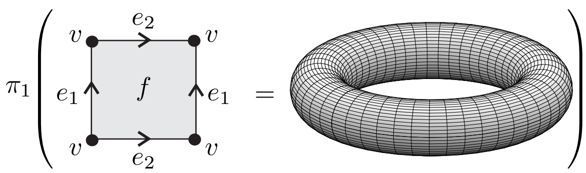
All of which leads to,

**Dehn's Algorithm** *Let $\Sigma_g$ be a surface group and $w$ a word in the generators $x_i, y_i$ with all spurs removed. Replace all subwords of $w$ consisting of $\geq 2g+1$ consecutive letters in some cyclic permutation of the relator word by the $\leq 2g-1$ consecutive letters remaining in the relator. Then $w$ represents the identity element of $\Sigma_g$ if and only if repeated applications of this process result in the empty word.*

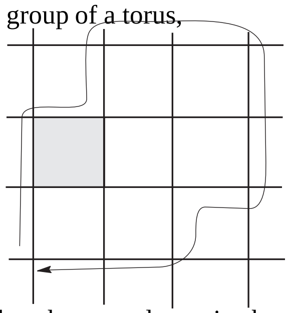



Clearly, if this process results in the empty word then $w$ must represent the identity. On the other hand, if $w$ represents the identity then by the above there must be a subword consisting of $\geq 2g+1$ consecutive letters in some cyclic permutation of the relator, ie: a subpath of the path $w$ consisting of more than half the boundary label of some face in the Cayley complex. Replacing this subword by the shorter part of the boundary label gives a word $w'$ involving fewer generators, but nevertheless, still representing the identity. The process can then be repeated with this shorter word, and must stop eventually with the empty word.

**(9.13)** There is no such algorithm for the presentation for fundamental group of a torus,




$\cong \langle x, y; [x,y] = 1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}.$



where the Cayley complex is shown on the right. The problem is that there are loops in the Cayley complex that do not travel along more than half the boundary of any face.

**(9.14)** Dehn's algorithm provides a very neat and efficient solution to the word problem in surface groups of genus $g \geq 2$. In general, we say that $G = \langle X; R\rangle$ *has a Dehn's algorithm* whenever there is some finite set of words $\{W_1, \ldots, W_n\}$, each representing the identity in $G$, and such that for any other word $w$ representing the identity, there is a $W_i = u_i\overline{u}_i$, with $u_i$ involving more generators than $\overline{u}_i$, and $w = w'u_iw''$.

The algorithm then runs as follows: for any word $w$, remove all occurrences of $x_\alpha x_\alpha^{-1}$ and $x_\alpha^{-1}x_\alpha$ and replace any subword involving more than half of a cyclic permutation of a $W_i$ with the remaining shorter part. Then $w$ represents the identity in $\langle X; R\rangle$ if repeated application of this results in the empty word.

We will see later that it doesn't matter which presentation is used for $G$, they will all have a Dehn's algorithm if any one of them does. Thus we will be able to speak of a group having a Dehn's algorithm.

But these are just details. The real question of course is which groups have a Dehn's algorithm? What is it about the surface groups of genus $\geq 2$ that gives such a nice solution to the word problem? Remarkably, it will turn out not to be an algebraic, but a *geometric* property of these groups that provides the key.

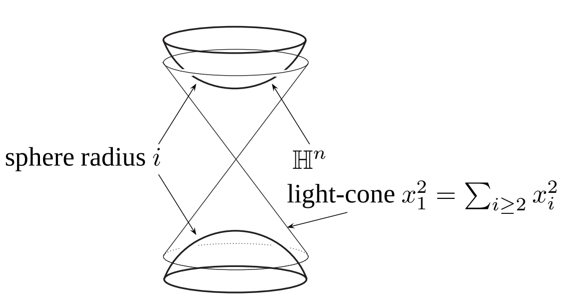
# 10 Hyperbolic Geometry

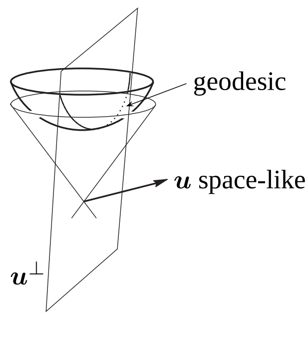
This section is quite sketchy as I imagine the reader is reasonably familiar with hyperbolic space.

**(10.1)** Hyperbolic space of dimension $n$ is dual to the $n$-sphere, both in its construction and properties. One obtains $S^n$ by taking the standard positive definite symmetric form on $\mathbb{R}^{n+1}$ and taking all the vectors of squared norm 1. To construct $\mathbb{H}^n$, consider $\mathbb{R}^{n+1}$ equipped now with the standard Hermitian form of signature $(1, n)$. Thus with respect to the standard basis we have the product of two vectors given by

$$(\boldsymbol{u}, \boldsymbol{v})_L = -u_1v_1 + \sum_{i\geq 2} u_iv_i.$$

The "$L$" stands for Lorentz. The (Lorentz) length of a vector is defined by the usual $||\boldsymbol{u}||_L = (\boldsymbol{u}, \boldsymbol{u})_L^{1/2}$. Because of the negative sign in the first coordinate of the Lorentz product, we can obtain vectors of length zero and vectors whose lengths are pure imaginary numbers. Call the vectors of $\mathbb{R}^{n+1}$ *space-like*, *time-like* and *light-like* if their lengths are real, complex or zero respectively. Hyperbolic space of dimension $n$, like $S^n$, is then a "sphere", only now a sphere of radius $i$ as shown below left:



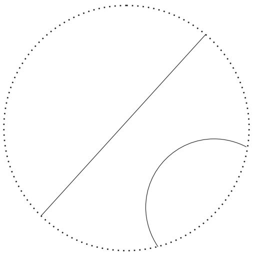



This "sphere" is disconnected, so we need to throw away one of the sheets. Alternatively, "projectivise" the time-like vectors, so that the points of $\mathbb{H}^n$ are the 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$ consisting entirely of time-like vectors.

**Aside 12** The terminology light-like, time-like, and so on, comes about as $\mathbb{R}^{n+1}$ with the Lorentz norm is a model for the Minkowski space-time of special relativity.

Geodesics on the $n$-sphere are the intersection of linear hyperplanes with $S^n$, and so for $\mathbb{H}^n$, where now the result can be parametrised as a curve of the form $(\cosh\theta, \sinh\theta)$. In particular, given a space-like vector $\boldsymbol{u}$, the orthogonal complement $\boldsymbol{u}^\perp$ (where orthogonal is interpreted with respect to the Lorentz product) is a hyperplane intersecting $\mathbb{H}^n$ in a geodesic, and all geodesics arise in this way.

**(10.2)** One can stereographically project $\mathbb{H}^n$ into $\mathbb{R}^n$ and so obtain the *Poincaré ball model* for hyperbolic space.

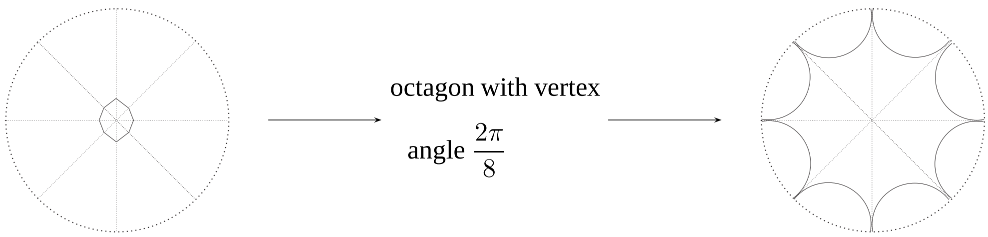


For the rest of this section we restrict for convenience to the 2-dimensional case, although everything we say is true in arbitrary dimensions. The space is now in an open disc of radius 1 (in $\mathbb{C}$ say, rather than $\mathbb{R}^2$), the geodesics being diameters and those portions of circles centered on the boundary. The element of arc length in this model is given by $|ds| = 2|dz|/(1-|z|^2)$, ie: a path $\gamma$ has length

$$||\gamma|| = \int_\gamma \frac{2|dz|}{1-|z|^2}$$

In particular, near the center of the disc ($|z|$ small) the metric in this model is "almost the same" as the Euclidean metric.

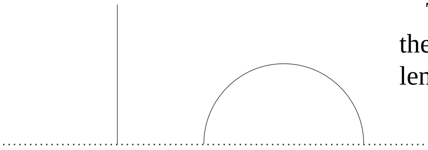
**(10.3)** An application of this last observation that is significant for us is that we can tessellate $\mathbb{H}^2$ by regular octagons with eight meeting at every vertex (and so the vertex angles are all $2\pi/8$). To see why, we first note that a regular Euclidean octagon has vertex angle $6\pi/8$, so as long as we are close enough to $0$, we can find a regular octagon in $\mathbb{H}^2$ with vertex angle $6\pi/8 - \varepsilon$ for any $\varepsilon > 0$. Now pull the vertices towards infinity at constant speed (so that the octagon remains regular at all times). In the limit we have the one below right where now the vertex angles are all zero.




Thus there must be an intermediary octagon with vertex angles $2\pi/8$. The sides of our octagon are all geodesics so we can reflect in these sides once we have suitably defined what we mean by a hyperbolic reflection: as each geodesic side is the intersection with $\mathbb{H}^2$ of a hyperplane $\boldsymbol{u}^\perp$, we take the linear map of $\mathbb{R}^3$ to itself that sends $\boldsymbol{u}$ to $-\boldsymbol{u}$ and fixes pointwise $\boldsymbol{u}^\perp$. When restricted to $\mathbb{H}^2$ this gives a hyperbolic reflection in the geodesic.

Now take the group generated by the eight reflections in the sides of this regular octagon. The images tessellate all of $\mathbb{H}^2$ without overlap in the manner desired.

**(10.4)** The other well known model for $\mathbb{H}^2$ is the upper half plane $\mathbb{C}^{\mathrm{Im}(z)>0}$.

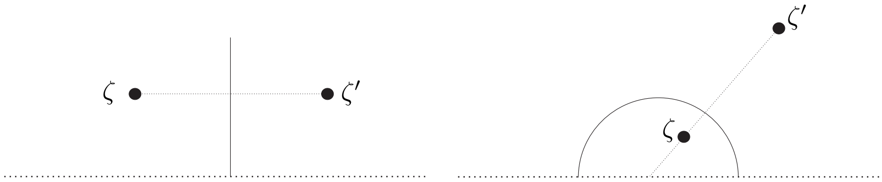


The geodesics are those portions of Euclidean lines perpendicular to the real axis and of circles centered on the real axis. The element of arc length is now $|ds| = |dz|/\text{Im}(z)$ so a parametrised curve $\gamma$ has length

$$||\gamma|| = \int_\gamma \frac{|dz|}{\text{Im}(z)}$$

The metric is thus compressed along the real axis so that it appears infinitely far away.

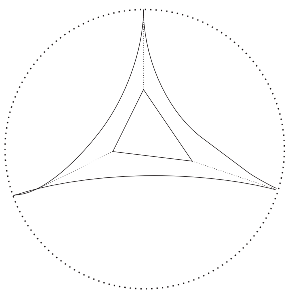
**(10.5)** It will be useful to have some isometries of the upper half plane model at our disposal. In fact, the isometries are generated by (in the sense that every isometry is a finite composition of) two fundamental types:



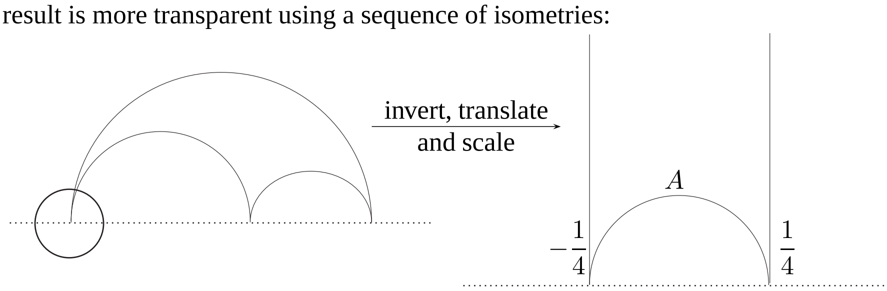

The first is Euclidean reflection in the geodesics of the first kind above. The second is *inversion* in the circles, where for any $\zeta$, its image $\zeta'$ under the inversion satisfies, and is uniquely defined by, $|\zeta||\zeta'| = r^2$, where $r$ is the radius of the circle.

**Exercise 28** By suitably combining Euclidean reflections and inversions, show that the following are isometries of the upper half plane: translation parallel to the real axis and scaling radially by a factor of $k$ from any point on the real axis.

**(10.6)** Juggling the models back and forth (this is a feature of arguments in this area) we can prove the following remarkable property of the hyperbolic plane: given any triangle, no matter how large, each side is contained in a $\ln(1+\sqrt{2})$-neighborhood of the union of the other two. This is to be contrasted strongly with the Euclidean plane, where a triangle can be scaled with one side getting arbitrarily far away from the other two. To see the claim, start with an arbitrary triangle and pull its vertices off to infinity. We show that this property holds for the resulting "ideal" triangle, in which case it will clearly also hold for the original one.



Now switch to the upper half plane model with our ideal triangle as below left, and move it into a position where the result is more transparent using a sequence of isometries:



Invert in the circle shown, which has the effect of sending that vertex of the triangle to $\infty$. Translate parallel to the real axis so the triangle is centered around $0$. Now scale radially by a suitable factor to get the result above right.

A point on the vertical side has distance $\frac{1}{2} < \ln(1+\sqrt{2})$ from the other vertical side, hence contained in a $\ln(1+\sqrt{2})$-neighborhood of the other two sides. This leaves the points on the bottom side. By scrutinizing the metric, one may see that the point $A$ is furthest from the vertical sides of any point on the bottom side. Thus it remains to check the length of a geodesic from $A$ to a vertical side, which we leave as an exercise.

# 11 Geometric Properties of Groups

The geometry of groups is the study of groups as metric spaces. The motivating example is the integers $\mathbb{Z}$, which is obviously a metric space as we know how to measure the distance between integers. Unfortunately, it is not a very interesting metric space. The reals $\mathbb{R}$ on the other hand *are* an interesting space, but in passing from $\mathbb{Z}$ to $\mathbb{R}$ too much of the group structure is lost.

Alternatively, we could study only those metric properties of $\mathbb{Z}$ that are unchanged by a passage to $\mathbb{R}$. Such a theory would need to be insensitive to "compact perturbations" such as squashing out compact sets (which is one way of getting from $\mathbb{R}$ to $\mathbb{Z}$). This is sometimes called "coarse geometry", as the very fine detail of the reals has disappeared to be replaced by the coarseness of the integers.

Another common term is "large scale geometry". As one moves further away from $\mathbb{Z}$ the apparent distance between the points decreases. In the limit, they have coalesced into a continuous whole. In other words, we study those metric properties that are so obvious they can still be seen from infinitely far away.

Gromov's original term was "asymptotic" group theory, and it is worth quoting from his seminal monograph [3],

> "This space may appear boring and uneventful to a geometers eye since it is discrete and the traditional local (eg: topological and infinitesimal) machinery does not run. To regain the geometric perspective one has to change one's position and move the observation point far away. Then the metric seen from the distance $d$ becomes the original distance divided by $d$ and for $d \to \infty$ the points coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy."

**(11.1)** We assume the reader is familiar with the notion of a metric $d$ on a set $X$. A *geodesic* between the points $x, y \in X$ is a continuous mapping $\gamma : [0, c = d(x,y)] \subset \mathbb{R} \to X$ such that $\gamma(0) = x, \gamma(c) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for any $t, t' \in [0, c]$. A metric space is a *geodesic space* iff there is a geodesic (not necessarily unique) between any two points.

**(11.2)** Our first example is the following metric on a group $G$ with generating set $S$. For any $g \in G$, define $d_S(1, g) = n$ iff $g$ can be expressed as a word $g = s_{i_1} \ldots s_{i_n}$ of length $n$ in the generators, but not as a word involving fewer than $n$ occurences of the generators. Define $d_S(g_1, g_2) = d_S(1, g_2 g_1^{-1})$, the *word metric* on $G$ with respect to the generating set $S$. The distance between two elements $g_1, g_2$ of the group is thus the smallest number of generators needed to express an element $h$ such that $hg_1 = g_2$. Notice that by definition, the $G$-action by right multiplication on $(G, d_S)$ is isometric.

**(11.3)** Here is another example, which when looked at from an infinite distance, will turn out to be the same as the previous one. Let $K^1 = K$ be a graph 2-complex. Each edge can be given a metric by replacing it with a copy of $[0, 1] \subset \mathbb{R}$ equipped with the usual metric. Given a path (using finitely many edges) between two points (now not necessarily vertices) define its length to be the sum of the lengths of the edges or partial edges in the path. The distance between two points is the infimum of the lengths of all paths between the two. If the points are just vertices then this is obviously just the minimum number of edges in a path connecting them.

Call this the *graph metric* on $K$.

**(11.4)** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A map (not necessarily continuous) $f : X \to Y$ is a $(\lambda, \varepsilon)$-*quasi isometric embedding* if and only if we have,

$$d_X(x, y) \leq \lambda d_Y(f(x), f(y)) + \varepsilon \text{ and } d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \varepsilon,$$

for some $\lambda \geq 1$ and $\varepsilon \geq 0$. Moreover, the map is a *quasi isometry* iff there is a $C \geq 0$ such that every point of $Y$ is in a $C$-neighborhood of $f(X)$.

**Exercise 29** Show that if a map $f$ allows us to bound the metrics according to $d_X \leq \lambda_1 d_Y + \varepsilon_1$ and $d_Y \leq \lambda_2 d_X + \varepsilon_2$, then this is a $(\lambda, \varepsilon)$-quasi isometry.

**Exercise 30**

1. Show that if there is a $(\lambda, \varepsilon)$-quasi isometry $f : X \to Y$ then there is a $(\lambda', \varepsilon')$-quasi isometry $f' : Y \to X$ and a constant $C \geq 0$ such that $d_Y(ff'(y), y) \leq C$ and $d_X(f'f(x), x) \leq C$.
2. Show that if $X \to Y$ is $(\lambda_1, \varepsilon_1)$-quasi isometry and $Y \to Z$ is $(\lambda_2, \varepsilon_2)$-quasi isometry then the composition is a $(\lambda_1\lambda_2, \lambda_2\varepsilon_1 + \varepsilon_2)$-quasi isometry $X \to Z$.

Thus quasi isometry is an equivalence relation on the class of all metric spaces.

**(11.5)** Our first example of a quasi-isometry is squashing out a compact set (or more generally, a set of finite diameter). Let $C$ be a compact subset of our space $X$ and define an equivalence relation on $X$ by letting $x \sim y$ if and only if either $x$ and $y$ both belong to $C$ or $x = y$. Using the same symbols for an equivalence class in $X/\!\sim$ as for the points of $X$, we define a metric $d'$ on the quotient space by

$$d'(x, y) = \begin{cases} d(x, y), x, y \notin C \\ d(x, C), y \in C \end{cases}$$

The quotient mapping (sending a point in $X$ to its equivalence class modulo $\sim$) is then a $(1, K)$-quasi isometry from $X$ to $X/\!\sim$, where $K$ is the diameter of the compact set $C$ (ie: the maximum distance between points in $C$).

**(11.6)** scaling by a factor of $\lambda$ is a $(\lambda, 0)$-quasi isometry;

**(11.7)** If $(X, d)$ a space and $Y \subset X$ is such that $X$ is contained in a $C$-neighborhood of $Y$ for some $C \geq 0$, then the inclusion map $(Y, d) \to (X, d)$ is a $(1, 0)$-quasi isometry, and so $X$ and $Y$ are quasi isometric. (note that of course the inclusion is an *isometry*, but the spaces are not isometric)

A very typical example of quasi isometric spaces is thus provided by "pixelisation". Let $X$ be a space tessellated by copies of a closed compact set $C$ (ie: the sets cover $X$, and two different copies of $C$ intersect only in their boundaries). Let $Y$ be a collection of points such that there is exactly one in each copy of $C$. Then $Y$ with the restricted metric to it is quasi isometric to $X$.

**(11.8)** Let $G$ be a group with generating set $S$ and $d_S$ the word metric on $G$. Let $K$ be the Cayley graph of $G$ with respect to the generators $S$ and equipped with the graph metric. Paths between vertices in $K$ correspond to words in the generators that multiply the element corresponding to the start vertex to obtain the element corresponding to the finish vertex. Thus the graph metric, restricted to the vertices of $K$ is just the word metric when the vertices are interpreted as elements of the group $G$. As any point on an edge of $K$ is distance $\leq \frac{1}{2}$ from a vertex we have that *the space $(G, d_S)$ and $K$ with the graph metric are quasi isometric*.

**(11.9)** Let $S, S'$ be two generating sets for a group $G$ with $d_S, d_{S'}$ the corresponding word metrics. To bound these two in terms of each other it suffices to consider distances of the form $d(1, g)$ as the metrics are defined in terms of these. Let,

$$\mu = \max_{s \in S}\{d_{S'}(1, s)\}, \text{ and } \mu' = \max_{s' \in S'}\{d_S(1, s')\}.$$

If $g = s_{i_1} \dots s_{i_k}$ a word in the generators $S$ of minimal length, then each $s_{i_j}$ can be written as a product of at most $\mu$ of the generators from $S'$. Thus $d_{S'}(1, g) \leq \mu n = \mu d_S(1, g)$. Similarly, $d_S(1, g) \leq \mu' n = \mu d_{S'}(1, g)$, and so *the spaces $(G, d_S)$ and $(G, d_{S'})$ are quasi isometric*.

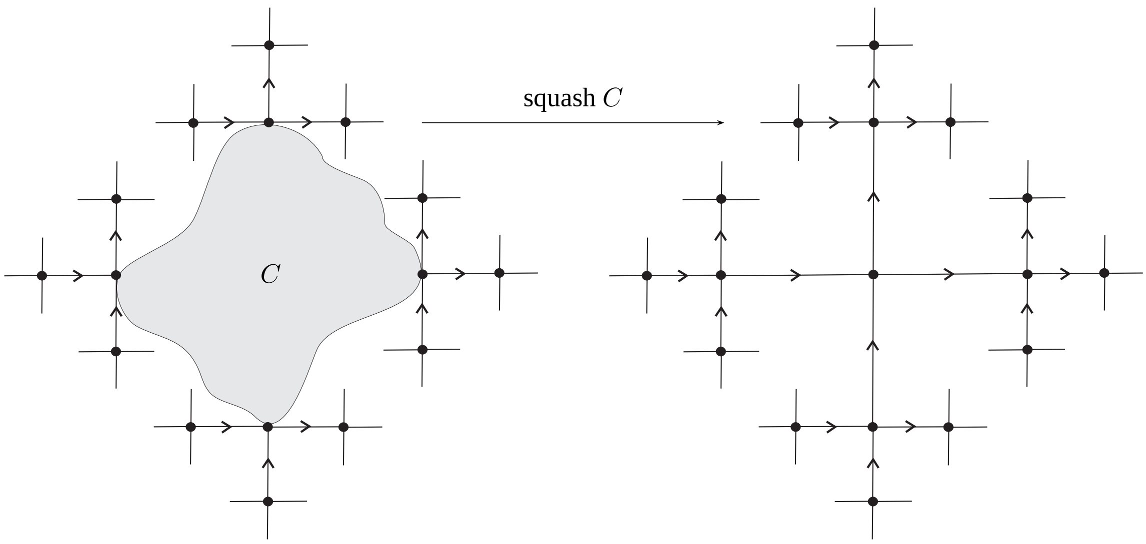
Thus also, *any two Cayley graphs for a group $G$ are quasi isometric*.

**(11.10)** In light of the last two observations, we say that a group $G$ and a space $X$ are quasi isometric if and only if $X$ is isometric to a Cayley graph of $G$ with respect to some set of generators. In particular, two groups are quasi isometric if any two (hence every two) Cayley graphs for the groups are quasi isometric.

**(11.11)** Here is a simple example of an asymptotic result about groups:

**Theorem 11.1** *Let $K(X;-)$ be a presentation 2-complex for a free group of finite rank and $\widetilde{K} \to K$ a covering of infinite degree with the corresponding subgroup finitely generated. Then $\widetilde{K}$ is quasi isometric to the universal cover of $K(X;-)$, where both are equipped with the graph metric.*

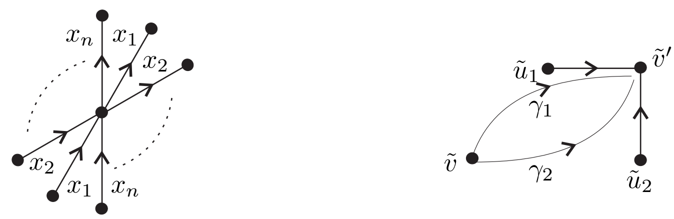
Indeed, if $K_1$ is the universal cover, then it is the Cayley complex of the free group with respect to the free generators $X$, and so is an infinite $2|X|$-valent tree. We show in fact that there is an integer $d$, such that if $C$ is the set of all points distance at most $d$ from the vertex corresponding to the identity in $K_1$, then outside of the set $C$, the graphs $K_1$ and $\widetilde{K}$ are identical:




**Proof:** Let $\tilde{v}$ be the basepoint for $\widetilde{K}$. By the finitely generated condition, and the exercises below, any spanning tree for $\widetilde{K}$ must contain all but finitely many of the edges of $\widetilde{K}$. In particular, we can construct a spanning tree inductively as follows: for every vertex distance $d$ from the basepoint $\tilde{v}$, choose precisely one of the vertices adjacent to the vertex having distance $d-1$ from $\tilde{v}$, and add the connecting edge to the tree.

By the definition of covering, each vertex of $\widetilde{K}$ has an incoming and an outgoing $x_i$-edge for each $x_i \in X$. If one of these edges is a loop or two of them start and end at the same vertex, then a spanning tree for $K^1$ cannot contain the loop, or can contain at most one of the two edges. Thus, this can happen at only finitely many vertices.

Thus, apart from these finitely many, the arrangement of edges around a vertex of $\widetilde{K}$ is as on the left:




with all the vertices shown distinct. Now suppose we have a vertex $\tilde{v}'$ of $\widetilde{K}$ such that there are geodesics $\gamma_1, \gamma_2$ from the basepoint that arrive at $\tilde{v}'$ across different edges from the (distinct) vertices $\tilde{u}_1$ and $\tilde{u}_2$. The two edges therefore cannot both be in the spanning tree constructed above, and therefore, there are only finitely many vertices having this "double geodesic" property. Let $d$ be big enough so that all the vertices with edge arrangements not like the above and all the vertices with this double geodesic property have distance $< d$ from the basepoint, and let $C$ be the collection of points distance $\leq d$ from $\tilde{v}$.

For any vertex outside of $C$, there are $2|X|$ distinct vertices adjacent to it, and any geodesic from $\tilde{v}$ to the vertex must arrive on a fixed one of these edges. This means that precisely one of the $2|X|$ vertices has distance $d-1$ from $\tilde{v}$ and all the others have distance $d+1$. These are hence also outside of the region $C$, and one may continue inductively, getting that the part of $\widetilde{K}$ outside $C$ is identical to that part of $K_1$ outside a ball of radius $d$ centered on the vertex corresponding to the identity. $\square$

**Exercise 31**

1. If $K^1$ is an infinite graph with each vertex incident with finitely many edges, and $T$ is a spanning tree for $K^1$ that contains all but finitely many edges of $K^1$, then *any* spanning tree for $K^1$ contains all but finitely many edges of $K^1$.
2. If $K$ is a 2-complex and $\pi_1(K)$ is finitely generated then there is a finite set of Schreier generators for $\pi_1(K)$.

**Aside 13** On any group there is a topology that encodes the finite index subgroup structure of the group, called the *profinite topology*. It has as a neighborhood basis at the identity the normal subgroups of finite index. In particular, a group $G$ is residually finite (see §7) if and only if the profinite topology on $G$ is Hausdorff. A natural question to ask is whether an arbitrary subgroup $H$ of $G$ is closed in this topology (if $H$ has finite index then this is easily seen to be so). Call $G$ *subgroup separable* or *LERF* iff every *finitely generated* subgroup is closed. Equivalently if $H$ is a finitely generated subgroup and $g \in G \setminus H$ there is a finite group $K$ and a homomorphism $\varphi : G \to K$ such that $\varphi(g) \notin \varphi(H)$.

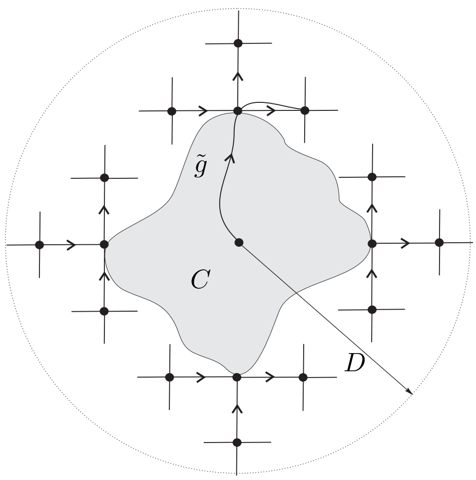
**(11.12)** We can put our asymptotic result to work. The following theorem was originally proved by Marshall Hall in the 1950's and subsequent proofs have been given by Stallings and Margolis.

**Theorem 11.2** *Free groups of finite rank are subgroup separable.*

**Proof:** If $H$ has finite index $n$ in the free group $G = \langle X; - \rangle$ there is a finite covering $\widetilde{K} \to K(X; -)$, where the elements of $H$ correspond to those paths in $K(X; -)$ that lift to loops at the basepoint of $\widetilde{K}$. Thus no word representing $g$ does so. The path lifting action gives a homomorphism $G \to \text{Aut}(\widetilde{K})$ to a finite group, where, by the above, the image of any element of $H$ acts by fixing the basepoint of $\widetilde{K}$, but the image of $g$ moves the basepoint. Thus $g$ and $H$ have different images as required.

The really interesting case is when $H$ has infinite index in $G$. We still have a covering $\widetilde{K} \to K(X; -)$, where now, by Theorem 11.1, we know that $\widetilde{K}$ looks like the infinite $2|X|$-valent tree $T$ outside of a compact set $C$ that contains the basepoint. As before, any word representing $g$ lifts to a non-closed path at the basepoint of $\widetilde{K}$. Let $D$ be large enough so that all the vertices in the region $C$ and all the vertices in this lifted path have distance $< D$ from the basepoint.

Let $T$ be the Cayley complex for the free group and consider the set of edges of the tree $T$ that go from a vertex distance $D$ from $\tilde{v}$ to a vertex distance $D+1$. The following is easily proven by induction: for each generator $x \in X$, there are the same number of $x$-edges in this set going from a distance $D$ vertex to a distance $D+1$ vertex as there are $x$-edges going the other way (ie: from a vertex distance $D+1$ to a vertex distance $D$).



This must then be true for $\widetilde{K}$, as $T$ and $\widetilde{K}$ are identical this far from the basepoint. Form a new *finite* complex $\widetilde{K}'$ as follows: the vertices of $\widetilde{K}'$ are those of $\widetilde{K}$ distance $\leq D$ from the basepoint. any edges of $\widetilde{K}$ inside this ball are also edges of $\widetilde{K}'$. That leaves the edges "on the boundary" of the ball to worry about. But, according to the previous paragraph they can be paired up as, as

We leave it to the reader to verify that $\widetilde{K}' \to K(X;–)$ is a covering, thus we have a homomorphism $G \to \text{Aut}(\widetilde{K}')$ to a finite group. That $g$ and $H$ have different images follows by observing that elements of $H$ act on $\widetilde{K}$ by fixing the basepoint, and as we are only interested in non-trivial ones, these loops can be assumed to stay inside the region $C$ (because of the tree like nature of $\widetilde{K}$ outside this region). Thus they act the same way on $\widetilde{K}'$, but $g$ acts on $\widetilde{K}'$ by sending the basepoint to $\tilde{u}$. $\square$

**(11.13)** A *geometric property* of groups is an invariant of quasi isometry (sometimes called an *asymptotic invariant*), in the sense that if groups $G_1$ and $G_2$ are quasi isometric then $G_1$ has the property if and only if $G_2$ has the property. Here is a (incomplete) list of some geometric properties, with only the first being obvious:

1. Being finite;
2. finitely presented;
3. virtually free;
4. the number of ends;
5. being hyperbolic (see the last section);
6. the $l_1$-homology and $l^\infty$ cohomology;

Obviously there is also a list of *non*-geometric properties, and a list where the status of the items is undecided.

**(11.14)** An isometry of a geodesic space is a map $X \to X$ preserving the distances between points, and a group $G$ acts by isometries on $X$ if we have a homomorphism $G \to \text{Isom}\,(X)$, where the isometries form a group in the obvious way. An action is *proper* if for any point $x \in X$ there is a $r_x > 0$ such that the set,

$$\{g \in G \,|\, B(x, r_x) \cap B(x, r_x)g \neq \emptyset\}$$

is *finite* ($B(x, r_x)$ is the ball center $x$ and radius $r_x$). An action is *cocompact* iff there is a compact subset $C$ whose images cover all of $X$, ie: $X = CG$.

**(11.15)** We can find presentations for groups acting in this way:

**Theorem 11.3 (Poincaré)** *Let $G$ act by isometries on the connected space $X$. If $U$ is an open set such that $X = UG$ then the set,*

$$S = \{g \in G \,|\, Ug \cap U \neq \emptyset\}$$

*generates $G$.*

**Proof:** Let $H$ be the subgroup generated by the elements of $S$, and $V = UH$, $V' = U(G\backslash H)$, both open as isometries are homeomorphisms. Suppose they are not disjoint, so that there are $h \in H$ and $h' \in G\backslash H$ with $Uh \cap Uh' \neq \emptyset$. But then $U \cap U(h'h^{-1}) \neq \emptyset$ so that $h'(h^{-1}) \in S$, ie: $h' \in SH \subset H$, a contradiction. Thus we have $X$ expressed as a disjoint union of open sets with $V$ non-empty, so connectedness gives that $V'$ must be empty, ie: $H = G$. $\square$

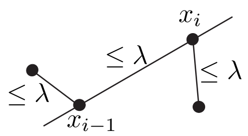
**(11.16)** If we have a group $G$ acting on a set $X$ then we can transfer the geometric properties of $X$ to $G$:

**Theorem 11.4 (Švarc-Milnor)** *Suppose $G$ acts properly cocompactly by isometries on the geodesic space $X$. Then $G$ is quasi-isometric to $X$.*

**Proof:** We have a $C \subset X$ compact with $CG = X$. Choose a basepoint $x_0 \in X$ and a $\lambda$ such that $C \subseteq B(x_0, \lambda)$. Thus every point of $X$ is within $\lambda$ distance of the set $\{x_0\}G$. Let $U = B(x_0, 3\lambda)$ so that the set

$$S = \{g \in G \,|\, U \cap Ug \neq \emptyset\}$$

generates $G$. We show that the map $g \mapsto (x_0)g$ is a quasi isometry from $(G, d_S)$ to the subspace $\{x_0\}G$ of $X$, and hence the result.




Let $g \in G$ and consider a geodesic in $X$ from $x_0$ to $x_0g$. Step off $\lambda$ lengths along the geodesic, with the last one possibly having length $\leq \lambda$. If $x_0, x_1, \ldots, x_n = x_0g$ are the points of the partition, then we have

$$n \leq \frac{1}{\lambda} d_X(x_0, x_0g) + 1.$$

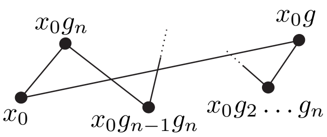
Choose elements $1 = g_0, g_1, \ldots, g_n = g$ of $G$ such that $x_0g_i$ is within distance $\lambda$ of $x_i$. Thus,

$$d_X(x_0g_{i-1}, x_0g_i) \leq 3\lambda,$$

so that $g_ig_{i-1}^{-1}$ moves $x_0$ a distance less than $3\lambda$ away from $x_0$. This gives that $g_ig_{i-1}^{-1}$ is one of our generators in $S$. But,

$$g = (g_ng_{n-1}^{-1})(g_{n-1}g_{n-2}^{-1}) \ldots (g_1g_0^{-1})g_0$$

a product of $n$ generators, so that $d_S(1, g) \leq n$. Thus $d_S(1, g) \leq (1/\lambda)d_X(x_0, x_0g) + 1$.




On the other hand, if $d_S(1, g) = n$ then we have a word $g = g_1 \ldots g_n$ in the generators $S$. Take a geodesic in $X$ from $x_0$ to $x_0g$ as before, and consider the points $x_0g_i \ldots g_n$. Then

$$d_X(x_0, x_0g) \leq \sum d_X(x_0g_{i-1} \ldots g_n, x_0g_i \ldots g_n).$$

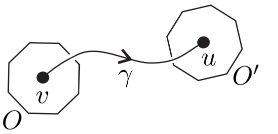
Hitting the segment between $x_0g_{i-1} \ldots g_n$ and $x_0g_i \ldots g_n$ with the isometry $g_n^{-1} \ldots g_{i-1}^{-1}$ gives a segment between $x_0$ and $x_0g_i$. Letting $\mu = \max_{g \in S}\{d_X(x_0, x_0g)\}$, we have

$$d_X(x_0, x_0g) \leq \mu n \leq \mu d_S(1, g).$$

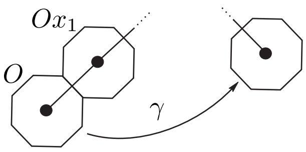
□

**(11.17)** Examples: $\mathbb{Z}$ is quasi isometric to $\mathbb{R}$ via a translation action; in general $\mathbb{Z}^n$ is quasi isometric to $\mathbb{R}^n$;

**(11.18)** We now construct a proper cocompact action by isometries of the surface group $\Sigma_2$ on the hyperbolic plane. Let $\widetilde{K}$ be the Cayley graph for $\Sigma_2$ constructed in §9. In §10 we realised $\widetilde{K}$ in the hyperbolic plane as a tessellation by regular octagons with eight meeting around a vertex. Indeed, if $c$ is the side length of an octagon, then this realisation is a $(c, 0)$-quasi isometry $\widetilde{K} \to$ this tessellation. From now on, when we mention $\widetilde{K}$ we mean this realisation of it in $\mathbb{H}^2$. Construct the dual tessellation $T$: place a new vertex at the center of each octagon, and join two such vertices by a geodesic segment iff they lie in octagons sharing an edge. It is easy to see that $T$ is also a tessellation of the hyperbolic plane by regular octagons with eight meeting at every vertex.



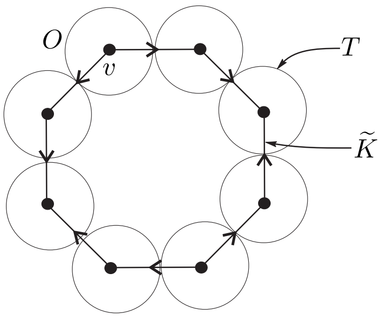

There is a distinguished octagon $O \in T$ containing the basepoint vertex $v$ of $\widetilde{K}$. For any $\gamma \in \Sigma_2$, the path-lifting action of $\Sigma_2$ on $\widetilde{K}$ gives an image vertex $u$ in $\widetilde{K}$. Let $O'$ be the octagon containing $u$ at its center. There is a unique isometry $\gamma'$ of $\mathbb{H}^2$ that sends $O \to O'$ in such a way that the edges of $\widetilde{K}$ meeting at $v$ match up with those meeting at $u$. Define $\Sigma_2 \to \text{Isom}^+\mathbb{H}^2$ by $\gamma \mapsto \gamma'$.




We show that this gives a proper cocompact isometric action of $\Sigma_2$ on the hyperbolic plane. Firstly, if

$$\gamma = x_1 \dots x_k$$

is a word in the generators then the composition $x_1' \dots x_k'$ sends $O$ to $O'$ with the edges matching, so by uniqueness, $\gamma' = x_1' \dots x_k'$. Thus for $\Sigma_2 \to \text{Isom}^+\mathbb{H}^2$ to be a homomorphism, we require that the images $a_1', a_2', b_1', b_2'$ of the generators satisfy the same relations in $\text{Isom}^+\mathbb{H}^2$ that the original $a_1, a_2, b_1, b_2$ generators do in $\Sigma_2$.




This amounts to showing that $\prod_i [a_i', b_i']$ acts as the identity map on the hyperbolic plane. But this word is the boundary label of a face in the Cayley complex $\widetilde{K}$, so by definition it maps the octagon $O$ to itself with the edges matching, so is the identity map. Thus $\Sigma_2 \to \text{Isom}^+\mathbb{H}^2$ is indeed a homomorphism. That the action is by isometries is by definition.

As $\widetilde{K}$ is connected and the octagons tessellate $\mathbb{H}^2$ we have that $O\Sigma_2 = \mathbb{H}^2$. If $O'$ is an octagon with $O \cap O'\gamma$ non-empty for some $\gamma \in \Sigma_2$, then there is an edge labelled $\gamma$ in $\widetilde{K}$ connecting $v$ to the vertex corresponding to $O'$. Thus $\gamma$ must be one of the generators for $\Sigma_2$, of which there are eight. Hence

$$\{\gamma \in \Sigma_2 \,|\, O \cap O'\gamma \neq \emptyset\},$$

is finite giving that the action is proper and cocompact (as $O$ is compact). Thus, we have,

**Proposition 11.1** *The surface group $\Sigma_2$ is quasi isometric to the hyperbolic plane.*

Indeed, any group acting properly cocompactly by isometries on the hyperbolic plane is quasi isometric to the plane, hence any two such groups are quasi isometric to each other.

**(11.19)** Just as a geodesic in the space $(X, d)$ is an isometric mapping of an interval $[0, c]$ into $X$, so a $(\lambda, \varepsilon)$-*quasi geodesic* is a $(\lambda, \varepsilon)$-quasi isometric mapping of $[0, c]$ to $X$.

The primary example is if we have two spaces $X$ and $Y$ and a quasi isometry $X \to Y$. Then clearly any geodesic in $X$ is mapped by the quasi isometry to a quasi geodesic in $Y$.

**(11.20)** One thinks of quasi geodesics as discrete approximations to geodesics. How good an approximation are they? Sometimes not very good: the same argument as above gives a quasi isometry between the group $\mathbb{Z} \oplus \mathbb{Z}$ and the Euclidean plane, so we get quasi geodesics in the plane by taking geodesic paths in the Cayley complex embedded in the plane as a tiling by squares. As can be readily seen, a quasi geodesic can get arbitrarily far away from a geodesic between the same two points. But it turns out that quasi geodesics *do* approximate geodesics well in spaces that "look like" the hyperbolic space of §10. We take this up in our final section.

# 12 Hyperbolic Groups

We saw at the end of §10 that the geometry of the hyperbolic plane has a remarkable property that distinguishes it from the Euclidean plane: there is an absolute constant $\delta = \ln(1 + \sqrt{2})$, such that a side of any triangle (no matter how large) is contained in a $\delta$-neighborhood of the other two sides. Thus, triangles in $\mathbb{H}^2$ are "thin", in that the points on the sides never get too far away from each other.

In this final section we study metric spaces with this property, and in particular, those groups which have a Cayley graph whose triangles are thin.

**(12.1)** Let $X$ be a geodesic metric space and $\delta > 0$ a fixed constant. We say that $X$ is *$\delta$-hyperbolic* iff every geodesic triangle in $X$ has the property that each side is contained in a $\delta$-neighborhood of the union of the other two. Say just *hyperbolic* if it is $\delta$-hyperbolic for some $\delta > 0$.

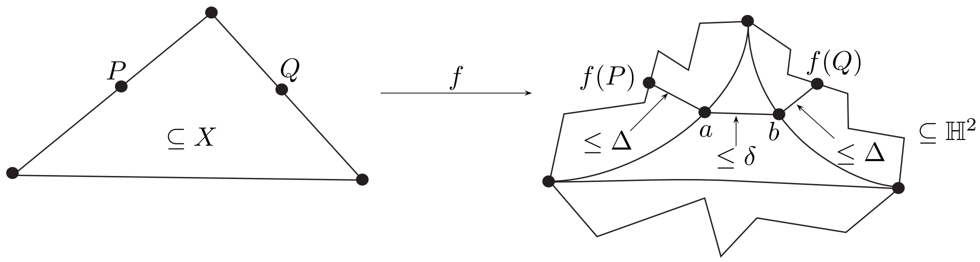
**(12.2)** Before giving some examples, we state,

**Theorem 12.1** *Let $X$ be a hyperbolic space. Then there is an absolute constant $\Delta \geq 0$ such for any two points $x, y \in X$ any quasi geodesic from $x$ to $y$ lies within a $\Delta$-neighborhood of any geodesic from $x$ to $y$.*

**(12.3)** An immediate consequence is

**Proposition 12.1** *If $X$ and $Y$ are geodesic metric spaces with $f : X \to Y$ a $(\lambda, \varepsilon)$ quasi-isometry and $Y$ $\delta$-hyperbolic, then $X$ is $(\lambda(2\Delta + \delta) + \varepsilon)$-hyperbolic, where $\Delta$ is the constant for $Y$ guaranteed by Theorem 12.1.*

**Proof:** Suppose we have a triangle in $X$ and consider a point $P$ on any one of its sides:




There is a $(\lambda, \varepsilon)$-quasi isometry $f : X \to Y$ which sends the geodesic sides of this triangle to quasi geodesics in $Y$ (the squiggly lines in the picture above right). Thus there is an absolute constant $\Delta$ with these quasi geodesics in a $\Delta$-neighborhood of a geodesic between the same two points. Drawing in these geodesics to give a triangle in $Y$, there is a point $a$ on the geodesic nearest $f(P)$ at most $\Delta$ distance from it. As triangles in $Y$ are $\delta$-thin, there is a point $b$ on another geodesic side of the triangle at most $\delta$ distance from $a$. Finally, the quasi geodesic closest contains a point $f(Q)$ at most $\Delta$ distance from $b$. The conclusion is that the point $Q$ back in $X$ mapping to $f(Q)$ satisfies,

$$d_{\widetilde{K}}(P, Q) \leq \lambda(2\Delta + \delta) + \varepsilon,$$

so that triangles in $X$ are $(\lambda(2\Delta+\delta)+\varepsilon)$-thin. Notice that everything on the right hand side is independent of the triangle in $X$ chosen. □

**(12.4)** The Proposition goves us our first non-trivial example of a hyperbolic metric space, namely we show that any Cayley graph for the surface group $\Sigma_2$ has this thin triangles property. At the end of the previous section we showed that the group could be made to act properly cocompactly by isometries on the hyperbolic plane $\mathbb{H}^2$, and so by the **Š**varc-Milnor Theorem, the group $\Sigma_2$ is quasi-isometric to the hyperbolic plane, which is itself $\ln(1 + \sqrt{2})$-hyperbolic. Thus every Cayley graph for $\Sigma_2$, equipped with the word metric, is a hyperbolic space by the Proposition above.

Indeed, any group acting properly cocompactly by isometries on the hyperbolic plane has Cayley graphs that are hyperbolic.

**(12.5)** Motivated by this example, a group is said to be *(δ-)hyperbolic* (or *word hyperbolic* or *hyperbolic in the sense of Gromov*) if some Cayley graph for it is a hyperbolic space with respect to the graph metric. This definition does not depend on which Cayley graph one chooses as we have already seen that any two Cayley graphs are quasi-isometric.

**(12.6)** We content ourselves with just stating some examples: finite groups (obviously, as the Caylay graphs have finite diameter so the thin triangles condition is rather trivially satisfied); free groups (their Cayley graphs are trees so triangles are definitely thin!); hyperbolic lattices, ie: lattices in the real Lie groups $\mathrm{SO}_{1,n}\mathbb{R} \approx \mathrm{Isom}^+\mathbb{H}^n$; in particular, if $M$ is a closed Riemannian manifold with constant sectional curvature $-1$, then $\pi_1(M)$ is hyperbolic.

**(12.7)** Hyperbolic groups have many very nice properties. Given more preparation time, these notes would have gone into some of them! Nevertheless, there is one very nice property that we state without proof as it answers the question we posed at the end of §9.

**Theorem 12.2** *For a hyperbolic group $G$ there is an absolute constant $C \geq 0$ such that any closed path in a Cayley complex for $G$ contains a sub-path of length $\leq C$ which is not a geodesic.*

That the closed path contains a non-geodesic sub-path is not the key bit: it is that this sub-path stays only so long, no matter how long the closed path taken.

This was the crucial property that the Cayley complex $\widetilde{K}$ for $\Sigma_2$ had: any closed path in the graph had to go "the long way" around one of the faces of the complex, hence contained a sub-path of length at most $8$ that was not a geodesic. As a consequence, and for pretty much the same reasons, we have,

**Theorem 12.3** *A group has a Dehn's algorithm if and only if it is hyperbolic.*

**(12.8)** We end with a result that shows, in some sense, that most finitely presented groups are hyperbolic. Consider the set of all presentations $\langle X; R\rangle$ where $|X|, |R|$ and the number of occurrences of the generators in each relator in $R$ is *fixed*. Let $N$ be the number of such presentations and $N_h$ be the number that are hyperbolic groups.

**Theorem 12.4 (Gromov, Ol'Shanskii)** *There is a $c > 0$ such that*

$$\frac{N_h}{N} = 1 - e^{-cn} + o(1),$$

*where $n$ is the length of the shortest relator.*

(here, as usual, $o(1)$ denotes a quantity that tends to $0$ as $n \to \infty$).

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