Coloured poset homology

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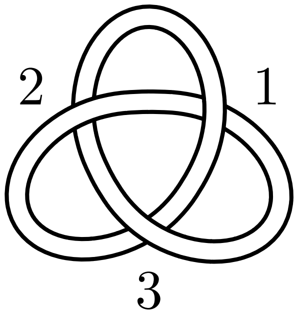
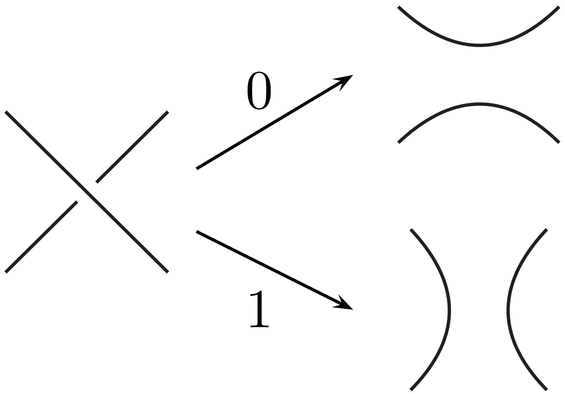
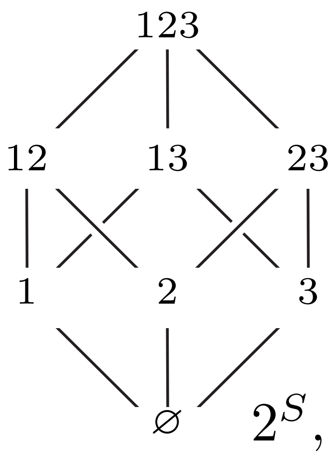
arXiv:0808.1686
0711.0103

Example: Jones polynomial





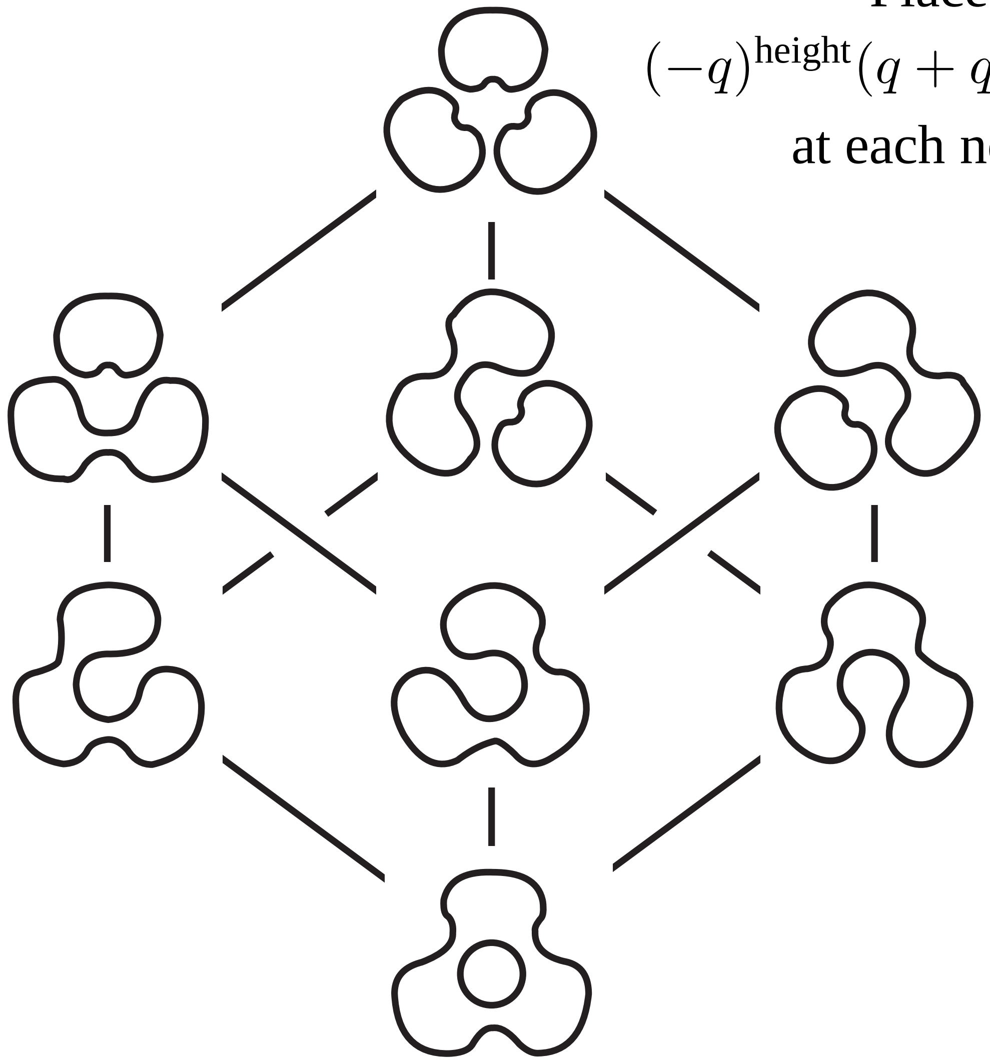


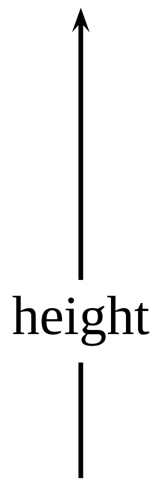
$2^S, S = \{1, 2, 3\}$

Place

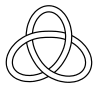
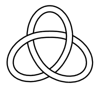
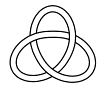
$(-q)^{\text{height}}(q + q^{-1})^{\#\text{circles}}$

at each node

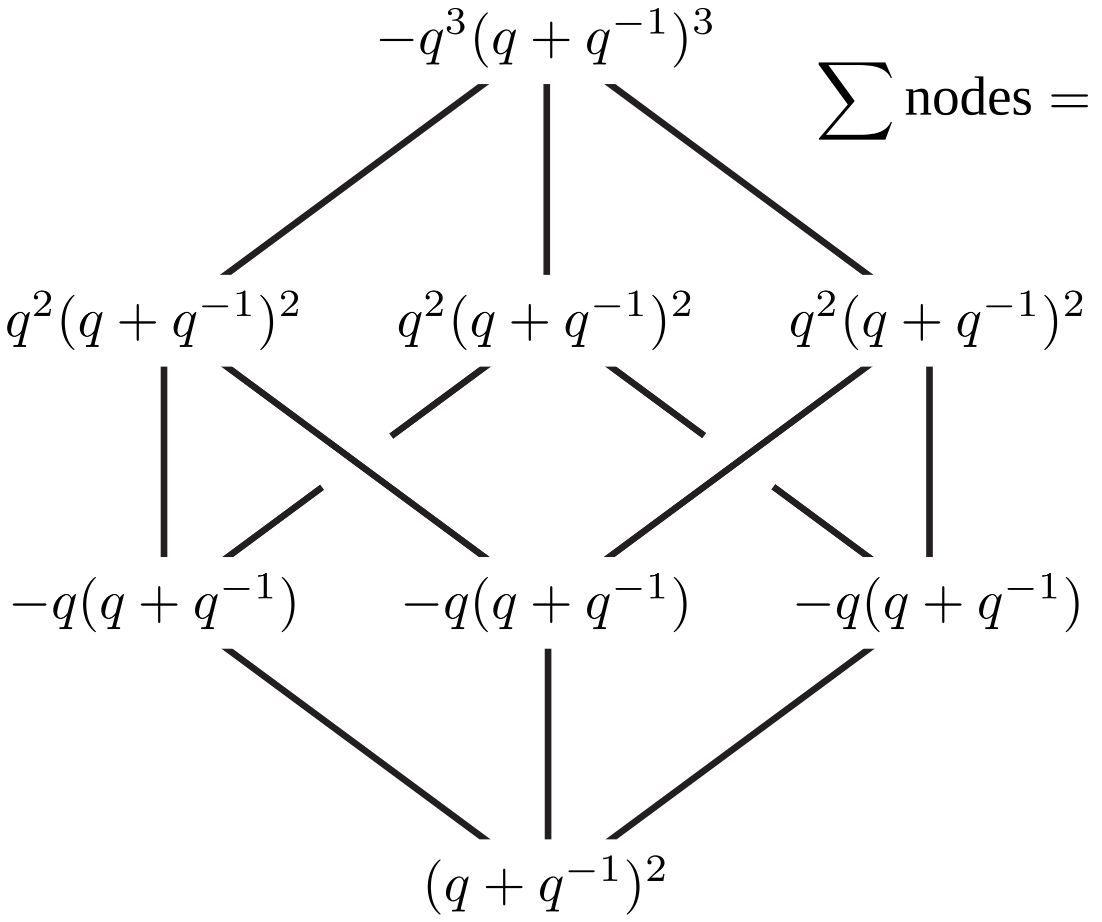


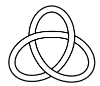


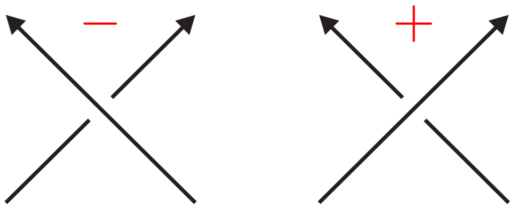
Example: Jones polynomial

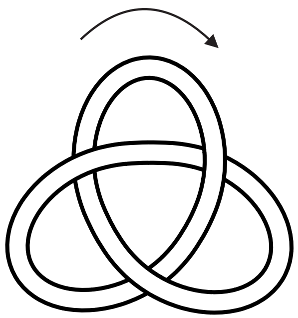
$J\Big($  $\Big) = \dfrac{1}{(q+q^{-1})}\widehat{J}\Big($  $\Big) \longleftarrow (-1)^{n_-}q^{n_+-2n_-}\Big\langle$  $\Big\rangle$

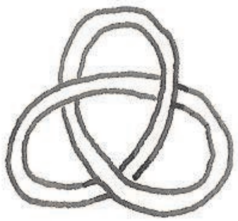
(Jones) (unnormalized Jones) (Kauffman bracket)



$\sum \text{nodes} = \Big\langle$  $\Big\rangle = -q^6+q^2+1+q^{-2}$





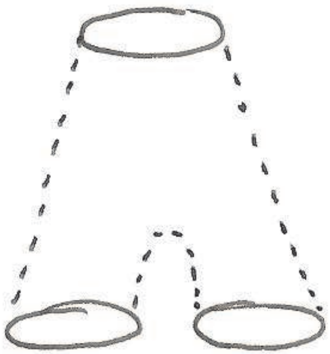


$\xrightarrow{\text{Jones}}$ $f \in \mathbb{Z}[q, q^{-1}]$

link L_2

$\uparrow$

link L_1



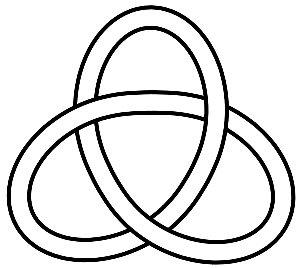
$\xrightarrow{\text{functor}}$

$$\begin{array}{c} \bigoplus_i \bigoplus_j M_{ij} \\ \uparrow \\ \bigoplus_i \bigoplus_j N_{ij} \end{array}$$

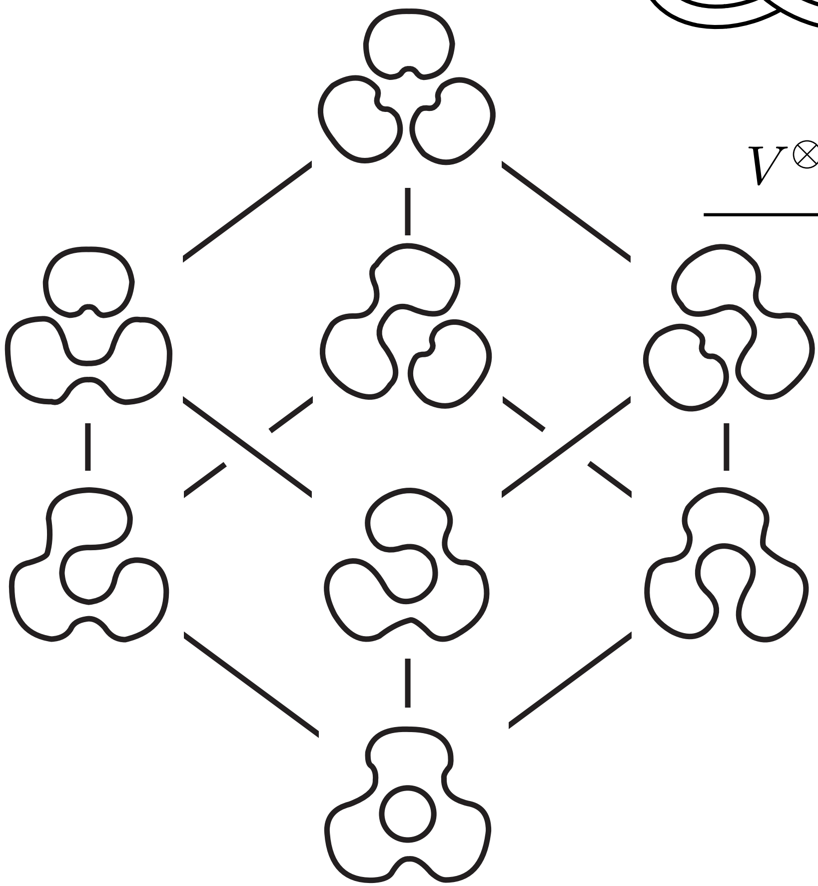
(bi-graded spaces)

$$\chi\left(\bigoplus_i \bigoplus_j N_{ij}\right) = \sum_i (-1)^i \mathrm{qdim}\left(\bigoplus_j N_{ij}\right) = \sum_i (-1)^i \sum_j (\dim N_{ij}) q^j = \text{“Jones”}$$

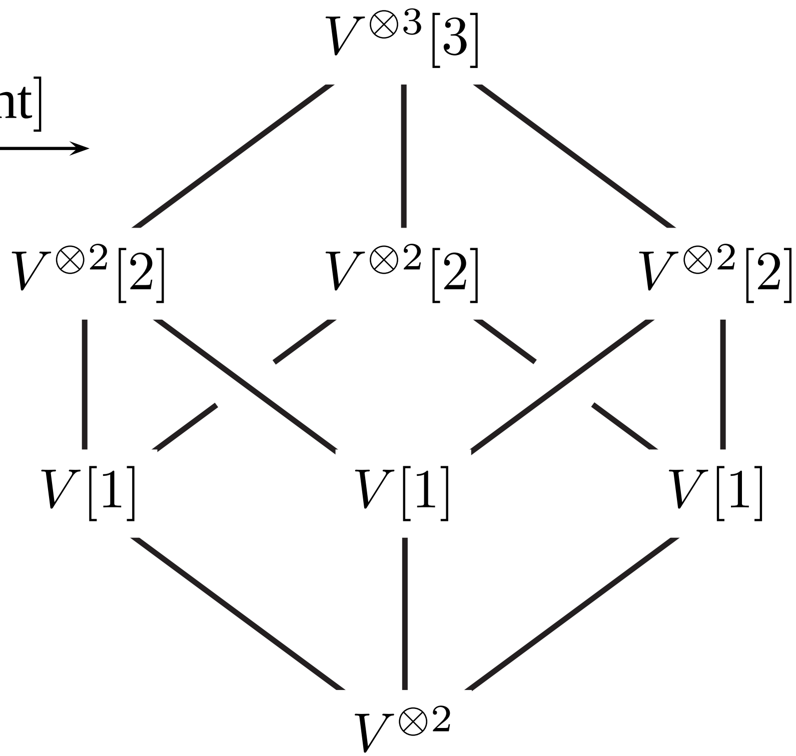
Example: Khovanov complex 1



$$V = \mathbb{Q} \oplus \mathbb{Q}$$
$$\qquad\; -1 \quad\; 1$$

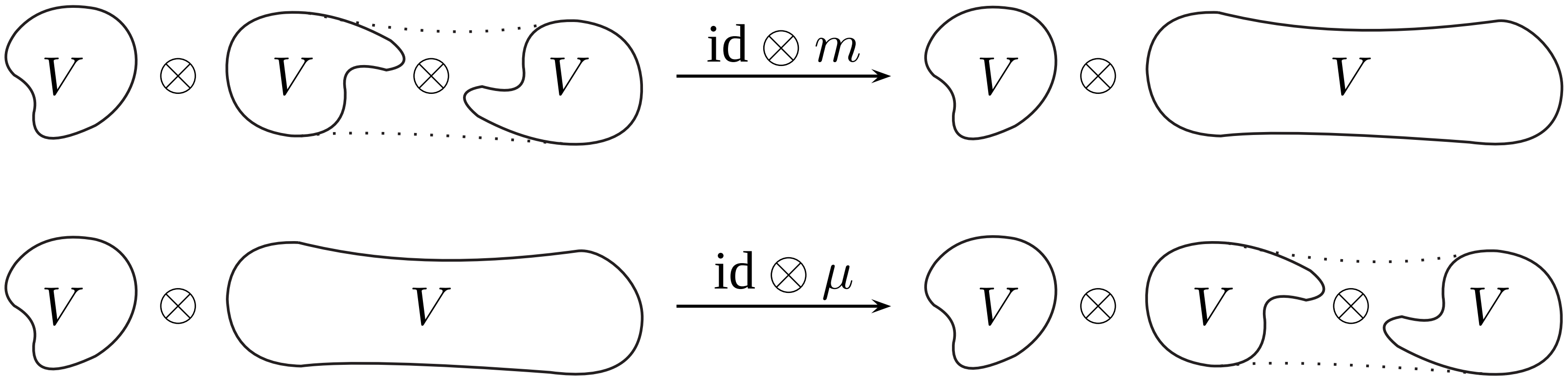


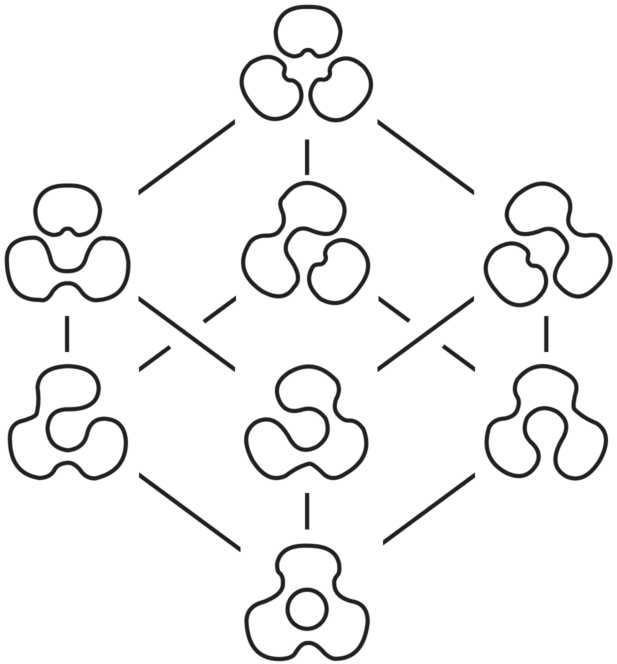
$$\xrightarrow{V^{\otimes \#\text{circles}}[\text{height}]}$$



$m : V \otimes V \to V$

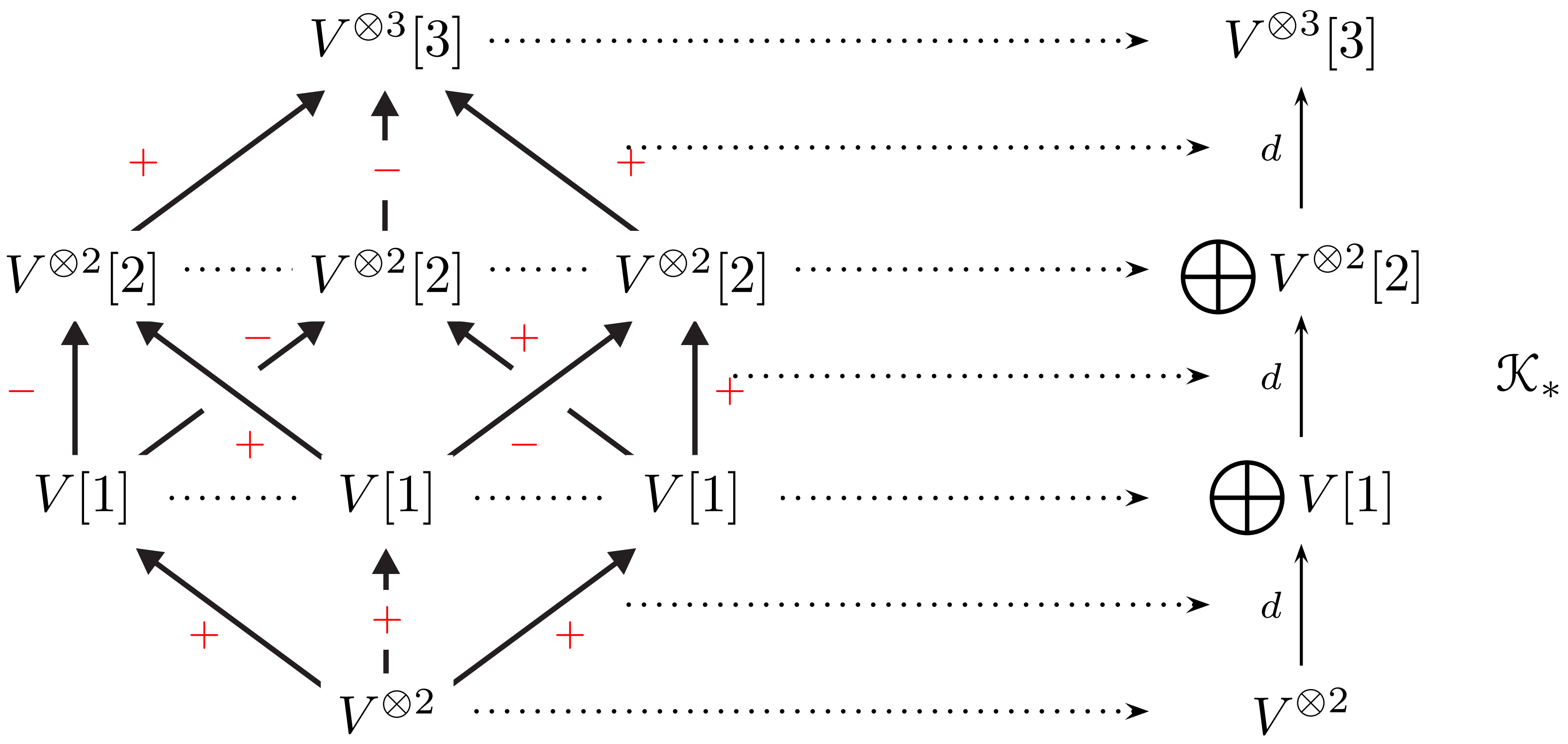
$\mu : V \to V \otimes V$





$$
\begin{tikzcd}
 & V^{\otimes 3}[3] & \\
V^{\otimes 2}[2] \arrow[ur, "\mathrm{id}\otimes\mu"] & V^{\otimes 2}[2] \arrow[u, "\mathrm{id}\otimes\mu"] & V^{\otimes 2}[2] \arrow[ul, "\mathrm{id}\otimes\mu"'] \\
V[1] \arrow[u, "\mu"] \arrow[ur, "\mu"] & V[1] \arrow[ul, "\mu"'] \arrow[ur, "\mu"] & V[1] \arrow[ul, "\mu"'] \arrow[u, "\mu"'] \\
 & V^{\otimes 2} \arrow[ul, "m"] \arrow[u, "m"] \arrow[ur, "m"'] &
\end{tikzcd}
$$

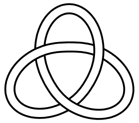
(squares *commute*)



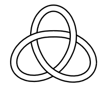
add $\pm$'s to edge maps so squares *anticommute*

Khovanov homology $KH_*\left(\quad , \mathbb{Q}\right) = H_*(\mathcal{K}_*)$

Example: Khovanov homology 1

	6	4	2	0	-2	$q\dim$
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1+q^{-2}$

Euler characteristic $\chi(\mathcal{K}_*)$

$= \sum(-1)^i q\dim KH_i\Big($  $,\mathbb{Q}\Big)$

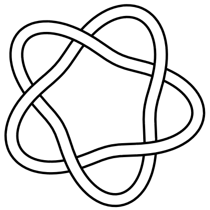
$= q^6 - q^2 - 1 - q^{-2}$

key property 1, essentially by construction:

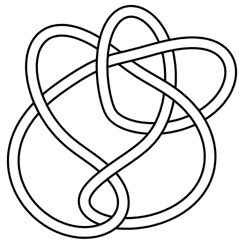
$$(-1)^{1+n_-}q^{n_+-2n_-}\chi(\mathcal{K}_*) = \widehat{J}(K)$$

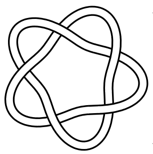
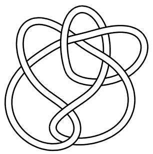
key property 2, and minor miracle: KH_* an invariant (after a bit of nudging)

$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$



$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$

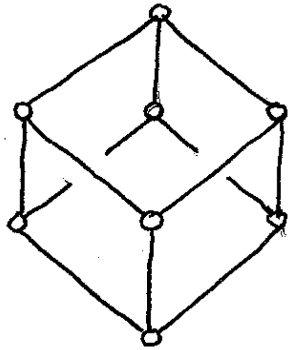


$J\Bigg($  $\Bigg) = J\Bigg($  $\Bigg)$

...similarly...

- Alexander polynomial: Heegaard-Floer homology (Ozsváth and Szabó)
- HOMFLY polynomial: Khovanov-Rozansky homology
- chromatic polynomial: graph homology (Helme-Guizon and Rong)

- Eg: X set, $\mathbb{B}(X) :=$ Boolean lattice on X



$X = \{1,2,3\}$

- Eg: (W,S) Coxeter group $:= \langle s_\alpha \in S \mid (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \rangle$

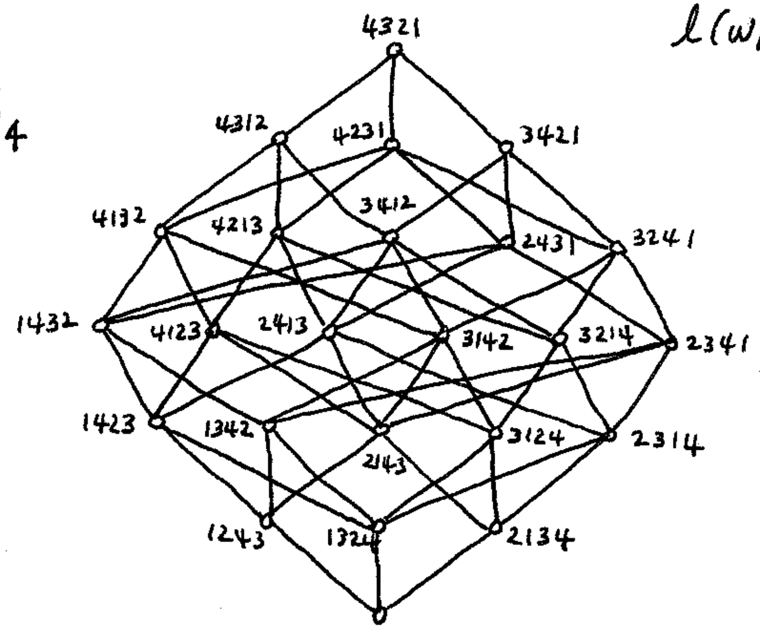
  $(m_{\alpha\beta} = m_{\beta\alpha},\ m_{\alpha\alpha} = 1 \iff \alpha = \beta)$

  Bruhat order: $w < w' \iff$

  $$w = w_0 \to w_1 \to \cdots \to \overbrace{w_i \to w_{i+1}}^{w_i t = w_{i+1}\ (t \in W^{-1}SW)} \to \cdots \to w_k = w'$$

  $$\underbrace{w_i \to w_{i+1}}_{\ell(w_i) < \ell(w_{i+1})}$$

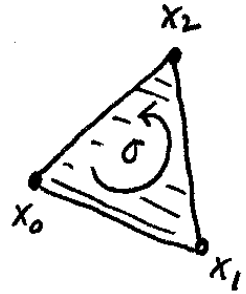
- Eg: $W = \mathfrak{S}_4$



$(\mathbb{B}(X) = \text{Bruhat poset for } \mathbb{Z}_2^{|X|})$

(oriented)
n-cells

- Poset $P \rightsquigarrow$ order (simplicial) complex $\Delta(P)$ $\quad \sigma = x_0 < \cdots < x_n$



$x_0 < x_1 < x_2$

- <u>poset</u> <u>homology</u> $H_*(P,R) := H_*(\Delta(P-\hat{0},\hat{1}),R)$

Folkman complex

- <u>Eg</u>: [Folkman] P finite geometric lattice rank r

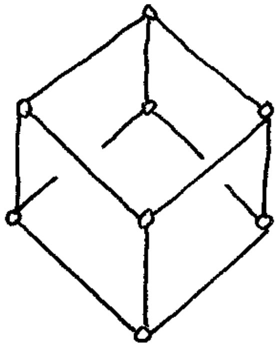
$$H_n(P,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^{|\mu(\hat{0},\hat{1})|}, & n=r-2 \\ 0, & \text{otherwise} \end{cases} \qquad (\mu = \text{Möbius } f^{\underline{cn}} \text{ on } P)$$

• (covariant) functor $\mathcal{F}: P \to R$-mod

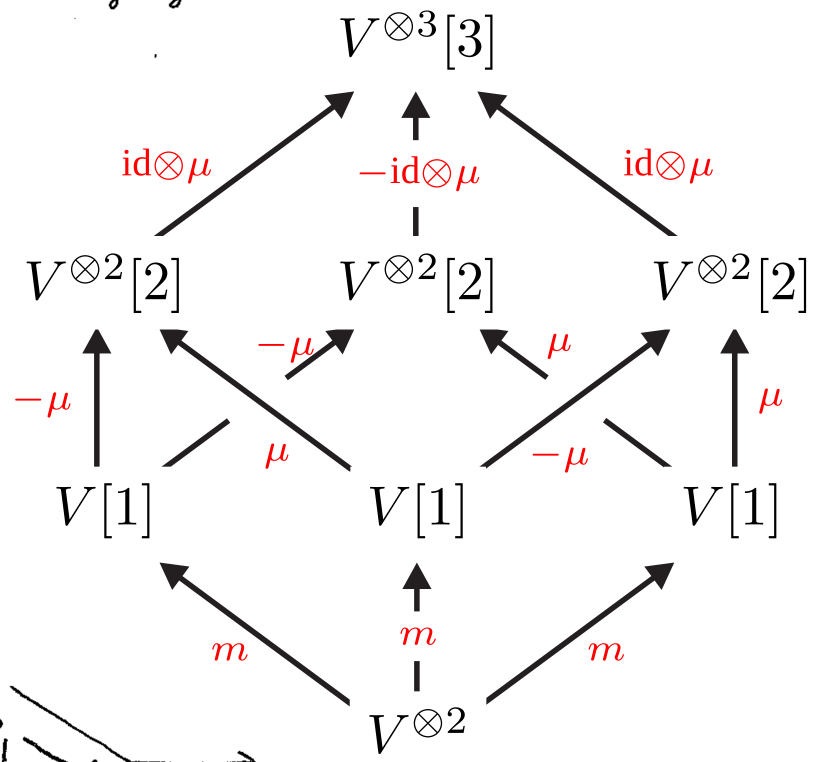
↳ poset as category

(... pre-sheaf modules over small category ...)

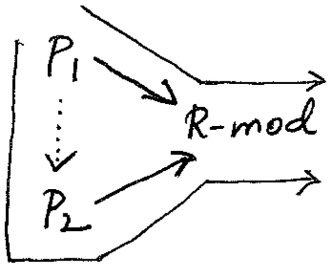
• Eg:



(a) Khovanov colouring



(category $\mathcal{CP}_R$ with morphisms natural transformations:

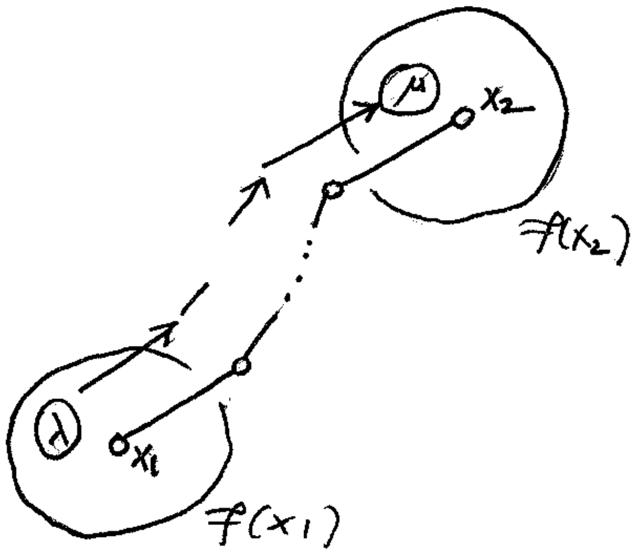


(all P have a $\hat{1}$)

- complex S_*: $S_k(P,\mathcal{F}) := \bigoplus_{x_1 \leq \cdots \leq x_k \leq \hat{1}} \mathcal{F}(x_1)$, $S_0(P,\mathcal{F}) := \mathcal{F}(\hat{1})$

<u>differential</u> $\lambda(x_1 \leq \cdots \leq x_k) \overset{d}{\longmapsto} \mu(\hat{x}_1 \leq x_2 \leq \cdots \leq x_k)$

$(\lambda \in \mathcal{F}(x_1))$ $\quad - \sum_{i=2}^{k} (-1)^i \lambda(x_1 \leq \cdots \leq \hat{x}_i \leq \cdots \leq x_k)$



- coloured poset homology $H_*(P,\mathcal{F}) := H(S_*(P,\mathcal{F}))$

(... homology with coefficients in the pre-sheaf ...)

Khovanov = sheaf

- $\underline{\text{Th}^{\underline{m}}\ 1}$ [E-T]: $KH_*(\mathbb{B},\mathcal{F}) \xrightarrow{\cong} H_*(\mathbb{B},\mathcal{F})$

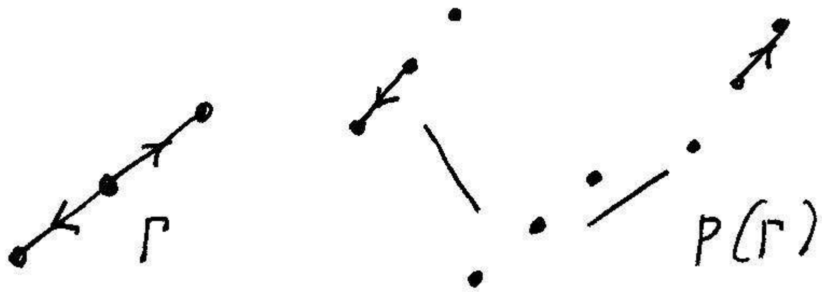
  Boolean

ie: $\left.\begin{array}{l}\text{Khovanov}\\ \text{Heegaard-Floer}\\ \cdots\end{array}\right\}$ homologies = sheaf homology

- extension: $\underline{\text{Th}^{\underline{m}}\ 2}$ [E-T]: $KH_*(P,\mathcal{F}) \xrightarrow{\cong} H_*(P,\mathcal{F})$

  $\in$ "large class" of posets including all Bruhat posets

- A (associative) R-algebra
- $P(\Gamma)$ = <u>quiver</u> poset of directed graph Γ



- $P(\Gamma) \underset{\mathcal{F}_A}{\overset{\text{colour}}{\rightsquigarrow}}$

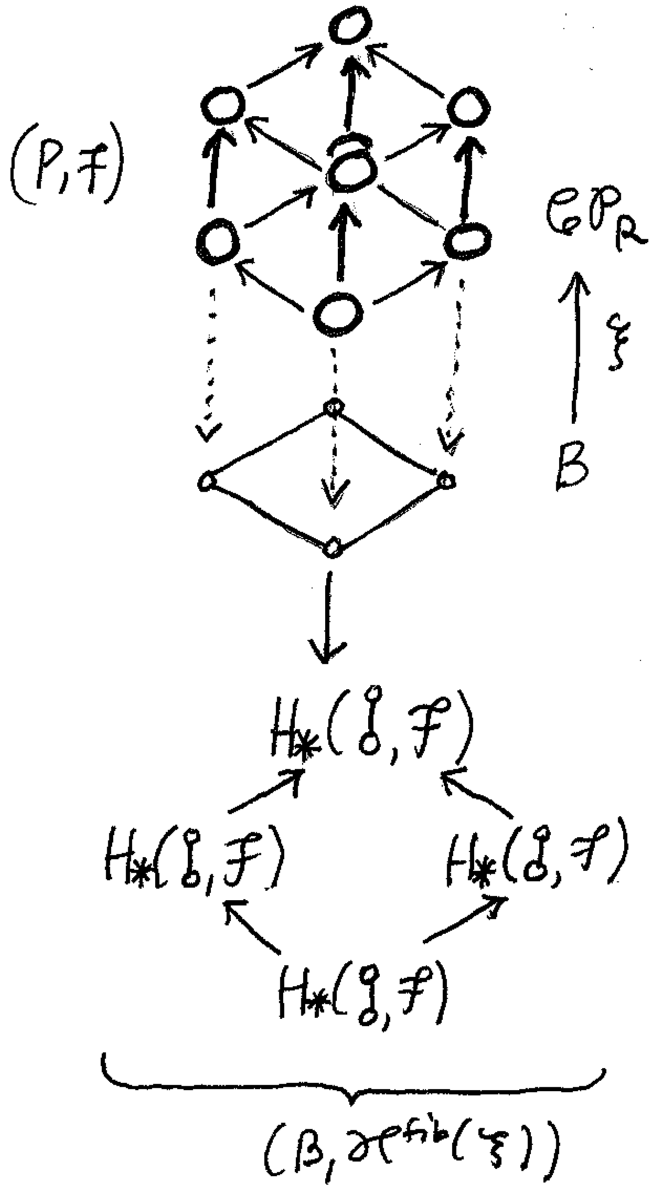
$$\begin{tikzcd} A\otimes A & & A\otimes A \\ & A\otimes A\otimes A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

- <u>Corollary</u> (T-Wagner):

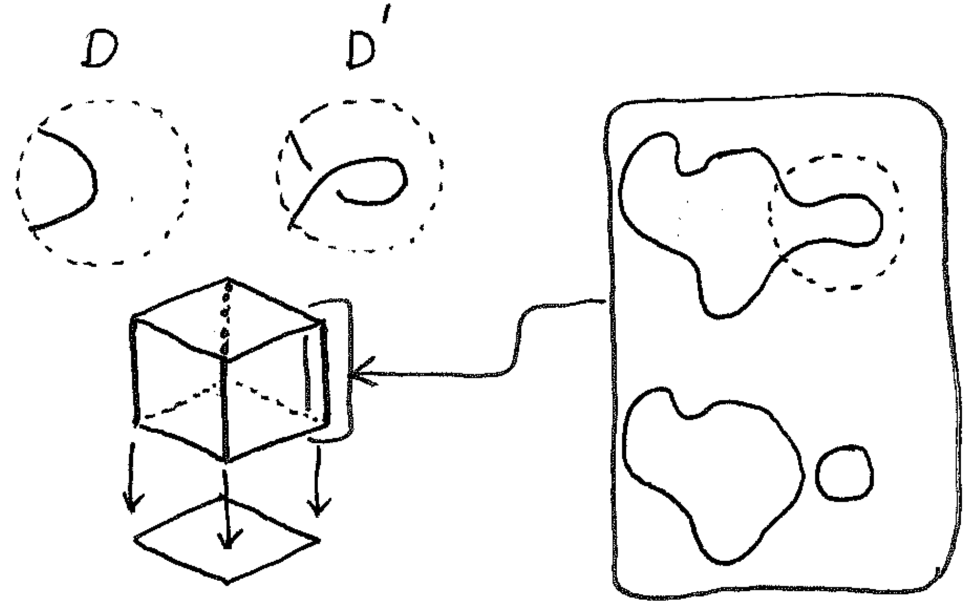
$$H_i(P(n\text{-gon}), \mathcal{F}_A) \cong HH_i(A, \mathbb{Z})$$

(Hochshild)

Bundles (of coloured posets)



• Eg:



• Thm 3 [E–T]: for a "large class" of bases B, there is a Leray–Serre style spectral sequence

$$E^2_{pq} = H_p(B, \mathcal{H}^{fib}_q(\xi)) \Rightarrow H_*(P, \mathcal{F})$$