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Coxeter groups and hyperbolic manifolds

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For my friend and colleague Colin Maclachlan on the occasion of his 65th birthday.

Abstract. The theory of Coxeter groups is used to provide an algebraic construction of finite volume hyperbolic manifolds. Combinatorial properties of finite images of these groups can be used to compute the volumes of the resulting manifolds. Three examples, in 4, 5 and 6-dimensions, are given, each of very small volume, and in one case of smallest possible volume.

1. Introduction

In the last quarter of a century, 3-manifold topology has been revolutionised by Thurston and his school, generating a huge literature on hyperbolic 3-manifolds that builds on the classical 2-dimensional case. Balanced against this is a relative scarcity of techniques and examples of hyperbolic n-manifolds for $n > 3$. Recent work of Ratcliffe and Tschantz has provided examples of smallest possible volume when $n = 4$ by identifying the faces of polytopes ([22], see [9] for a seminal example). On the other hand, in a construction more algebraic in nature, they computed the covolumes of the principal congruence subgroups of level p a prime in $\text{PO}_{1,n}\mathbb{Z}$, giving an infinite family of manifolds, with known volume, in every dimension [23] . In a similar vein, [10] allows one to construct manifolds in arbitrary dimensions by considering the kernels of representations of Coxeter groups onto finite classical groups. However, it turns out that the volumes of the manifolds arising from both constructions are not particularly small (for example in [23], the smallest 6-dimensional manifold has Euler characteristic $|\chi| = 44226$). Thus, there seem to be no constructions in arbitrary dimensions that provide examples, in low dimensions at least, of very small volume.

This paper is an attempt to redress this. A construction is given that essentially makes algebraic the classical geometric idea of identifying the faces of a polytope. Algebraic is meant in the following sense: any finite volume hyperbolic manifold arises as a quotient $\mathbb{H}^n/\Pi$ by the action of a group Π acting properly, freely and cofinitely by isometries on $\mathbb{H}^n$. The fundamental group of the manifold is then

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isomorphic to Π. Moreover, by Mostow rigidity [17, 18, 20], finite volume hyperbolic n-manifolds, for $n \geq 3$, are completely determined by their fundamental groups, in that two such manifolds are isometric precisely when their fundamental groups are isomorphic. Thus, to construct $M = \mathbb{H}^n/\Pi$, we construct Π.

The geometric notions of a free proper cofinite action can be replaced by the algebraic conditions that Π is a torsion free lattice in the Lie group $\text{Isom}\,\mathbb{H}^n \cong \text{PO}_{1,n}\mathbb{R}$. The task can then be spilt into two parts. Start with a Coxeter group Γ embedded as a lattice in $\text{Isom}\,\mathbb{H}^n$. Fortunately, there already exists an extensive literature on such hyperbolic Coxeter groups (see [29] and the references there). By Selbergs lemma [24], such Γ are virtually torsion free (as indeed is any abstract Coxeter group), so the second step is to find the desired torsion free Π as a subgroup of finite index in Γ. The machinery needed to control this second step occupies most of this paper–it is combinatorial in nature and uses only the algebraic characterisitcs of Γ. Basic properties of the resulting manifolds, such as volume and orientability, can then be determined from the structure of Γ.

The paper is organised as follows: §2 contains preliminary material about Coxeter groups; §3 describes the construction and the important role played by the conjugacy classes of finite Coxeter groups; these classes are determined in §4, where the results of Roger Carter's lengthy investigation [5] are summarized in a form amenable to our purposes; computing volumes is achieved via the (standard) techniques of §5, and §6 contains three examples, one in each of 4, 5 and 6-dimensions. These three are either the smallest possible ($n = 4$) or would appear to be the smallest[1] known ($n = 5, 6$).

2. Coxeter groups

Let Γ be a group and $S \subset \Gamma$ finite. The pair (Γ, S) is called a Coxeter system (and Γ a Coxeter group) if Γ admits a presentation with generators $s_\alpha \in S$, and relations,

$$(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1,$$

where $m_{\alpha\beta} \in \mathbb{Z}^+ \cup \{\infty\}$ and $m_{\alpha\alpha} = 1$ (see [1, 15] for basic facts about Coxeter groups). It is customary to omit relations for which $m_{\alpha\beta} = \infty$. Associated to any Coxeter group is its symbol, with nodes indexed by the s_α, where nodes s_α and s_β are joined by an edge labelled $m_{\alpha\beta}$ if $m_{\alpha\beta} \geq 4$, an unlabelled edge if $m_{\alpha\beta} = 3$ and a dotted edge if $m_{\alpha\beta} = \infty$. It is a standard abuse of notation to denote a Coxeter group and its symbol by the same letter.

Coxeter groups satisfy a Freiheitssatz, in the sense that for any $S' \subseteq S$, the subgroup of Γ generated by the $s_\alpha \in S'$ is also a Coxeter group with symbol

[1] After this paper was written, John Ratcliffe and Steven Tschantz constructed a 5-dimensional manifold with $\frac{1}{8}$ the volume of our example [21].

obtained from Γ by removing those nodes not in S' and their incident edges. In general, a subsymbol of Γ is some subset of the vertices and their incident edges. If a symbol Γ is disconnected with connected components $\Gamma_1, \dots, \Gamma_k$, then Γ is isomorphic to $\Gamma_1 \times \cdots \times \Gamma_k$. A group with connected symbol is called irreducible.

It should come as no surprise that in finding torsion free subgroups of Coxeter groups the finite, or spherical, Coxeter groups play a central role. The finite irreducible Coxeter groups are well known to be: the Weyl groups of simple Lie algebras over $\mathbb{C}$, the dihedral groups, the group of symmetries of a regular dodecahedron, and the group of symmetries of the regular 4-dimensional polytope, the 120-cell. The Killing-Cartan notation will be used throughout this paper, in which the Weyl groups fall into three infinite classical families,

$$A_n (n \geq 1),\ B_n (n \geq 2) \text{ and } D_n (n \geq 4),$$

as well as five exceptional groups, G_2, F_4, E_6, E_7 and E_8. The non-Weyl groups are the dihedral $I_2(m)$ for $m \geq 2$, and the dodecahedron and 120-cell symmetry groups H_3 and H_4 (note that there are the exceptional isomorphisms $A_1 \times A_1 \cong I_2(2)$, $A_2 \cong I_2(3)$, $B_2 \cong I_2(4)$ and $G_2 \cong I_2(6)$). The symbols for the Weyl groups are in Tables 1-2, and the non-Weyl groups in Proposition 5 and Table 3. Given an arbitrary (not necessarily connected) Coxeter symbol, it represents a finite Coxeter group if and only if each of its connected components is in the list above.

A smaller role is reserved for the parabolic Coxeter (or affine Weyl) groups: the irreducible ones are $\widetilde{A}_n (n \geq 1)$, $\widetilde{B}_n (n \geq 3)$, $\widetilde{C}_n (n \geq 2)$, $\widetilde{D}_n (n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, and $\widetilde{G}_2$. For their symbols, consult [15, §2.5]. In general, a Coxeter group is parabolic (or Euclidean) if and only if its symbol has connected components corresponding to the groups in this list.

A convex polytope P in $\mathbb{H}^n$ is an intersection,

$$P = \bigcap_{s \in S} H_s^-,$$

of closed half spaces H_s^- bounded by hyperplanes H_s. We will assume that S is finite, and to avoid degenerate cases, that P contains a non-empty open subset of $\mathbb{H}^n$. A Coxeter system (Γ, S) is *hyperbolic* of dimension n if and only if there is a $P \subset \mathbb{H}^n$ such that assigning $s_\alpha \in S$ to the reflection in the hyperplane H_{s_α} induces an isomorphism between Γ and the group generated by reflections in the faces of P. Call Γ cocompact (resp. cofinite) if P is compact (resp. of finite volume) in $\mathbb{H}^n$.

Given an abstract Coxeter system (Γ, S), when can one realise it as a cocompact/cofinite hyperbolic Coxeter group? Dealing first with hyperbolicity, a Gram matrix $G(\Gamma)$ for Γ has rows and columns indexed by S: if $m_{\alpha\beta} \in \mathbb{Z}^+$, set the $\alpha\beta$-th and $\beta\alpha$-th entries of $G(\Gamma)$ to be $-\cos(\pi/m_{\alpha\beta})$, and if $m_{\alpha\beta} = \infty$, choose the $\alpha\beta$-th and $\beta\alpha$-th entries to be some real $c_{\alpha\beta}$ with $c_{\alpha\beta} \leq -1$. Recalling that a symmetric matrix has signature (p, q) if it has precisely p eigenvalues that are < 0 and q that are > 0,

Theorem 1 ([29], Theorem 2.1). *A Coxeter group is n-dimensional hyperbolic if there exist $c_{\alpha\beta}$'s such that $G(\Gamma)$ has signature $(1, n)$.*

Note that in [15, §6.8–6.9], the condition of Theorem 1, together with the condition that the Gram matrix be non-degenerate, is taken as the *definition* of hyperbolic Coxeter group, and so a more restricted class of groups is obtained, and indeed, classified. A classification in the general case remains a difficult problem.

The configuration of the bounding hyperplanes of P can be described as follows. If $m_{\alpha\beta} \in \mathbb{Z}^+$ then the corresponding hyperplanes intersect with dihedral angle $\pi/m_{\alpha\beta}$; if $c_{\alpha\beta} = -1$ they are parallel, that is, intersect at infinity; and if $c_{\alpha\beta} < -1$ they are ultraparallel (they do not intersect in the closure of $\mathbb{H}^n$) with a common perpendicular geodesic of length $\eta_{\alpha\beta}$, where $-\cosh \eta_{\alpha\beta} = c_{\alpha\beta}$. Because of these last two, some authors prefer to embellish the Coxeter symbol, replacing the dotted edges by thick solid ones if $c_{\alpha\beta} = -1$, or by ones labelled $\eta_{\alpha\beta}$ when $c_{\alpha\beta} < -1$.

To determine if (Γ, S) is realised by a cocompact/cofinite $P \subset \mathbb{H}^n$, let $\mathscr{F} = \mathscr{F}(\Gamma, S)$ be the collection of finite subgroups of Γ generated by subsets $S' \subset S$ (including $S' = \emptyset$, which generates the trivial group). Partially order $\mathscr{F}$ by inclusion. Similarly $\overline{\mathscr{F}}$ is the poset obtained by taking both the finite and the parabolic subgroups. The poset of an abstract combinatorial polytope P is the set of cells of P, partially ordered by inclusion.

Theorem 2 ([29], Proposition 4.2). *An n-dimensional hyperbolic Coxeter group (Γ, S) is cocompact (resp. cofinite) iff $\mathscr{F}$ (resp. $\overline{\mathscr{F}}$) is isomorphic as a partially ordered set to the poset of some n-dimensional abstract combinatorial polytope.*

A slight variation yields a very useful Γ-complex. Let $\Gamma\mathscr{F} = \Gamma\mathscr{F}(\Gamma, S)$ be the set of all right cosets in Γ of the subgroups in $\mathscr{F}$. Partially order this set by inclusion and let Σ be the affine complex realising $\Gamma\mathscr{F}$: the k-cells of Σ are chains $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ with the $\sigma_i \in \Gamma\mathscr{F}$. The group Γ acts cellularly on Σ by right multiplication. If Γ is hyperbolic then Σ can be identified with the barycentric subdivision of the tessellation of $\mathbb{H}^n$ by congruent copies of the polytope P. But in fact for any Coxeter system, the complex turns out to be negatively curved:

Theorem 3 ([19], see [8] Corollary 6.7.5). *The complex Σ can be equipped with a piecewise Euclidean metric, where each k-cell is isometric to a regular k-dimensional Euclidean simplex, and such that the resulting space is CAT(0) with the Γ-action by right multiplication isometric.*

3. Torsion free subgroups of Coxeter groups

The construction of torsion free subgroups of a Coxeter group requires answers to the following two questions: where exactly is the torsion, and given a subgroup, when does it avoid it?

Theorem 4. *Let* (Γ, S) *be a Coxeter system. Then any element of finite order in* Γ *is conjugate to an element of a finite subgroup generated by some* $S' \subseteq S$.

This is exercise V.4.2 in Bourbaki [1], and an algebraic proof using root systems appears in [4]. We provide a geometrical proof that is presumably well known, but does not appear to be in the literature:

Proof. Consider the isometric Γ-action by right multiplication on the complex Σ above. Any torsion element γ in a group of isometries of a CAT(0) space has a fixed point [2, Corollary II.2.8], and since the Γ-action is cellular, γ must fix a k-cell $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ with $\sigma_i = \langle S_i \rangle g$ for $g \in \Gamma$ and $S_i \subset S$ with $\langle S_i \rangle$ finite. Thus, $g\gamma g^{-1}$ fixes the k-cell $\langle S_0 \rangle < \cdots < \langle S_k \rangle$, giving $g\gamma g^{-1} \in \langle S_0 \rangle$. □

To address the second question above, it will turn out to be convenient to use the language of permutation modules rather than subgroups. Let Ω be a finite set and U a vector space over $\mathbb{C}$ with basis Ω. An action of any group Γ on Ω can be extended to a linear action on U, thus giving U the structure of a (permutation) $\mathbb{C}\Gamma$-module. Call the module transitive if it arises via a transitive Γ-action on Ω.

Conjugacy classes of subgroups Π of index $|\Omega|$ in a group Γ are in one to one correspondence with equivalence classes of transitive representations $\Gamma \to \mathrm{Sym}\Omega$. These in turn are in one to one correspondence with isomorphism classes of transitive permutation $\mathbb{C}\Gamma$-modules. From now on we will just say $\mathbb{C}\Gamma$-module in the understanding that all modules are permutation modules.

Let $\mathscr{C}$ be a conjugacy class of torsion elements in Γ. We will say that the $\mathbb{C}\Gamma$-module U *avoids* $\mathscr{C}$ when the corresponding conjugacy class of subgroups has empty intersection with $\mathscr{C}$; U is *torsion free* when it avoids the conjugacy classes of all torsion elements. The following is well known and easily proved:

Proposition 1. *Let* U *be a* $\mathbb{C}\Gamma$*-module for some group* Γ *with basis* Ω. *Then* U *is torsion free exactly when it avoids the conjugacy classes of elements of prime order. Moreover, if* $\mathscr{C}$ *is such a class, and* $\gamma \in \mathscr{C}$ *any element, then* U *avoids* $\mathscr{C}$ *exactly when* γ *fixes no point of* Ω.

So, if we had a list of representatives for the conjugacy classes of prime order torsion in Γ, verifying that a given U is torsion free would become the simple matter of checking that no element in the list fixed a point of Ω. By Theorem 4, such a list could be compiled in particular for (Γ, S) a Coxeter group by listing representatives of the conjugacy classes in the finite subgroups generated by $S' \subseteq S$. Indeed, we can restrict to the finite subgroups generated by S' that are maximal with the property that they generate a finite subgroup. The list of torsion elements so obtained will very probably include redundancies, but will certainly be complete, which is clearly all that matters for the task at hand. A discussion of conjugacy in finite Coxeter groups is the subject of §4.

Rather than dealing with an indigestible whole, we can divide the task of finding a torsion free U into manageable pieces. Given $\mathbb{C}\Gamma$-modules $U_i, i = 1, 2$, with bases Ω_i, let Γ act on $\Omega_1 \times \Omega_2$ via $(\mathbf{u}, \mathbf{v})^\gamma = (\mathbf{u}^\gamma, \mathbf{v}^\gamma)$. Extending this action linearly to the complex vector space with basis $\Omega_1 \times \Omega_2$ gives a permutation $\mathbb{C}\Gamma$-module that may be identified with $U_1 \otimes U_2$. The problem is that $U_1 \otimes U_2$ may not be transitive. By an abuse of notation, we will use $U_1 \otimes U_2$ to denote the $\mathbb{C}\Gamma$-module with basis some designated orbit of Γ on $\Omega_1 \times \Omega_2$ (in the examples in §6, $U_1 \otimes U_2$ will always transitive).

The following follows by definition and Proposition 1

Lemma 1. *Let $U_i, i = 1, 2$, be $\mathbb{C}\Gamma$-modules and $\mathscr{C}$ a conjugacy class of torsion elements in the group Γ. Then $U_1 \otimes U_2$ avoids $\mathscr{C}$ if and only if at least one of the U_i does.*

Thus we may build a torsion free $\mathbb{C}\Gamma$-module U by finding $U_1, \dots, U_k$ such that each conjugacy class of torsion elements is avoided by at least one of the U_i, and then letting $U = \otimes U_i$.

In order to determine the volume of the resulting manifold we will need to be able to show that $\otimes U_i$ is a transitive module. The remaining results in this section give us the tools to do this, with the first one just an elementary result in group theory rewritten in our language,

Lemma 2. *Let $U_i, i = 1, 2$ be transitive $\mathbb{C}\Gamma$-modules with bases Ω_i, and Ω the designated orbit in $\Omega_1 \times \Omega_2$ yielding the $\mathbb{C}\Gamma$-module $U_1 \otimes U_2$. Then lcm$|\Omega_i|$ divides $|\Omega|$, and $|\Omega| \leq \prod |\Omega_i|$, with equality if the $|\Omega_i|$ are relatively prime.*

Proof. Intersect Ω with the sets $\{u\} \times \Omega_2$ as u ranges over Ω_1. As the Γ-action on Ω_1 is transitive there are $|\Omega_1|$ such sets, each of the same size, thus $|\Omega_1|$ divides $|\Omega|$, and similarly for Ω_2, giving that lcm$|\Omega_i|$ divides $|\Omega|$. The upper bound is immediate, as is the final statement, for under the conditions given, lcm$|\Omega_i| = \prod |\Omega_i|$. □

Recall that a Γ-action on Ω is imprimitive if $\Omega = \bigcup \Omega_i$, a disjoint union of blocks with $1 < |\Omega_i| < |\Omega|$, and $\Omega_i^\gamma = \Omega_j$ for all i and $\gamma \in \Gamma$ (ie: the Γ-action on Ω induces a Γ-action on the blocks). Call a module U imprimitive if the Γ-action on its basis is.

Lemma 3. *Let U be an imprimitive $\mathbb{C}\Gamma$-module with basis Ω and $\overline{\Omega}$ the set of blocks of imprimitivity with the corresponding $\mathbb{C}\Gamma$-module $\overline{U}$ transitive (and hence each block having the same size μ). Let F be a finite subgroup of Γ with a fixed point in $\overline{\Omega}$ but no non-identity element fixing a point in Ω. Then $|F|$ divides μ.*

Proof. If B is the block corresponding to the fixed point in $\overline{\Omega}$ of F, then it is a disjoint union of F-orbits, each of which has the form $\{\gamma(x) \mid \gamma \in F\}$ for some $x \in B$. The condition on the Γ-action on Ω gives that each F-orbit has size $|F|$, which thus divides the size of B. □

Notice that we can draw the same conclusion if we merge Ω and $\overline{\Omega}$: if U is transitive and F acts without fixed points on Ω then $|F|$ divides $|\Omega|$. Consequently, if $\mathscr{L}(\Gamma)$ is the lowest common multiple of the orders of the finite subgroups of Γ, then $\mathscr{L}(\Gamma)$ divides the size of the basis for any torsion free module U. In §6 we will use Lemma 3 in the following special case,

Proposition 2. *Let $U_i, i = 1, 2$ be transitive $\mathbb{C}\Gamma$-modules with bases Ω_i, F a finite subgroup of Γ with a fixed point in Ω_1 but no non-identity element fixing a point in Ω_2. If Ω is an orbit in $\Omega_1 \times \Omega_2$ arising from the $\mathbb{C}\Gamma$-module $U_1 \otimes U_2$, then $|\Omega_1||F|$ divides $|\Omega|$.*

Proof. The Γ-action on Ω is imprimitive with blocks of imprimitivity the intersections with Ω of the sets $\{u\} \times \Omega_2$ as u ranges over Ω_1. Letting $\overline{\Omega}$ be the set of blocks, the Γ-action on $\overline{\Omega}$ is transitive as the action on Ω_1 is. If F fixes $u_0 \in \Omega_1$ then it fixes the block $\Omega \cap \{u_0\} \times \Omega_2$ in $\overline{\Omega}$, whereas if an element of F fixes a point of Ω then restricting to the second coordinate gives a fixed point in Ω_2. Thus by Lemma 3, $|F|$ divides the size of each block in $\overline{\Omega}$ of which there are $|\Omega_1|$. □

Finally, U is *orientable* whenever there is some $\mathbf{v} \in \Omega$ that is not fixed by any word w involving an odd number of the generators S (if U is transitive, then *no* vertex will be fixed by such a word as w involves an odd number of generators precisely when $w_1 w w_1^{-1}$ does for any w_1).

Proposition 3. *1. If U_1 is orientable then so is $U_1 \otimes U_2$.*
2. If U is transitive and torsion free, with Π the (conjugacy class of the) corresponding subgroup, then the manifold $\mathbb{H}^n/\Pi$ is orientable if and only if U is orientable.

Proof. The first part is clear and the second as $\mathbb{H}^n/\Pi$ is orientable precisely when Π acts orientation preservingly on $\mathbb{H}^n$, which is to say, Π is entirely composed of elements expressible as words involving an even number of the generators (it is an elementary property of Coxeter groups that the parity of any word is an invariant of the corresponding element). □

4. Conjugacy in finite Coxeter groups

We saw in §3 that a list of representatives for the conjugacy classes of torsion elements in (Γ, S) could be obtained by listing the conjugacy class representatives of the finite subgroups generated by maximal $S' \subseteq S$. If any such finite subgroup is reducible, then its conjugacy classes are easily obtained from those of its irreducible components. Thus, we need only consider conjugacy in irreducible finite Coxeter groups.

There are then two possibilities to consider: the Weyl groups and the non-Weyl groups, and these are dealt with separately. Many of the Weyl groups have alternative descriptions in terms of well-known finite groups. For example, those of type A_n are isomorphic to the symmetric groups $\mathfrak{S}_{n+1}$; those of type B_n arise as the split extension

$$1 \longrightarrow \bigoplus^{n} \mathbb{Z}_2 \longrightarrow B_n \longrightarrow \mathfrak{S}_n \longrightarrow 1,$$

where for $\sigma \in \mathfrak{S}_n$ and x_k the $\mathbb{Z}_2$-tuple with 1 in the k-th position and zeroes elsewhere, we have $\sigma^{-1} x_k \sigma = x_{\sigma(k)}$ (the group of so-called signed permutations). Similarly for those of type D_n but with $\mathbb{Z}_2^n$ replaced by $\mathbb{Z}_2^{n-1}$ (the even signed permutations). Thus their conjugacy classes can be determined on a case by case basis. Alternatively, [5] determines them conceptually using their structure as reflection groups, and it is the results from here that we use.

4.1. The Weyl groups

The description is couched in terms of root systems (see [15, Chapters 1 and 2]). Let V be a Euclidean space with positive definite bilinear form $\langle\ ,\ \rangle$ and orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. For any $\mathbf{v} \in V$, let $s_{\mathbf{v}}$ be the linear reflection in the hyperplane orthogonal to $\mathbf{v}$. Tables 1–2 give the standard representations of the Weyl groups acting on V: the subgroup of the orthogonal group $O(V)$ generated by the $s_{\mathbf{v}}$ where the $\mathbf{v}$ label the nodes of the symbol, is isomorphic to the abstract Coxeter group having that symbol. These $\mathbf{v}$ are then a set of simple roots.

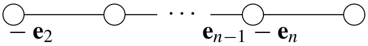
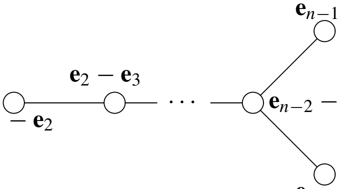
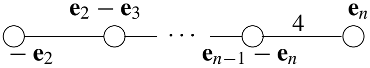
If (Γ, S) is an irreducible Weyl group, identify its symbol with the appropriate one in Tables 1–2, thus identifying S with the $s_{\mathbf{v}}$. It turns out [5] that the conjugacy classes in Γ are parametrised by certain diagrams, closely related to the Coxeter symbol. We now proceed to give the diagrams, and, with §3 in mind, a method for obtaining a representative for the corresponding conjugacy class in terms of the generating reflections for Γ.

The diagrams for each group are given in Theorems 5-8 below. For each one, label the nodes (if they do not come prelabelled) by roots from the appropriate root system in Tables 1–2, in such a way that if the nodes labelled $\mathbf{u}$ and $\mathbf{v}$ are connected by an edge labelled m, then

$$\frac{4\langle \mathbf{v}, \mathbf{u}\rangle\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{u}, \mathbf{u}\rangle} = m - 2.$$

In fact, we always have $m = 2$ or 3, and the diagrams use the same labelling conventions as for Coxeter symbols. If the diagram can be identified with a subsymbol of the Coxeter symbol, then a labelling is easily obtained–take a labelling

Table 1. Root systems for the classical Weyl groups [15, §2.10]

Type and order	Root system	Coxeter symbol and simple system
A_n $(n+1)!$	$\{\mathbf{e}_i - \mathbf{e}_j \ (1 \le i \ne j \le n+1)\}$	
D_n $2^{n-1}n!$	$\{\pm\mathbf{e}_i \pm \mathbf{e}_j \ (1 \le i < j \le n)\}$	
B_n $2^n n!$	$\{\pm\mathbf{e}_i \ (1 \le i \le n),$ $\pm\mathbf{e}_i \pm \mathbf{e}_j \ (1 \le i < j \le n)\}$	

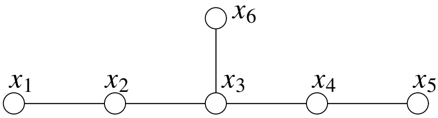
by simple roots off the Coxeter symbol. Otherwise, it can be harder, as in the example below.

Colour the nodes of the labelled diagram black and white with a node of one colour joined only to nodes of the other. Let $\mathscr{B}$ and $\mathscr{W}$ be the sets of nodes of the two colours. Then a representative of the conjugacy class corresponding to the diagram is given by,

$$\prod_{\mathbf{v}\in\mathscr{B}} s_{\mathbf{v}} \prod_{\mathbf{u}\in\mathscr{W}} s_{\mathbf{u}}.$$

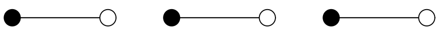
We use the algebraists convention of reading such expressions from left to right. If $\mathbf{v}$ isn't a simple root, then $s_{\mathbf{v}}$ is not identified with one of the generating reflections of Γ. In this case find a simple $\mathbf{v}'$ and a word $w \in \Gamma$ in the generators, such that the image of $\mathbf{v}$ under w is $\mathbf{v}'$. Then $s_{\mathbf{v}} = w s_{\mathbf{v}'} w^{-1}$, an expression in terms of the generating reflections.

For example, suppose we have a group of type E_6 with generators,



Consult the root system for E_6 in Table 2 and make the identifications $x_1 = s_{\mathbf{e}_5-\mathbf{e}_4}$, $x_2 = s_{\mathbf{e}_4-\mathbf{e}_3}$, $x_3 = s_{\mathbf{e}_3-\mathbf{e}_2}$, $x_4 = s_{\mathbf{e}_2-\mathbf{e}_1}$, $x_5 = s_{\mathbf{v}'}$ and $x_6 = s_{\mathbf{e}_1+\mathbf{e}_2}$ where $\mathbf{v}' = \mathbf{e}_1 + \mathbf{e}_8 - \frac{1}{2}\sum_{i=1}^{8} \mathbf{e}_i$.

By Theorem 7, E_6 has a conjugacy class of elements of order three corresponding to the diagram $\coprod_{i=1}^{3}$ ○——○. A labelling and a colouring is given by,



There can be no labelling entirely by simple roots as the diagram is not a subsymbol of the Coxeter symbol for E_6. Thus, the roots labelling the nodes of the coloured diagram correspond, from left to right, to the reflections x_5, x_4, x_2, x_1, x_6 and $s_{\mathbf{v}}$ for $\mathbf{v} = -\mathbf{e}_6 - \mathbf{e}_7 + \frac{1}{2}\sum_{i=1}^{8} \mathbf{e}_i$. This last one is not a simple root, but if $w = x_6x_3x_2x_1x_4x_3x_2x_6x_3x_4$, then in the action of Γ on V, $\mathbf{v}$ is sent by w to $\mathbf{v}'$ above. When performing these calculations it is helpful to remember that the reflection $s_{\mathbf{e}_i-\mathbf{e}_j}$ permutes the basis vectors by transposing $\mathbf{e}_i$ and $\mathbf{e}_j$ and fixing all others. Thus $s_{\mathbf{v}} = ws_{\mathbf{v}'}w^{-1} = wx_5w^{-1}$ and a representative of the conjugacy class corresponding to this diagram is

$$x_5x_2x_6x_4x_1wx_5w^{-1}.$$

In [5] a product of this form is read from right to left, so the inverse of "our" element is obtained (the $s_{\mathbf{v}}$ are involutions). But all the diagrams below have the form

$$\overset{\mathbf{v}_0}{\bigcirc}\!\!-\!\!\!-\!\!\!-\!\!\overset{\mathbf{v}_1}{\bigcirc}\cdots\overset{\mathbf{v}_{k-1}}{\bigcirc}\!\!-\!\!\!-\!\!\!-\!\!\overset{\mathbf{v}_k}{\bigcirc},$$

so that the subgroup generated by the $s_{\mathbf{v}_i}$ is isomorphic to $\mathfrak{S}_{k+1}$, and the two elements obtained are conjugate in Γ anyway. Alternatively, although this is far less trivial, any element of a Weyl group is conjugate to its inverse [5, Corollary to Theorem C].

Theorem 5 (types A and B). *The conjugacy classes of prime order in the Weyl groups of types A_n and B_n correspond to the diagrams:*

A_n. *order* $p \geq 2 : \coprod^{k} \underbrace{\bigcirc\!\!-\!\!\!-\!\!\bigcirc\cdots\bigcirc\!\!-\!\!\!-\!\!\bigcirc}_{p-1}$ *for all* $k \geq 1$ *with* $kp \leq n+1$.
B_n.

$$\textit{order } 2 : \coprod^{k} \overset{\mathbf{e}_i-\mathbf{e}_{i+1}}{\bigcirc} \quad \coprod^{m} \overset{\mathbf{e}_j}{\bigcirc} \quad \textit{and order } p \geq 3 : \coprod^{k} \overbrace{\bigcirc\!\!-\!\!\!-\!\!\bigcirc\cdots\bigcirc\!\!-\!\!\!-\!\!\bigcirc}^{p-1},$$

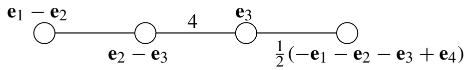
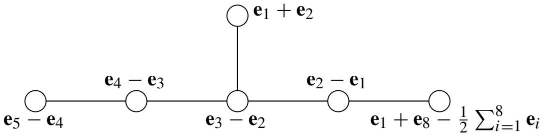
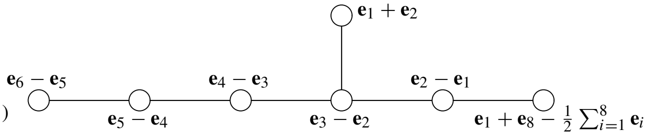
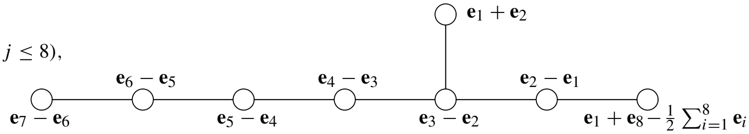
where the order 2 *diagrams are for all* $k+m > 0$ *with* $2k+m \leq n$; *the order* $p \geq 3$ *diagrams are for all* $k \geq 1$ *with* $kp \leq n$.

This result summarises Propositions 23 and 24 of [5]. Note that all the unlabelled graphs above can be obtained as subsymbols of the Coxeter symbols for A_n and B_n.

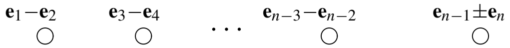
Theorem 6 (type D). *The conjugacy classes with prime order in D_n correspond to the diagrams: (a) order* 2 :

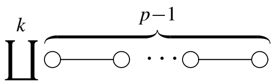
$$\coprod^{k} \overset{\mathbf{e}_i-\mathbf{e}_{i+1}}{\bigcirc} \quad \coprod^{m} \overset{\mathbf{e}_j-\mathbf{e}_{j+1}}{\bigcirc} \quad \overset{\mathbf{e}_j+\mathbf{e}_{j+1}}{\bigcirc}$$

Table 2. Root systems for the exceptional Weyl groups [15, §2.10]

Type, order and Root system	Coxeter symbol and simple system
F_4 $2^7\,3^2$. $\{\pm\mathbf{e}_i \pm \mathbf{e}_j\ (1 \le i < j \le 4)$, $\pm\mathbf{e}_i\ (1 \le i \le 4)$, $\frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4)\}$;	
E_6 $2^7\,3^4\,5$. $\{\pm\mathbf{e}_i \pm \mathbf{e}_j\ (1 \le i < j \le 5)$, $\frac{1}{2}\sum_{i=1}^{8} \epsilon_i \mathbf{e}_i\}$; $(\epsilon_i = \pm 1, \prod \epsilon_i = 1, \epsilon_8 = -\epsilon_7 = -\epsilon_6)$	
E_7 $2^{10}\,3^4\,5\,7$. $\{\pm\mathbf{e}_i \pm \mathbf{e}_j\ (1 \le i < j \le 6)$, $\pm(\mathbf{e}_7 - \mathbf{e}_8)$, $\frac{1}{2}\sum_{i=1}^{8} \epsilon_i \mathbf{e}_i\}$; $(\epsilon_i = \pm 1, \prod \epsilon_i = 1, \epsilon_8 = -\epsilon_7)$	
E_8 $2^{14}\,3^5\,5^2\,7$ $\{\pm\mathbf{e}_i \pm \mathbf{e}_j\ (1 \le i < j \le 8)$, $\frac{1}{2}\sum_{i=1}^{8} \epsilon_i \mathbf{e}_i\}$; $(\epsilon_i = \pm 1, \prod \epsilon_i = 1)$	

for all $k + m > 0$ with $2(k + m) \le n$, except for when n is even and $m = 0$, where $\coprod_{i=1}^{n/2} \bigcirc$ corresponds to two classes with associated diagrams:



(b) order $p \ge 3$: 

for all $k \ge 1$ with $kp \le n$.

See [5, Proposition 25]. Note the diagram corresponding to two different conjugacy classes–hence the need for different labellings. This exception arises from the fact that D_n is isomorphic to the subgroup of B_n consisting of the even signed permutations: it permutes the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ while changing the sign of an even number of them. From the proof of Proposition 25 in [5], if w_1, w_2 are the elements corresponding to the two labellings in part (a) above, and $ww_1w^{-1} = w_2$, then w must change the sign of an odd number of the basis vectors, so the w_i are not conjugate in D_n although they are in B_n.

Theorem 7 (type E). *The conjugacy classes with prime order in the Weyl groups of type E_6, E_7 and E_8 correspond to the diagrams:*

E_6. *(a) order* 2 : $\coprod^{k} \circ$, $(k = 1, \ldots, 4)$;

(b) order 3 : $\coprod^{k} \circ\!\!-\!\!\!-\!\!\circ$, $(k = 1, \ldots, 3)$;

(c) order 5 : $\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$.

E_7. *(a) order* 2 : $\coprod^{k} \circ$, $(k = 1, \ldots, 7)$, *except for* $k = 3, 4$, *each of which correspond to two classes with associated diagrams:*

$$\overset{\mathbf{e}_1-\mathbf{e}_2}{\circ} \quad \overset{\mathbf{e}_3-\mathbf{e}_4}{\circ} \quad \overset{\mathbf{e}_5\pm\mathbf{e}_6}{\circ}\,, \qquad \overset{\mathbf{e}_1-\mathbf{e}_2}{\circ} \quad \overset{\mathbf{e}_3-\mathbf{e}_4}{\circ} \quad \overset{\mathbf{e}_5-\mathbf{e}_6}{\circ} \quad \overset{\mathbf{e}_7-\mathbf{e}_8}{\circ}$$

$$\text{and} \quad \overset{\mathbf{e}_1+\mathbf{e}_2}{\circ} \quad \overset{\mathbf{e}_3+\mathbf{e}_4}{\circ} \quad \overset{\mathbf{e}_5+\mathbf{e}_6}{\circ} \qquad \circ\ \mathbf{e}_1+\mathbf{e}_3+\mathbf{e}_5+\mathbf{e}_7-\tfrac{1}{2}\textstyle\sum_{i=1}^{8}\mathbf{e}_i$$

(b) order 3 : $\coprod^{k} \circ\!\!-\!\!\!-\!\!\circ$, $(k = 1, \ldots, 3)$; *(c) order* 5 : $\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$,

(d) order 7 : $\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$.

E_8. *(a) order* 2 : $\coprod^{k} \circ$, $(k = 1, \ldots, 8)$, *except for* $k = 4$ *which corresponds to two classes with associated diagrams:*

$$\overset{\mathbf{e}_1-\mathbf{e}_2}{\circ} \quad \overset{\mathbf{e}_3-\mathbf{e}_4}{\circ} \quad \overset{\mathbf{e}_5-\mathbf{e}_6}{\circ} \quad \overset{\mathbf{e}_7\pm\mathbf{e}_8}{\circ}\ ;$$

(b) order 3 : $\coprod^{k} \circ\!\!-\!\!\!-\!\!\circ$, $(k = 1, \ldots, 4)$;

(c) order 5 : $\coprod^{k} \circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$, $(k = 1, 2)$;

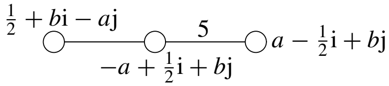
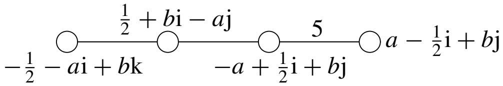
(d) order 7 : $\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$.

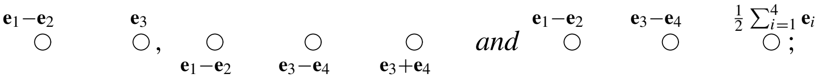
Both this result and Theorem 8 below arise from a careful examination of the results from [5, §8]. These groups do not have convenient descriptions as the classical groups do. In [5], a collection of non-conjugate elements is formed, and a counting argument gives that the list is complete. As with the groups of type D_n, there are some pairs of conjugacy classes corresponding to the same (unlabelled) diagram. That the different labellings given in the two theorems yield non-conjugate elements is checked by observing that they fix a different number of roots in the systems from Table 2.

Theorem 8 (type F). *The conjugacy classes with prime order in the Weyl group of type* F_4 *correspond to the diagrams: (a) order* 2 :

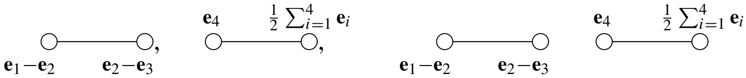
$$\overset{\mathbf{e}_1-\mathbf{e}_2}{\circ}\,, \quad \overset{\mathbf{e}_1}{\circ}\,, \quad \overset{\mathbf{e}_1-\mathbf{e}_2}{\circ} \quad \overset{\mathbf{e}_3-\mathbf{e}_4}{\circ}\,,$$

Table 3. Root systems for the type H non-Weyl groups [15, §2.13]

Type and order	Coxeter symbol and simple system
H_3 $2^3\,3\,5.$	
H_4 $2^6\,3^2\,5^2.$	



(b) order 3 :



4.2. The non-Weyl groups

Identify a 4-dimensional Euclidean space V with the division ring $\mathbf{H}$ of quaternions, so that the bilinear form of §4.1 becomes $\langle\mathbf{u},\mathbf{v}\rangle = \frac{1}{2}(\mathbf{u}\bar{\mathbf{v}}+\mathbf{v}\bar{\mathbf{u}})$ where $\bar{\mathbf{v}} = c_1 - c_2\mathrm{i} - c_3\mathrm{j} - c_4\mathrm{k}$ is quaternionic conjugation. If $\mathbf{v}\in\mathbf{H}$ has norm 1 then the reflection $s_\mathbf{v}$ is given by $s_\mathbf{v}(\mathbf{u}) = -\mathbf{u}\bar{\mathbf{v}}\mathbf{u}$. Let the split extension of $\mathbb{Z}_2^4$ by the alternating group $\mathfrak{A}_4$ act on $\mathbf{H}$, with $\mathfrak{A}_4$ permuting the coordinates and the $\mathbb{Z}_2^4$ generated by sign changes. If

$$\phi = \frac{1+\sqrt{5}}{2},$$

is the golden number and $a = \frac{1}{2}\phi$ and $b = \frac{1}{2}(\phi-1)$, then the 120 images under this action of 1, $\frac{1}{2}(1+\mathrm{i}+\mathrm{j}+\mathrm{k})$ and $a+\frac{1}{2}\mathrm{i}+b\mathrm{j}$ form a root system for H_4 ([15, §2.13]). The 30 roots orthogonal to k give a root system for H_3. Coxeter symbols and simple systems are given in Table 3.

Proposition 4 (type H). *The conjugacy classes with prime order in the groups of type H_3 and H_4 have representatives:*

H_3. *(a) order* $2 : x_1, x_1x_3$ *and* $(x_2x_1x_3)^5$; *(b) order* $3 : x_1x_2$; *(c) order* $5 : x_2x_3$ *and* $(x_2x_3)^2$.

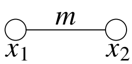
H_4. *(a) order* $2 : x_1, x_1x_4, x_2x_1x_3x_2x_1w^{12}\bar{w}x_4x_3x_4$ *and* $x_1x_2x_1x_3x_2x_1w^{12}\bar{w}x_4x_3x_4$; *(b) order* $3 : x_1x_2$ *and* x_2x_3; *(c) order* $5 : x_3x_4$, $(x_3x_4)^2$, $x_3w^3\bar{w}w^2$, $x_3w^9\bar{w}w^2$ *and* $x_1x_2x_1x_3w^8\bar{w}x_4\ x_3x_4$, *where $x_i = s_{\mathbf{v}_i}$ with $\mathbf{v}_i$ the i-th simple root from the left in the Coxeter symbol, $\bar{w} = x_4x_3x_2$ and w is the Coxeter element $x_4x_3x_2x_1$.*

Proof. The 34 conjugacy classes in H_4, and a representative transformation of $\mathbf{H}$ in each, are given in [12, Table 3]. By observing their effect on 1, i, j, k the order of these transformations can be computed to give four of order 2, two of order 3 and five of order 5, the others non-prime. Thus it remains to show that the elements stated in the Proposition are non-conjugate. For the order 2 and 3 elements this is most easily done by computing the number of roots fixed by each–a tedious but finite task that gives for example in the order 3 case, two roots fixed by x_1x_2 and six by x_2x_3, Three of the order five elements fix no roots, but nevertheless have different traces.

Similar calculations give the H_3 classes. Alternatively, the group is well known to be isomorphic to $\mathbb{Z}_2 \times \mathfrak{A}_5$, with $\mathfrak{A}_5$ the rotations of a dodecahedron, and the center $\mathbb{Z}_2$ generated by the antipodal map. □

Finally we have the, obviously well known,

Proposition 5 (type I). *The conjugacy classes with prime order in the dihedral groups,*



of type I have the following representatives for all $l \leq k$ with $m/\gcd(m, l)$ prime: if $m = 2k + 1$, x_1 and $(x_1x_2)^l$; and for $m = 2k$, x_1, x_2 and $(x_1x_2)^l$;

5. Volume

Let Π be a group acting properly, freely, cofinitely by isometries on $\mathbb{H}^n$ and let $M = \mathbb{H}^n/\Pi$. If n is even then the Gauss-Bonnet (-Hirzebruch) formula [11, 13, 14, 26], gives

$$\text{vol}(M) = \kappa_n \chi_{\text{top}}(M),$$

with $\chi_{\text{top}}(M)$ the Euler characteristic of M and $\kappa_n = 2^n(n!)^{-1}(-\pi)^{n/2}(n/2)!$. As Π is of finite homological type and torsion free the Euler characteristic, $\chi(\Pi) = \sum_i (-1)^i \text{rank}_{\mathbb{Z}} H_i(\Pi)$ is defined [3, §IX.6], and since M is a $K(\Pi, 1)$ space ($\mathbb{H}^n$ being contractible) the homologies $H_*(M) \cong H_*(\Pi)$, so $\chi_{\text{top}}(M) = \chi(\Pi)$.

Now suppose Π arises via a torsion free transitive $\mathbb{C}\Gamma$-module U as in §3, with Γ a hyperbolic Coxeter group and $\dim U = m$ (which is thus the index in Γ of Π). Then $\chi(\Gamma)$ is defined and $\chi(\Pi) = m\chi(\Gamma)$. But calculating the Euler characteristic of Coxeter groups turns out to be a simple task:

Theorem 9. *Let* (Γ, S) *be a Coxeter group. For any* $\sigma \in \mathscr{F}$, *let* $|\sigma|$ *be the order of this group. Then,*

$$\chi(\Gamma) = \sum_{\substack{\sigma_0 < \cdots < \sigma_k \\ \sigma_i \in \mathscr{F}}} \frac{(-1)^k}{|\sigma_0|}, \tag{1}$$

where the sum is over all chains $\sigma_0 < \cdots < \sigma_k$ *with* $\sigma_i \in \mathscr{F}$.

A similar result appears in [6] for the Euler characteristic defined in [7].

Proof. Let Σ be the affine Γ-complex from §2, metrised as there to be a CAT(0) space. For each k-cell $\sigma = (\sigma_0 < \sigma_1 < \cdots < \sigma_k)$ let Γ_σ be the isotropy group of σ and Y the k-cells $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ where each $\sigma_i \in \mathscr{F}$. Then Y is a set of representatives of the cells of Σ modulo the Γ-action. Moreover, Σ is contractible (as indeed is any CAT(0) space) so by [3, Proposition IX.7.3(e′)],

$$\chi(\Gamma) = \chi_\Gamma(\Sigma) = \sum_{\sigma \in Y} (-1)^{\dim \sigma} \chi(\Gamma_\sigma),$$

with the right hand side the Γ-equivariant Euler characteristic of Σ. If $\sigma = (\sigma_0 < \cdots < \sigma_k) \in Y$ then $\Gamma_\sigma = \sigma_0$ and $\chi(\Gamma_\sigma) = 1/|\Gamma_\sigma| = 1/|\sigma_0|$ as these groups are finite. □

If Γ is a Coxeter symbol and Δ a subsymbol, let $\Gamma \setminus \Delta$ be the subsymbol obtained by removing from Γ the vertices of Δ and their incident edges. The following is then sometimes useful.

Theorem 10. *Let* Γ *be a Coxeter symbol and* Ψ *a subsymbol. Let* Σ_Ψ *be the sum (1) restricted to those chains* $\sigma_0 < \cdots < \sigma_k$ *with* $\Psi < \sigma_0$. *Then,*

$$\Sigma_\Psi = \sum_{\Delta \in \mathscr{P}(\Psi)} (-1)^{v(\Delta)} \chi(\Gamma \setminus \Delta),$$

where $\mathscr{P}(\Psi)$ *is the set of all subsymbols of* Ψ, *and* $v(\Delta)$ *the number of nodes of* Δ.

Proof. Apply the inclusion-exclusion principle as in [27, §2.1] □

If Ψ is the symbol for an infinite Coxeter group, then $\sum_\Psi = 0$, a sum over an empty set, so in particular if Γ is infinite we obtain as a Corollary that,

$$\sum_{\Delta \in \mathscr{P}(\Gamma)} (-1)^{v(\Delta)} \chi(\Gamma \setminus \Delta) = 0,$$

which is Serre's closed form for the Euler characteristic [25, Proposition 16]

For n odd, the Euler characteristic of an n-dimensional hyperbolic Coxeter group is zero, and there is in general no simple method for computing volumes. Of course, $\text{vol}(\mathbb{H}^n/\Pi) = m\text{vol}(P)$ where Π is an index m subgroup of Coxeter group Γ with fundamental polytope P. For certain Γ the volumes of such P have been determined (see for example [16]).

6. Examples

We give three manifolds as examples of the construction, one in each $\mathbb{H}^n$ for $n = 4, 5, 6$. We have let volume be our guiding principle, so that the three are either the smallest possible ($n = 4$) or would appear to be the smallest known ($n = 5, 6$). As mentioned in the Introduction, a large family of 4-manifolds with $\chi = 1$ have recently been constructed by Ratcliffe and Tschantz [22] using a computer.

We first explain the notation. In each case we have a Coxeter group Γ acting on a collection of finite sets Ω_i, so that if U_i is the $\mathbb{C}$-vector space with basis Ω_i, then $\otimes U_i$ is a torsion free $\mathbb{C}\Gamma$-module.

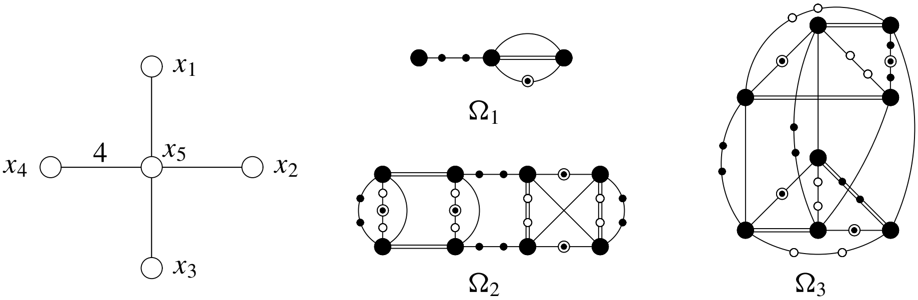
The actions of Γ on the Ω_i are depicted using two notations. If Ω_i has solid black nodes (as in 6.1, 6.2 diagrams $\Omega_1 - \Omega_3$ and 6.3 diagrams Ω_1 and Ω_2) then generator x_j of Γ acts by swapping two nodes if they are connected by an edge labelled according to the scheme: —— for x_1; ══ for x_2; —⊙— for x_3; —○—○— for x_4; —•—•— for x_5; —•— for x_6 and —○— for x_7 (or even combinations that can be interpreted unambiguously like ═⊙═ for an x_2 and x_3 edge). Nodes not so connected are fixed by the corresponding x_j.

The other notation, used in §6.2 diagram Ω_4 and §6.3 diagrams Ω_3 and Ω_4, is convenient when the action of Γ is imprimitive and we want to compress rather a lot of information (so the resulting diagrams require a certain amount of decoding). The numbered vertices in the diagrams are the blocks of imprimitivity of an imprimitive Γ-action; the action on the blocks is depicted using the notation above.

To recover the action on the original set Ω, suppose that the block Ω_i is given by $\Omega_i = \{m_{i1}, m_{i2}, \dots, m_{ik}\}$ with $m_{i1} < m_{i2} < \cdots < m_{ik}$. We then write $(i, j)_k = \sigma$ to mean m_{il} goes to $m_{j\sigma(l)}$ under the action of the generator x_k. An absence of such an indication corresponds to σ being the identity. If the block Ω_i is fixed by x_j, then the action of x_j on the m_{il} is described with each diagram.

6.1. *A 4-manifold with $\chi = 1$.*

Let Γ be the Coxeter group with symbol on the left,

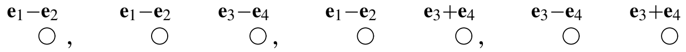


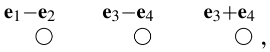
which appears in [15, §6.9], although it is easily checked that the Gram matrix $G(\Gamma)$ has signature $(1, 4)$ and that the poset $\overline{\mathscr{F}}$ is isomorphic to the poset of a combinatorial 4-simplex. Thus, by Theorems 1–2, Γ acts cofinitely on $\mathbb{H}^4$ with fundamental region a cusped 4-simplex.

From $\mathscr{F}$ and Theorem 9 we have

$$\chi(\Gamma) = 1/\mathscr{L}(\Gamma),$$

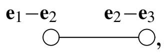
where $\mathscr{L}(\Gamma) = 2^6\,3$, the lowest common multiple of the order of the elements of $\mathscr{F}$, is found using Tables 1–3. There are five finite subgroups generated by maximal $S' \subseteq S$: a D_4, three B_3's and an $A_1 \times A_1 \times A_1 \times A_1$, and conjugacy class representatives in these groups can be listed using the results of §4. For example, x_1, x_2, x_3 and x_5 generate a Weyl group of type D_4. Consult the root system in Table 1 and make the identifications $x_2 = s_{\mathbf{e}_1-\mathbf{e}_2}, x_5 = s_{\mathbf{e}_2-\mathbf{e}_3}, x_1 = s_{\mathbf{e}_3-\mathbf{e}_4}$ and $x_3 = s_{\mathbf{e}_3+\mathbf{e}_4}$. By Theorem 6 the order two torsion corresponds to the diagrams,





and


with resulting representatives x_2, x_1x_2, x_2x_3, x_1x_3, $x_1x_2x_3$ and $x_2s_{\mathbf{e}_1+\mathbf{e}_2}x_1x_3$, where $s_{\mathbf{e}_1+\mathbf{e}_2} = wx_3w^{-1}$ with $w = x_5x_1x_2x_5$; the order three torsion consists of the single class



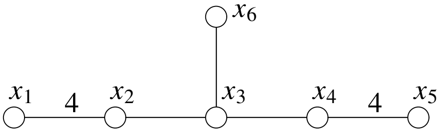
with corresponding representative x_2x_5. Perform this process for the four other groups.

The three other figures above give actions of the x_j on sets Ω_i using the notation from the beginning of the section. By checking that each relator word $(s_\alpha s_\beta)^{m_{\alpha\beta}}$ acts as the identity on Ω_i, it can be seen that the diagrams give actions of Γ on Ω_i. Let U_i be the resulting $\mathbb{C}\Gamma$-module, transitive in each case as the diagrams are connected.

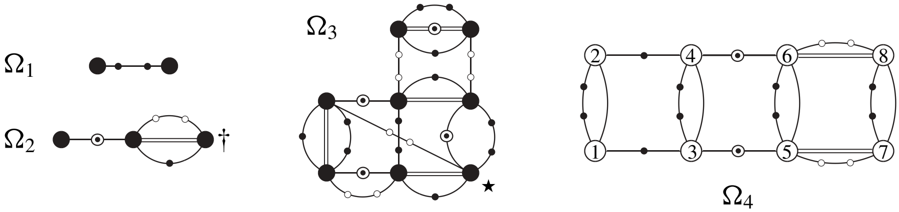
One now checks, using Proposition 1, that each representative of torsion in Γ is avoided by at least one of the U_i: for instance, those listed above are avoided by U_2, U_2, U_2, U_3, U_3, U_2 and U_1 respectively. Thus $U = \otimes U_i$ is torsion free. The index in Γ of the corresponding subgroups Π is at most $\prod |\Omega_i| = 2^6\,3 = \mathscr{L}(\Gamma)$. On the otherhand, by the comments following Lemma 3 the index must be a multiple of $\mathscr{L}(\Gamma)$, hence the index is $\mathscr{L}(\Gamma)$. We thus obtain a (non-compact) hyperbolic 4-manifold $M = \mathbb{H}^4/\Pi$, with $\chi(M) = \chi(\Pi) = \mathscr{L}(\Gamma) \times \chi(\Gamma) = 1$.

6.2. A 5-manifold with volume $14\zeta(3)$

The Coxeter symbol,



corresponds to a 5-dimensional hyperbolic group acting cofinitely on $\mathbb{H}^5$ with fundamental region a cusped 5-simplex ([15, §6.9] or Theorems 1–2). The four diagrams,



give $\mathbb{C}\Gamma$-modules U_i as in §6.1, with the Γ-action on Ω_4 imprimitive with eight blocks, and each block having size eight. If x_j fixes the block i, we use the notation $\mathbf{i} : j\sigma, \ldots$ to mean the action of x_j on the eight points in the block is as a permutation σ given by,

1:2(2,4)(5,7),3(3,5)(4,6),4(2,5)(4,7)	**4**:2(1,3)(6,8),4(2,4)(5,7)	**7**:3(3,6)(4,7),6(5,7)(6,8)
2:2(1,3)(6,8),3(3,4)(5,6),4(2,4)(5,7)	**5**:6(4,6)(5,7)	**8**:3(1,5)(4,8),6(5,7)(6,8)
3:2(2,4)(5,7),4(2,5)(4,7)	**6**:6(3,4)(7,8)	

The permutations $(i, j)_k$ not equal to the identity are,

$(1,2)_1,(3,4)_1=(4,5)$	$(7,8)_1,(7,8)_5=(1,3)(5,6)(7,8)$	$(6,8)_2=(2,3,5)(4,7,6)$
$(1,2)_5,(3,4)_5=(1,2)(3,5,6,4)(7,8)$	$(3,5)_3=(3,4,5,6,7,8)$	$(5,7)_2=(2,3)(4,5,6,7,8)$
$(5,6)_1=(3,6,4)(5,7,8)$	$(4,6)_3=(5,7)(6,8)$	$(5,7)_4=(1,2,4,5,6,7,8,3)$
$(5,6)_5=(1,2)(3,5)(4,7)(6,8)$	$(6,8)_4=(1,2,4,8,7,5)(3,6)$	

Listing the torsion in Γ using the results of §4, one can check that the U_i collectively avoid all torsion, so that $U = \otimes U_i$ is a torsion free $\mathbb{C}\Gamma$-module.

We now compute the volume of the manifold M^5 using the results of §3. By lemma 2, we have that $U_2 \otimes U_3$ is transitive, so in particular the size of any Γ-orbit in $\Omega_2 \times \Omega_3 \times \Omega_4$ is divisible by 3. On the other hand, using the isomorphism between $\langle x_2, x_3, x_4\rangle$ and the symmetric group $\mathfrak{S}_4$ given by $x_i \mapsto (i-1, i)$, and letting

$$w_1 = x_2x_4,\ w_2 = x_3x_2x_3,$$

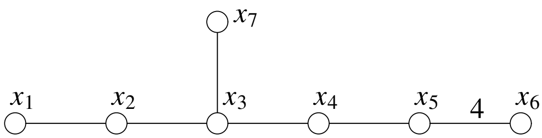
we have that $\langle w_1, w_2 \rangle$ is a subgroup of Γ of order eight. No non-identity element of it fixes a point of Ω_3, hence of $\Omega_2 \times \Omega_3$. However, the subgroup $\langle x_2, x_3, x_4 \rangle$ fixes a point in the first block of imprimitivity of Ω_4. Proposition 2 thus gives that $2^3|\Omega_4| = 2^9$ divides the size of any Γ-orbit in $\Omega_2 \times \Omega_3 \times \Omega_4$, hence $2^9\,3 = |\Omega_2 \times \Omega_3 \times \Omega_4|$ does too, and $U_2 \otimes U_3 \otimes U_4$ is a transitive $\mathbb{C}\Gamma$-module.

The subgroup of order two generated by $x_1x_3x_5$ acts on Ω_1 with no non-identity element fixing a point. On the other hand, it fixes the point $(\dagger, \star, *)$ of $\Omega_2 \times \Omega_3 \times \Omega_4$ where $\dagger$ and $\star$ are as shown, and $*$ is a point in the seventh block of imprimitivity of Ω_4. Thus $2|\Omega_2 \times \Omega_3 \times \Omega_4| = \prod |\Omega_i|$ divides the size of a Γ-orbit giving that $U = \otimes U_i$ is transitive.

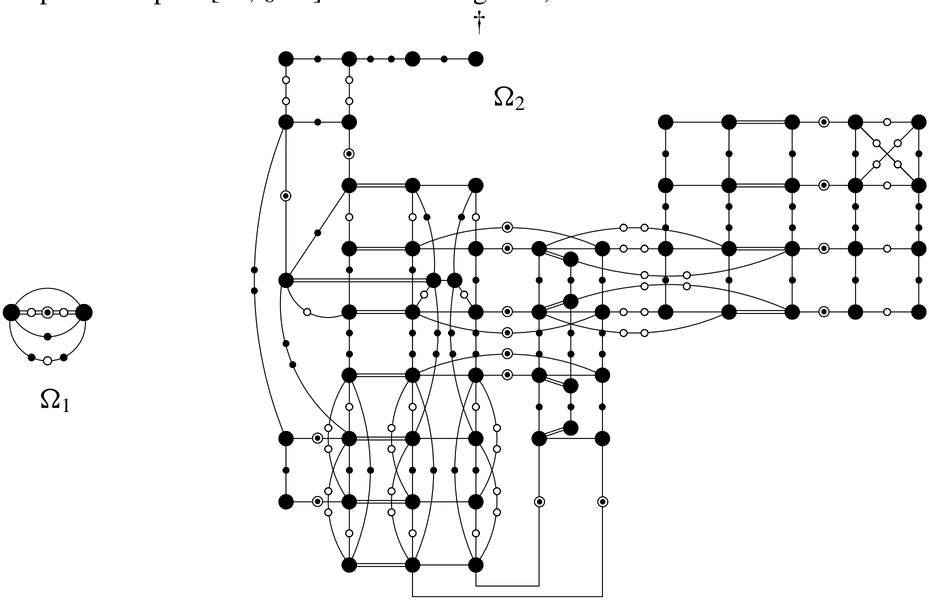
There is no Euler characteristic in 5-dimensions, but by [16], a cusped 5-simplex fundamental region for Γ has volume $7\zeta(3)/2^9\,3$, with ζ the Riemann-zeta function. Thus M^5 has volume $7\zeta(3)\prod|\Omega_i|/2^9\,3 = 14\zeta(3)$.

6.3. A 6*-manifold with* $\chi = -16$

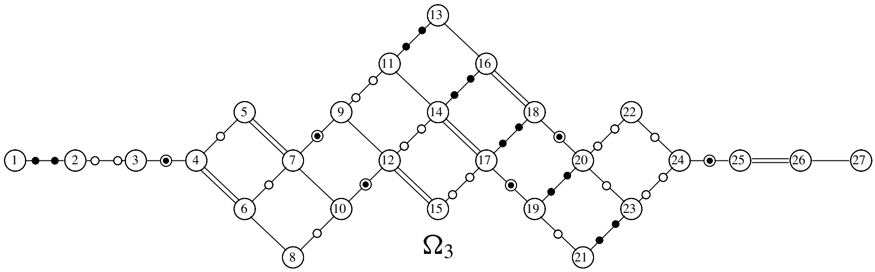
The Coxeter symbol,



is, as the others, that of a Γ acting cofinitely on $\mathbb{H}^6$ with fundamental region a cusped 6-simplex [15, §6.9]. The two diagrams,



give primitive Γ-actions on 2 and 64 points. Diagram Ω_3 below depicts an action with 27 blocks of imprimitivity, each of size six[2]:



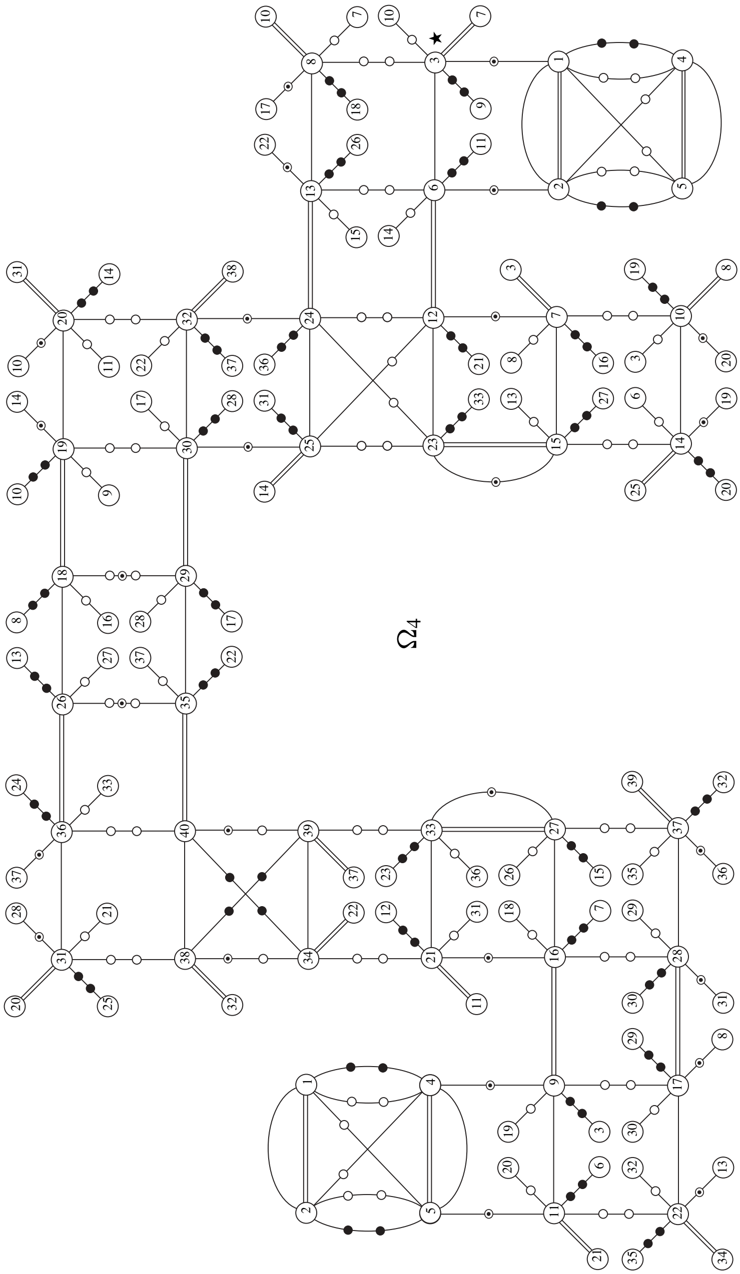
Ω_3

If x_j fixes the block i, then the action on the six points of the block is a product of three disjoint 2-cycles (although the generator x_6 in fact fixes all 162 points of Ω_3). Partially order the elements of $\mathfrak{S}_6$ by $\sigma < \tau$ if and only if $\sigma(i) < \tau(i)$, where i is the smallest element of $\{1, \dots, 6\}$ on which σ and τ differ. Restrict this order to the permutations of type three disjoint 2-cycles. The x_j action on block i is then as,

1:1(3)2(4)3(8)4(1)7(1)	**10**:2(3)4(8)5(1)	**19**:1(1)2(8)4(3)
2:1(3)2(4)3(8)7(1)	**11**:2(1)3(8)7(4)	**20**:1(1)2(8)
3:1(3)2(4)5(1)7(1)	**12**:5(1)7(4)	**21**:1(1)2(8)3(4)4(3)
4:1(3)4(8)5(1)	**13**:2(1)3(8)4(1)7(4)	**22**:1(1)2(8)3(1)5(3)
5:1(3)3(1)4(8)5(1)	**14**:3(8)7(4)	**23**:1(1)2(8)3(4)
6:3(4)4(8)5(1)	**15**:1(1)3(3)5(1)7(4)	**24**:1(1)2(8)5(3)
7:4(8)5(1)	**16**:3(8)4(1)7(4)	**25**:1(1)4(4)5(3)7(1)
8:2(3)3(4)4(8)5(1)	**17**:1(1)7(4)	**26**:3(8)4(4)5(3)7(1)
9:2(1)5(1)7(4)	**18**:1(1)4(1)7(4)	**27**:2(1)3(8)4(4)5(3)7(1)

using the notation $\mathbf{i} : j(k)\dots$ to mean x_j acts on the i-th block as the k-th permutation of the form three disjoint 2-cycles. The permutations $(i, j)_k$ are all the identity for Ω_3.

[2] Nikolai Vavilov has pointed out to me the curious fact that Ω_3 is the weight diagram of the representation of the Chevalley group of type E_6 with highest weight (see for example [28]).


Ω_4

Finally, diagram Ω_4 gives a Γ-action on 40 blocks of imprimitivity, each of size eight, where repeated vertices and edges are to be thought of as being identified. No block is fixed by any generator except for x_6, which, as for Ω_3, fixes all 320 points. The permutations $(i, j)_k$ not equal to the identity are given by,

$(1,2)_2,(1,4)_4,(12,24)_4=(2,3)(4,6)(5,7)$	$(2,4)_7=(2,4)(3,5)(6,7)$	$(1,5)_7,(27,16)_1,(8,13)_1=(3,6,4,7,5)$
$(2,5)_4,(4,5)_2=(2,4,6,3,5)$	$(1,4)_5,(1,2)_1=(3,6,5,4)$	$(5,11)_3,(10,19)_5=(6,7)$
$(11,20)_7=(5,6)$	$(9,19)_7,(15,23)_2,(14,25)_2,(6,14)_7=(3,4)$	$(11,21)_2,(11,22)_4=(2,3,4)(6,8,7)$
$(9,17)_4=(2,3)(5,8,7,6)$	$(22,35)_5,(21,33)_1=(2,3,4)(5,6,7)$	$(22,34)_2,(21,34)_4=(3,4,5)(6,7)$
$(22,17)_1,(10,3)_7=(3,4)(5,7,6)$	$(17,30)_7=(5,6,7,8)$	$(2,5)_5,(4,5)_1,(37,28)_1=(2,3,4)(5,7,6)$
$(4,9)_3,(38,31)_4=(6,7,8)$	$(9,11)_1,(30,32)_1=(2,3,4)(5,6,8,7)$	$(22,32)_7,(20,19)_1,(23,15)_3=(2,3)(6,7,8)$
$(17,28)_2,(10,20)_3=(4,5)(6,7,8)$	$(28,30)_5=(1,3)(5,6,7)$	$(28,31)_3=(1,2)(3,4)(5,6)$
$(28,29)_7=(1,2,3)(4,6,5)(7,8)$	$(16,28)_4,(29,17)_5,(23,12)_1=(2,3)(4,5,6)$	$(27,37)_4,(26,36)_2=(4,5)$
$(16,18)_7=(1,2)(3,5,4)$	$(16,21)_3=(3,5)(4,6,7)$	$(27,33)_2,(26,35)_4=(3,5)(4,6)(7,8)$
$(27,33)_3,(26,35)_3=(2,3,5,6,7,4)$	$(21,31)_7=(2,4,3)(7,8)$	$(33,36)_7,(35,37)_7=(1,2)(3,4)(5,6,7,8)$
$(33,39)_4,(35,40)_2=(3,4,6,5)(7,8)$	$(34,39)_1,(34,40)_5=(2,4,6,3,5)(7,8)$	$(38,39)_5,(38,40)_1=(1,2,3)(5,7,6)$
$(34,38)_7=(2,4)(3,5)(6,7,8)$	$(34,38)_3=(1,2,4,5,6,3)(7,8)$	$(39,40)_7=(1,2)(3,6)(5,7)$
$(39,40)_3=(1,3)(2,5)(6,7)$	$(31,36)_1=(1,2,3)(5,7)(6,8)$	$(36,37)_3=(1,3)(2,4)(6,8)$
$(2,6)_3=(7,8)$	$(26,27)_7=(1,2)(3,6)(4,7)$	$(29,35)_1,(13,24)_2=(4,5,6)$
$(18,29)_3=(2,3,4)(6,7)$	$(18,29)_4=(3,4,5)(7,8)$	$(18,19)_2,(8,10)_2=(1,2)(4,5)$
$(29,30)_2=(1,3,2)(7,8)$	$(20,31)_2,(6,3)_1=(5,7,6)$	$(19,30)_4=(2,3,4))$
$(20,32)_4=(3,4)(5,6)$	$(32,38)_2=(3,4)(7,8)$	$(32,37)_5=(1,2,3,4)(5,7,8,6)$
$(25,30)_3=(1,2,4,3)(6,7,8)$	$(24,32)_3=(1,2,3)(6,7)$	$(25,31)_5=(2,3,4)(5,7,8,6)$
$(24,36)_5=(3,4)(5,7)$	$(24,25)_1=(1,2)(5,6,7)$	$(23,25)_4=(1,2)(4,5)(6,7)$
$(23,24)_7=(1,2)(4,5,6)$	$(12,25)_7,(12,6)_2=(2,3)(5,6,7)$	$(23,33)_5=(4,6)(7,8)$
$(12,21)_5=(3,4)(6,7,8)$	$(7,12)_3=(4,5,6,8,7)$	$(7,8)_7=(1,2)(3,6)(4,5,7)$
$(15,27)_5=(3,4,5)$	$(7,16)_5=(3,7,5,6,4)$	$(14,15)_4,(10,7)_4=(1,2)(3,5)(6,7)$
$(14,20)_5=(2,3)(7,8)$	$(14,19)_3=(2,3)(4,5)(6,8,7)$	$(13,22)_3=(3,5,6,7,4)$
$(8,17)_3,(7,3)_2=(3,5,6,4)$	$(13,15)_7=(1,2)(3,4)$	$(13,26)_5=(3,5,6)(4,7)$
$(6,13)_4=(4,5)(6,7)$	$(3,8)_4=(3,5,4,7,6)$	$(6,11)_5=(2,3,4)(5,6)(7,8)$
$(3,9)_5=(5,7)$	$(1,3)_3=(3,5,4)(6,7,8)$	

As per usual, let $U = \otimes U_i$, having checked that the U_i collectively avoid the torsion in Γ, so that we obtain a (non-compact) 6-dimensional manifold M^6.

To compute the volume of M^6, let Ω be a Γ-orbit on $\Omega_2 \times \Omega_3$ arising from $U_2 \otimes U_3$. By Lemma 2 we have that $2^6\, 3^4$ divides $|\Omega|$, while on the other hand, $|\Omega| \leq |\Omega_2||\Omega_3| = 2^7\, 3^4$. The element x_1 of order 2 fixes no point of Ω_3 but fixes the point $\dagger$ of Ω_2, thus the subgroup F generated by x_1 does so. By Proposition 2, we have that $2|\Omega_2| = 2^7$ divides $|\Omega|$, giving $|\Omega| = 2^7\, 3^4 = |\Omega_2| \times |\Omega_3|$. Thus $U_2 \otimes U_3$ is a transitive $\mathbb{C}\Gamma$-module.

Let $\overline{\Omega}_4$ be the blocks of imprimitivity of the Γ-action on Ω_4. The element $\gamma = x_1 x_3 x_2 x_4$ of order 5 fixes the point $(\dagger, *)$ of $\Omega_2 \times \Omega_3$ but acts on $\overline{\Omega}_4$ as a product of eight 5-cycles ($*$ is a point of Ω_3 in the first block of imprimitivty). Thus $F = \langle \gamma \rangle$ satisfies the conditions of Proposition 2, hence the Γ-orbits on $\Omega_2 \times \Omega_3 \times \overline{\Omega}_4$ have size divisible by $5(|\Omega_2| \times |\Omega_3|) = 2^7\, 3^4\, 5$. The elements,

$$\gamma = (x_5 x_4)^2 x_5 x_1 \text{ and } \delta = x_3 x_2 x_1 x_5 x_7 x_3 x_5 x_3 x_4 x_1 x_2 x_3,$$

are contained in the finite subgroup $\langle x_1, x_2, x_3, x_4, x_5, x_7 \rangle$ isomorphic to E_6 and fix the point $(\dagger, *)$ of $\Omega_2 \times \Omega_3$ (where $*$ is in the first block of imprimitivity). The faithful action of γ and δ on the roots of E_6 are as permutations satisfying $\gamma^2 = \delta^2 = (\gamma\delta)^4 = 1$ so that $F = \langle \gamma, \delta \rangle$ has order divisible by 8. Moreover, γ and δ act on $\overline{\Omega}_4$ as a product of twenty 2-cycles and $\gamma\delta$ as a product of ten 4-cycles, giving that F satisfies the conditions of Proposition 2. Thus $2^3(|\Omega_2| \times |\Omega_3|)$ divides the size of any Γ-orbit on $\Omega_2 \times \Omega_3 \times \overline{\Omega}_4$. Hence $2^{10}\, 3^4\, 5$ divides this size which in turn is $\leq 2^{10}\, 3^4\, 5$, and so we have equality. Thus $U_2 \otimes U_3 \otimes \overline{U}_4$ is transitive.

Similarly $x_1x_2x_3x_4x_7$ has order 8, acting on Ω_4 as fourty 8-cycles while fixing the point $(\dagger, *, \star)$ of $\Omega_2 \times \Omega_3 \times \overline{\Omega}_4$, hence $U_2 \otimes U_3 \otimes U_4$ is transitive ($*$ as before is in the first block of imprimitivity of Ω_3). Finally,

$$\mu = x_3x_4x_5x_6x_5x_4x_3x_7x_3x_4x_5x_6x_5x_4x_3x_4x_5x_6x_5x_4x_5x_6x_5x_6x_7,$$

is an element of finite order, being in the B_5 generated by $x_3, \ldots, x_7$. As it involves an odd number of generators, it acts as a 2-cycle on Ω_1, so in particular is non-trivial (actually it turns out to be an involution). Thus $\langle\mu\rangle$ has finite order ≥ 2. As x_6 fixes all the points in both Ω_3 and Ω_4, the action of μ on these sets is the same as that of the element obtained by removing the occurances of x_6 from it. As the resulting word collapses to the identity we have that μ fixes every point of Ω_3 and Ω_4. As it fixes the point $\dagger$ of Ω_2 also, we have $\langle\mu\rangle$ fixing a point in $\Omega_2 \times \Omega_3 \times \Omega_4$. Proposition 2 gives $2|\Omega_2 \times \Omega_3 \times \Omega_4| = \prod |\Omega_i|$ divides the size of a Γ-orbit, and so $U = \otimes U_i$ is transitive.

Thus M^6 has Euler characteristic $\chi(\Gamma) \prod |\Omega_i| = (-1/\mathscr{L}(\Gamma)) \times 16\mathscr{L}(\Gamma) = -16$. As U_1 is clearly an orientable module, U is as well by Proposition 3, so that M^6 is orientable.

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