The homotopy theory of Khovanov homology

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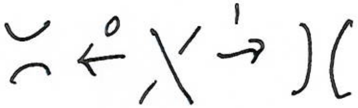
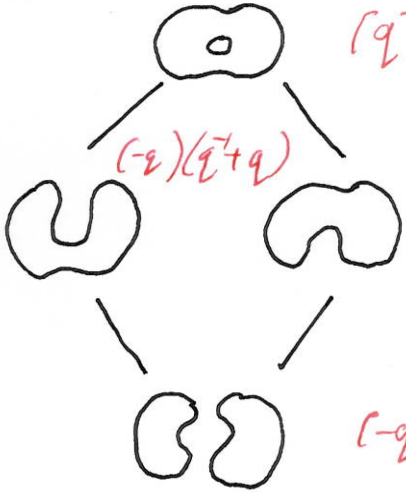
(joint with Paul Turner (Geneva))

① Khovanov homology:



Kauffman $\longrightarrow$

$X \longmapsto (-q)^{|X|}(q^{-1}+q)^{\#S^1}$



$\Sigma = q^{-2}+q^0+q^2+q^4$

$= \langle$  $\rangle$

$X \longmapsto V^{\otimes \# S^1}[|X|]$

$\left(V = \underset{-1}{\mathbb{Q}} \oplus \underset{1}{\mathbb{Q}} \quad V[1] = \underset{0}{\mathbb{Q}} \oplus \underset{2}{\mathbb{Q}} \right.$

$\left. \dim = q^{-1}+q \quad \dim = q \cdot \dim V \right)$

oriented D $\begin{cases} s & \text{(number of negative crossings)} \\ t & \text{(number of positive crossings)} \end{cases}$

$$\begin{array}{ccccc} & & V^{\otimes 2} & & \\ & \overset{m}{\swarrow} & & \overset{m}{\searrow} & \\ V[1] & & & & V[1] \\ & \underset{\Delta}{\searrow} & & \underset{-\Delta}{\swarrow} & \\ & & V^{\otimes 2}[2] & & \end{array} \rightsquigarrow$$

complex	$\overline{Kh}^{i,q}$	dim
C^0	$\underset{-2}{\mathbb{Q}} \oplus \underset{0}{\mathbb{Q}}$	$q^{-2}+q^0$
C^1	0	
C^2	$\underset{2}{\mathbb{Q}} \oplus \underset{4}{\mathbb{Q}}$	q^2+q^4

χ

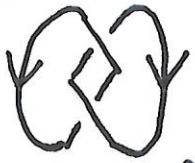
Jones

$(-1)^s (q)^{t-2s} \langle D \rangle$

$\downarrow \qquad \uparrow \chi$

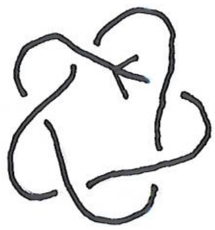
$\overline{Kh}^{i+s,\, q+2s-t}(D)$

Khovanov



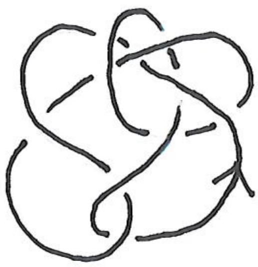
$q^{-6} + q^{-4} + q^{-2} + q^{0}$

1	1		
		1	1



$-q^{-15} + q^{-7} + q^{-5} + q^{-3}$

1						
		1				
		1				
				1		
					1	1

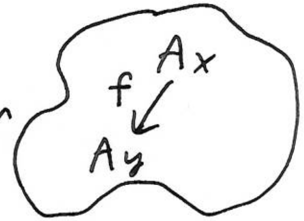


$-q^{-15} + q^{-7} + q^{-5} + q^{-3}$

1							
		1					
		1					
			1	1			
			1		1		
					2		
							1
						1	1

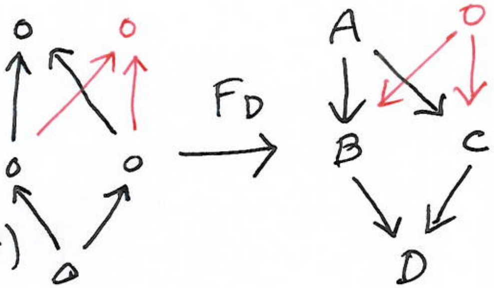
② (pre)Sheaves (alg. mumbo-jumbo):

presheaf: $C \xrightarrow{F} Ab$ (contravariant)

$\varprojlim_C F = \varprojlim$ 

$= \{(\ldots, a_x, \ldots, a_y, \ldots) \mid a_x \overset{f}{\longmapsto} a_y, \text{etc}\ldots\}$

Eg:



(poset)

$\varprojlim F_D = \ker\begin{pmatrix} & A & \\ \swarrow & & \searrow \\ B & & C \end{pmatrix} = \overline{Kh}^0\Big($  $\Big)$

$\left(\substack{\text{gp.}\\ \mathbb{Z}/2}\right) \circ\circlearrowleft \xrightarrow{F_G} A \circlearrowright f \quad (f^2 = 1)$

$\mathbb{Z}/2$-module

$\varprojlim F_G = \{a \mid f(a) = a\} = H^0(\mathbb{Z}/2, A)$

$0 \to H \to F \to G \to 0$ (SES presheaves)

$\rightsquigarrow$

$0 \to \varprojlim H \to \varprojlim F \to \varprojlim G \to \varprojlim^1 H \to \varprojlim^1 F \to \cdots$

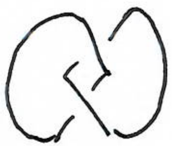
$\varprojlim^i F$ "higher limits" $(i \geq 0)$

Theorem 1: D link diagram, F_D Khovanov presheaf

$\varprojlim^i F_D \cong \overline{Kh}^i(D)$

(c.f. $\varprojlim^i F_G \cong H^i(G, A)$)
gp. cohomology

③ Homotopy:

 $\longrightarrow$

$$\begin{array}{ccc} A & & 0 \\ \downarrow & \times & \downarrow \\ B & & C \\ & \searrow\ \swarrow & \\ & D & \end{array}$$

presheaf

$\xrightarrow[\text{EM functor}]{K(-,n)}$

$$\begin{array}{ccc} X_n & & * \\ \downarrow & \times & \downarrow \\ Y_n & & Z_n \\ & \searrow\ \swarrow & \\ & W_n & \end{array}$$

diagram of spaces

$\lim^i \downarrow$ (from D) $\quad$ holim $\downarrow$ (from W_n)

$\overline{Kh}^i(\;) \xleftarrow[(n \gg 0)]{\pi_{n-i}} \overline{S}_n(\;)$

space (spectrum)

$Kh^i(D) = \overline{Kh}^{i+s}(D)$

$\downarrow \qquad \uparrow \pi_{n-i}$

$S_n(D) = \Omega^{-s}\overline{S}_n(D)$

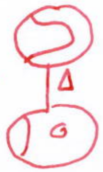
<u>Theorem 2</u>: the homotopy type of $S_n(D)$ is an invariant

<u>Eg</u>: Reidemeister move: $(\;) \longleftrightarrow (\;)$

$Kh^i(\;) \text{ "=" } \overline{Kh}^{i+1}(\;) \cong \pi_{n-(i+1)}\overline{S}_n(\;) = \pi_{n-(i+1)}\Omega\overline{S}_n(\;) \cong \pi_{n-i}\overline{S}_n(\;)$

$\cong \overline{Kh}^i(\;)$

$\text{ "=" } Kh^i(\;)$

$\overline{S}_n(\;) = \text{holim}$  $\simeq \Omega\, \text{holim}$  $= \Omega\overline{S}_n(\;)$