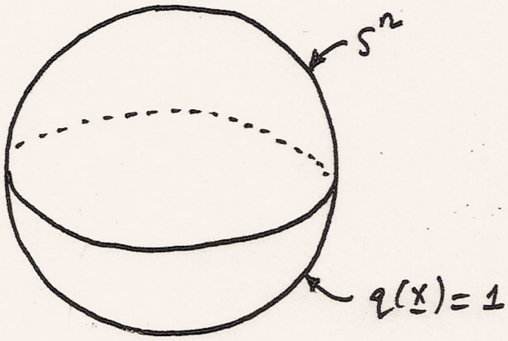
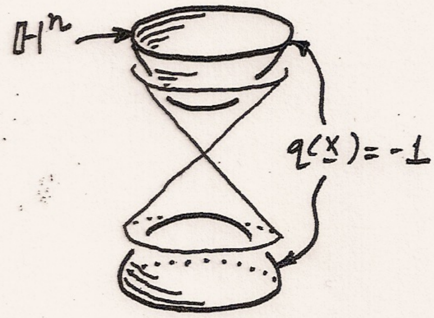
Geometries: $X = S^n, \mathbb{E}^n, \mathbb{H}^n$.

$$q(\underline{x}) = \sum x_i^2 \longleftarrow (\mathbb{R}^{n+1}, q) \longrightarrow q(\underline{x}) = -x_{n+1}^2 + \sum x_i^2$$

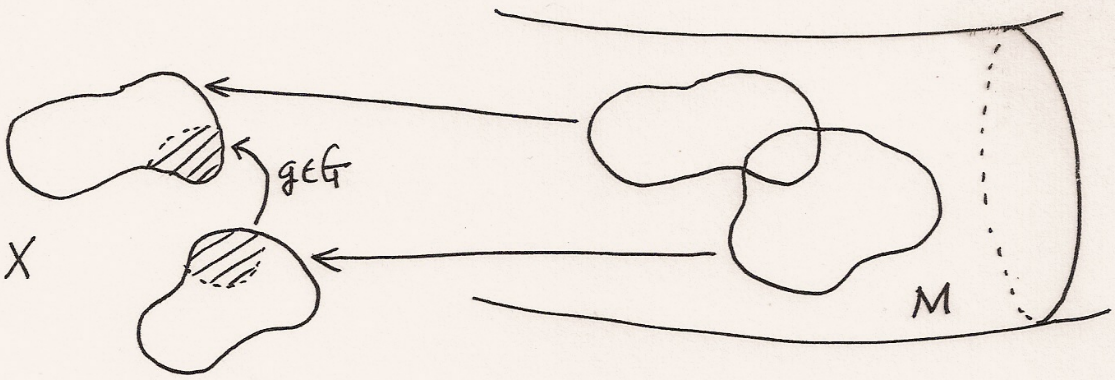




Geometric manifolds:

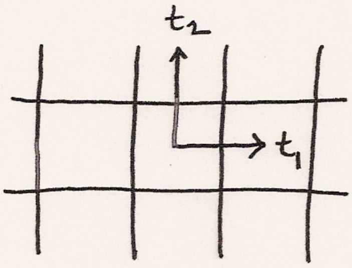
(X,G)-manifold M $\left\{ \begin{array}{l} X = S^n, \mathbb{E}^n, \mathbb{H}^n \\ G = \operatorname{Isom} X \end{array} \right.$

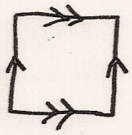
($\equiv X$-manifold)

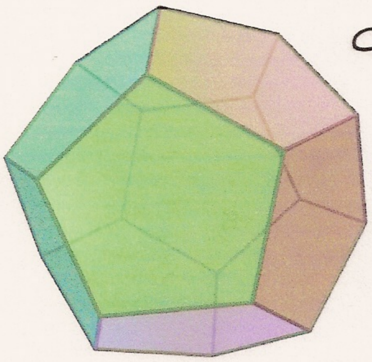


Killing-Hopf: X-manifolds = Clifford Klein space forms X/Π

= X/Π with $\Pi \subset \mathrm{Isom}\,X$ discrete, torsion free.



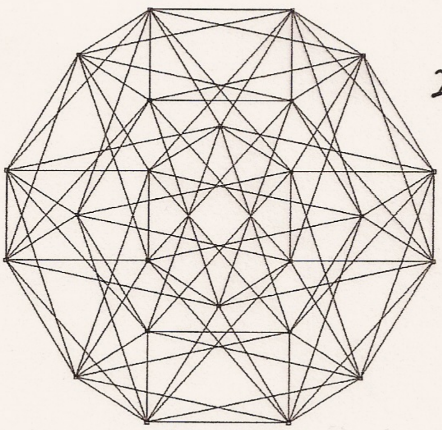
$\mathbb{E}^2/\langle t_1, t_2\rangle =$  flat torus



$\subset S^3$

$S^3/\Pi =$

Poincaré homology sphere



24-cell $\subset \mathbb{H}^4$

$\mathbb{H}^4/\Pi =$

Ratcliffe-Tschantz manifold

All are "reflection manifolds" $\overset{\text{def}}{\Longleftrightarrow}$ $\Pi \subset$ reflection group acting on X.

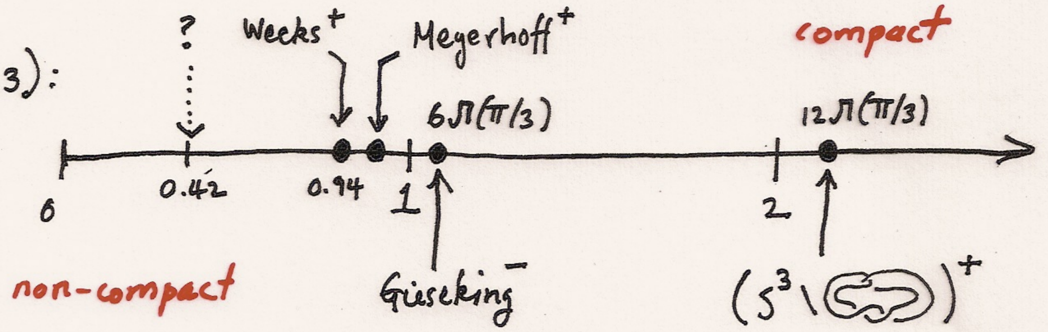
Mostow: $X = \mathbb{H}^n \ (n \neq 2) \Rightarrow$ isometry type of $M = \mathbb{H}^n/\Pi$ determined by Π.

$M =$ n-dimensional hyperbolic manifold $\begin{cases} \text{complete} \\ \text{finite volume} \\ \text{without } \partial \end{cases}$

$\left.\begin{array}{l} \text{Jørgensen} \\ \text{Thurston} \\ \text{Wang} \end{array}\right\} \Rightarrow$ for fixed n, set of volumes of such well-ordered subset of $\mathbb{R}$ (discrete for $n \neq 3$)

(weak) Siegel problem: for fixed n, what is the minimum volume of a hyperbolic (n-manifold) n-orbifold?

Eg ($n=3$):

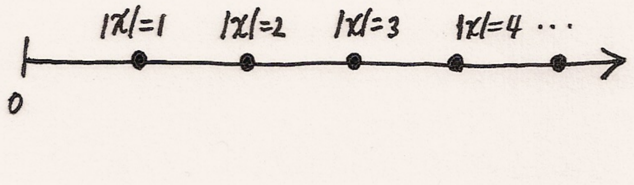


($n>3$):

n even

Gauss-Bonnet-Chern:

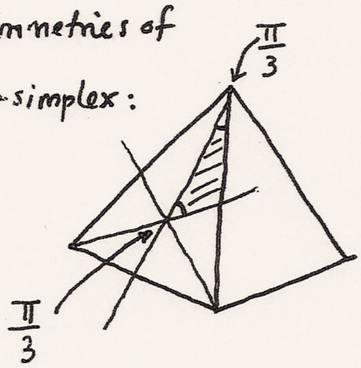
$$\mathrm{vol}(M) = \frac{(-1)^{n/2}}{2}\mathrm{vol}(S^n)\chi(M)$$

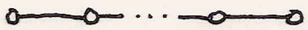


n odd

estimates for Margulis constant $\varepsilon(n)$.

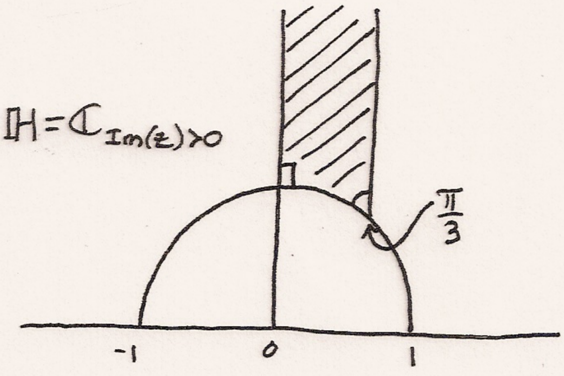
symmetries of n-simplex:





$(\cong \mathfrak{S}_{n+1}$, Weyl group of $\mathfrak{sl}_{n+1}\mathbb{C})$

$\mathbb{H} = \mathbb{C}_{\mathrm{Im}(z)>0}$



$(\cong GL_2\mathbb{Z}/\text{center})$

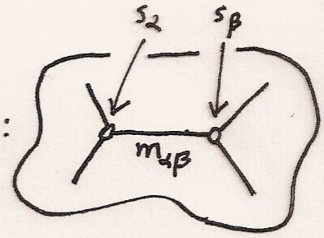
Coxeter groups:

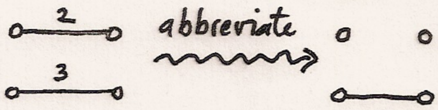
$$\underset{\text{finite}}{S} \subseteq \underset{\text{group}}{W}$$

$$(W,S) = \langle s_\alpha \in S \,;\, (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \rangle$$

$$m_{\alpha\beta} \in \mathbb{Z}^{\geq 1} \cup \{\infty\}, \quad m_{\alpha\beta} = 1 \Leftrightarrow \alpha = \beta.$$

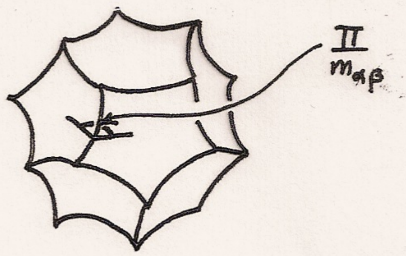
symbol:





Coxeter polytopes:
in $X = S^n, \mathbb{E}^n, \mathbb{H}^n$

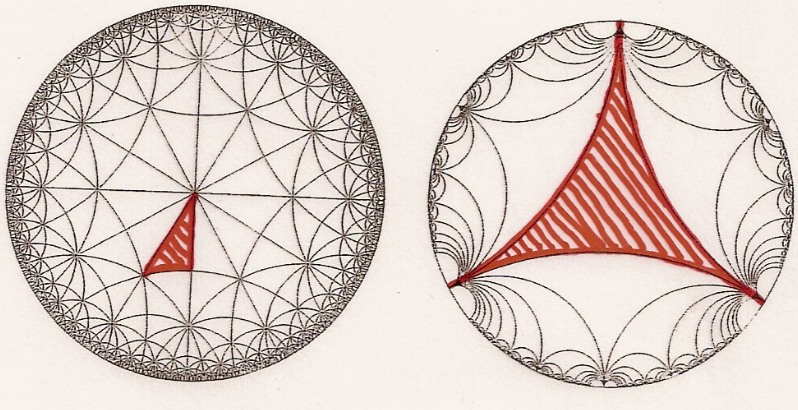
$\bigcap$ half spaces:



$\underline{X = S^n, \mathbb{E}^n}$

exist for all n

$\underline{X = \mathbb{H}^n}$



compact
$n \leq 29$ (8)

non-compact (finite volume)
$n \leq 997$ (21)

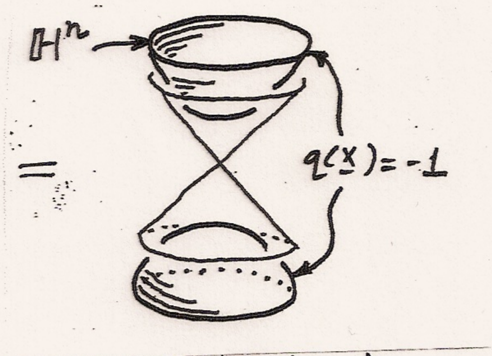
Eg: of a construction using Coxeter groups

$I_{n,1} = \mathbb{Z}^{n+1}$ with

$\langle \cdot , \cdot \rangle : \mathbb{Z}^{n+1} \to \mathbb{Z}$

$\langle \underline{x}, \underline{y} \rangle = -x_1 y_1 + \sum_{i>1} x_i y_i$

$\Rightarrow \Lambda \otimes \mathbb{R} =$



$q(\underline{x}) = \langle \underline{x}, \underline{x} \rangle$

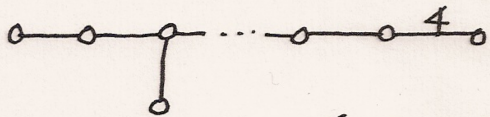
(Lorentzian lattice = free R-module with $(1,n)$-Hermitian form)

ring of integers in N° field

$I_{n,1}$ = the odd self-dual Lorentzian lattice rank $n+1$ {Serre-Milnor

$\mathrm{Aut}(I_{n,1}) = I_{n,1} \to I_{n,1}$ preserving $\langle \cdot , \cdot \rangle$

Acts on $\mathbb{H}^n$ as a Coxeter group:



$= W$

$(4 \leq n \leq 9)$

① find $W \xrightarrow{\varphi} F$ (finite)

$\ker\varphi$ torsion free

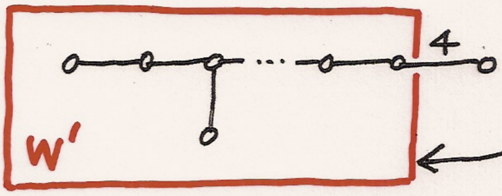
$\widehat{M} = \mathbb{H}^n / \ker\varphi$

② find $H \subset F$ such that

$g \in W$ finite order $\Rightarrow \varphi(g) \notin H$.

$H \subset \mathrm{Aut}(\widehat{M})$ acting freely

$\widehat{M} \to \widehat{M}/H$

$W =$ 

Weylgroup :

n	4	5	6	7	8
W'	A_4	D_5	E_6	E_7	E_8

---

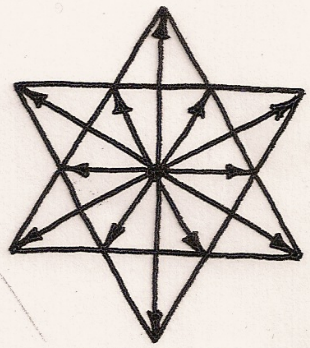
finding $W \longrightarrow F$ (finite) $\Longleftrightarrow$ construct finite $F = \langle s_1, \ldots, s_k \rangle$

with s_i satisfying W-relations

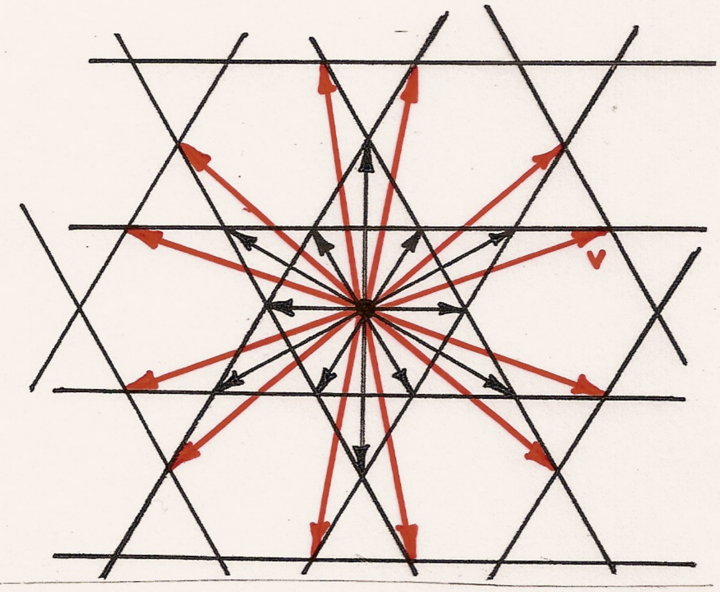
Weylgroup = symmetries of a

<u>rootsystem</u> $\Phi \subset \mathbb{R}^n$.

$L = \mathbb{Z}$-span Φ the <u>root lattice</u> in $\mathbb{R}^n$.



G_2 root system $\subset \mathbb{R}^2$



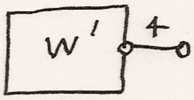
$\Lambda = \mathbb{Z}$-span$\{W'$-orbit of $v\}$

red vectors

$\cong \mathbb{Z}^n$

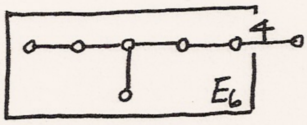
W' acts on $\Lambda / 2\Lambda$ $(\cong \mathbb{Z}_2^n)$

$\Rightarrow$ form $W' \ltimes (\Lambda / 2\Lambda)$

φ :  $\longrightarrow W' \ltimes (\Lambda/2\Lambda) = \langle W', v \rangle$

(actually F slightly bigger...)

Eg $(n=6)$:

$W =$  $\xrightarrow{\varphi} \mathbb{Z}_2 \ltimes (E_6 \ltimes \mathbb{Z}_2^6)$

$$\hat{M} = \mathbb{H}^6 / \ker \varphi$$

$H \subset \mathrm{Aut}(\hat{M})$ acting freely on $\hat{M}$

$$M = \hat{M} / H$$

How big?

N° theory

$$\mathrm{vol}(M) = 2^3 |E_6| \, \mathrm{covol}(I_{6,1})$$

$$\mathrm{covol}(I_{n,1}) = \frac{2^{\frac{n}{2}-1}}{n!} \pi^{\frac{n}{2}} \prod_{k=1}^{\frac{n}{2}} |B_{2k}|$$

(Siegel)

Topology

$$\mathrm{vol}(M) = \frac{(-1)^{\frac{n}{2}}}{2} \mathrm{vol}(S^n) \boxed{\chi(M)}$$

$$\boxed{\chi(M) = 2^3 |E_6| \boxed{\chi(W)}}$$

$$\boxed{\sum_{\Delta} (-1)^{|\Delta|} \chi(\Gamma \setminus \Delta) = 0}$$

(Serre)

$$\mathrm{vol}(M) = \frac{8}{15}\pi^3 \quad \text{or} \quad \chi(M) = -1$$