Images of hyperbolic groups

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$E$ elliptic curve over $\mathbb{Q}$:

\[ Y^2Z = 4X^3 + aXZ^2 + bZ^3 \]

$\ a, b \in \mathbb{Q}$

**Taniyama-Shimura:**

\[ E \cong (\mathcal{H} \cup \mathbb{P}^1)/\Gamma, \]

$\Gamma \leq \text{SL}_2\mathbb{Z}$ defined by congruences:

- there is $m \in \mathbb{Z}^{>0}$, $A_1, \ldots, A_k \in \text{SL}_2\mathbb{Z}$
- with $A_1 = \text{Id}$, such that
- $\Gamma = \{A \in \text{SL}_2\mathbb{Z} : A \equiv A_i \pmod{m} \text{ for some } i\}$.

\[ \Rightarrow \text{FLT} \]

**obstruction:** is every $\Gamma \leq_n \text{SL}_2\mathbb{Z}$ defined by congruences?

**Weil-Belyi**

\[ E \cong (\mathcal{H} \cup \mathbb{P}^1)/\Gamma \text{ where } \Gamma \leq_n \text{SL}_2\mathbb{Z} \]
Arithmetic groups and congruence subgroups

$V_Q$ a $\mathbb{Q}$-space, $V = V_Q \otimes \mathbb{R}$.

$G \subseteq GL(V)$ a real algebraic $\mathbb{Q}$-group.

$\Lambda (\cong \mathbb{Z}^n)$ free $\mathbb{Z}$-module in $V_Q$,

$$G_\Lambda = \{g \in G : g(\Lambda) = \Lambda\}.$$

$\Gamma \leq G$ arithmetic $\Leftrightarrow$ $\Gamma$ commensurable with $G_\Lambda$ for some $\Lambda$.

$\Gamma \cap G_\Lambda$ finite index in $\Gamma$ and $G_\Lambda$.

$\Gamma \leq G$ semisimple real Lie group ($< \infty$ components) is arithmetic $\Leftrightarrow$ there is a semisimple $G$ as above, and

$$\begin{array}{rcl}
\tilde{G}^o & \xrightarrow{\text{epimorphism } \psi} & G^o \\
p | & \text{covering} & (1). \ker \psi \text{ compact;}
\end{array}$

$\psi(p^{-1}(G_\Lambda^o))$ commensurable with $\Gamma$.

$\Gamma$ arithmetic; $\Gamma(m) = \{\gamma \in \Gamma : \gamma \equiv \text{Id} \pmod{m}\}$;

A subgroup of $\Gamma$ is congruence iff contains a $\Gamma(m)$ for some $m$.

**Congruence Subgroup Problem.** Let $\Gamma \subseteq G$ be arithmetic. Are all finite index subgroups of $\Gamma$ congruence subgroups?
Many positive solutions for $\Gamma \subseteq G$, $\text{rk}_\mathbb{R} G > 1$.

[Serre, Bass, Milnor, Matsumoto, Raghunathan, Platino, ...]

*On the other hand:* $V_\mathbb{K}(n + 1)$-dim. space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

$$B(u, v) = -u_1 \overline{v_1} + \sum_{i \geq 2} u_i \overline{v_i}$$

Hermitian form signature $(1, n)$.

$$\mathbb{K}H^n = \{v \in V_\mathbb{K} \mid B(v, v) < 0\}/v \sim \lambda v,$$

("Projectivised time-like vectors")

gives $n$-dim. $\mathbb{K}$-hyperbolic space.

... the ($\mathbb{R}$-) rank 1 symmetric spaces.

$\text{SU}_{1,n}(\mathbb{K}) = f \in \text{SL}(V_\mathbb{K})$ preserving form $B$,

... gives $\mathbb{R}$-rank 1 simple groups,

$\text{SO}_{1,n}, \text{SU}_{1,n}, \text{Sp}_{1,n}$ and $F_4$. 
Some negative solutions \((\text{rk}_\mathbb{R} G = 1)\)

Conjecture (Serre 1970). \(\Gamma\) arithmetic \(\subseteq G\) simple, \(\text{rk}_\mathbb{R} G = 1\), then \(\Gamma\) fails to have the congruence subgroup property.

Millson’s property: \(\Gamma' \leq \Gamma\) with \(b_1 = \text{rk}_\mathbb{Z} H_1(\Gamma') \neq 0 \Rightarrow \Gamma\) fails CSP.

(i). arithmetic \(\Gamma \subseteq \text{SO}_{1,n}(\mathbb{R}), n \neq 3, 7\), then all have Millson’s property; [Millson, Li, Raghunathan, Venkataramana, Lubotzky, . . . ]

(ii). arithmetic \(\Gamma \subseteq \text{SU}_{1,n}(\mathbb{C})\), all have Millson’s property; [Kazhdan, Shimura, Borel, Wallach]

(iii). \(\Gamma \subseteq\) other \(\text{rk}_\mathbb{R} = 1\) groups \(\Rightarrow\) \(\text{rk}_\mathbb{Z} H_1(\Gamma) = 0\).

Subgroup growth: If \(\Gamma\) to have congruence subgroup property then asymptotically,

\[
\log \sigma_n(\Gamma) \sim \frac{\log n}{\log \log n}
\]

number of subgroups of index \(\leq n\)

But, eg: if \(\Gamma \subseteq \text{SO}_{1,n}, n = 2, 3\) then \(\sigma_n(\Gamma)\) grows quicker than this!
**Property (A).** \( \Gamma \) surjects infinitely many alternating groups \( A_n \).

\[ \cdots \text{arose historically from } \cdots \]

**The Hurwitz Problem:** \( M \) orientable compact (resp. non-compact, finite volume) \( \mathbb{R}H^n \)-manifold; then

\[
|\text{Aut}^+(M)| \leq \frac{\text{vol}(M)}{\text{vol}(\mathbb{R}H^n/\Gamma)},
\]

\( \Gamma \) = uniform (resp. non-uniform) lattice in \( \text{Isom}^+ \mathbb{R}H^n \) of smallest volume.

Which finite (simple) \( G = \text{Aut}(M) \) for \( M \) achieving (1)? \[ \Leftrightarrow \]

Which finite (simple) \( G \) arise as \( 1 \rightarrow \pi_1(M) \rightarrow \Gamma \rightarrow G \rightarrow 1 \)?

<table>
<thead>
<tr>
<th>( n = 2 )</th>
<th>compact</th>
<th>non-compact</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma = \text{triangle group} )</td>
<td>( \Gamma = \text{PSL}_2 \mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z}) )</td>
<td>( \text{&quot;classical Hurwitz problem&quot; [Conder]} )</td>
</tr>
<tr>
<td>( \Gamma = (\mathbb{Z}/2\mathbb{Z}) \rtimes \text{tetrahedral group} )</td>
<td>( \Gamma = \text{tetrahedral group} ) (5) [Everitt]</td>
<td>(6) [Everitt]</td>
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<tr>
<td>( n \geq 4 )</td>
<td>?</td>
<td>?</td>
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Some groups with property (A)

[Pyber-Müller] $A \ast B$ for $A, B$ finite, non-trivial, not both $\cong \mathbb{Z}/2\mathbb{Z}$, surjects almost all $A_n$.

[Everitt, conjectured by G. Higman c1969] Every lattice in $\text{PSL}_2 \mathbb{R} (\cong \text{SO}_{1,2}(\mathbb{R}))$ surjects almost all $A_n$.

$\Rightarrow$ eg: given $p, q, r \in \mathbb{Z}^{>0}$ prime, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, there is $N \in \mathbb{Z}^{>0}$ such that $A_n$ is $(p, q, r)$-generated for all $n \geq N$.

To show $\Gamma = \langle X; R \rangle$ has property (A):

$$
\Gamma \rightarrow \text{Sym}\Omega \quad \cong \quad \downarrow \text{finite covering of } CW\text{-complexes}.
$$

\ldots where $\pi_1(K_0) \cong \Gamma$, eg:

Look for complexes that can be “pasted” together:

$$
\left\{ \begin{array}{c}
K_i \\
\downarrow \\
K_0
\end{array} \right\}_{i=1\ldots n} \quad \longrightarrow \quad [[K_1 \ldots K_n]] \\
\downarrow \\
K_0
$$

$$
\{ \Gamma \rightarrow \text{Sym}\Omega_i \}_{i=1\ldots n} \quad \Gamma \rightarrow \text{Sym}(\cup_i \Omega_i)
$$

Finally, use classical “recognition theorems” for $A_n$. 
\( \Gamma \subseteq G \) arithmetic with property (A) \( \Rightarrow \Gamma \) fails to have the congruence subgroup property.

**Question.** Does every lattice \( \Gamma \subseteq G \), simple, \( \text{rk}_\mathbb{R} = 1 \) have property (A)?

<table>
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<tr>
<th>(A)</th>
<th>CSP</th>
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<td>( \text{SO}_{1,2}(\mathbb{R}) )</td>
<td>all ( \Gamma )</td>
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<tr>
<td>( \text{SO}_{1,n}(\mathbb{R}), n \geq 3 )</td>
<td>some examples</td>
</tr>
<tr>
<td>( \text{SU}_{1,n}(\mathbb{C}) )</td>
<td>?</td>
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<tr>
<td>( \text{Sp}_{1,n}(\mathbb{H}) )</td>
<td>?</td>
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<tr>
<td>( F_4 )</td>
<td>?</td>
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\( \text{(*) some exceptions when } n = 7 \text{) } \)